

LECTURE 2

World lines of extremal length

0. Reminder: The Twin Paradox,

I. The WHY of extremal length

II. Generalization

III. The Variational Principle,

IV. Parametrization

V. Torsionless metric compatible

transport

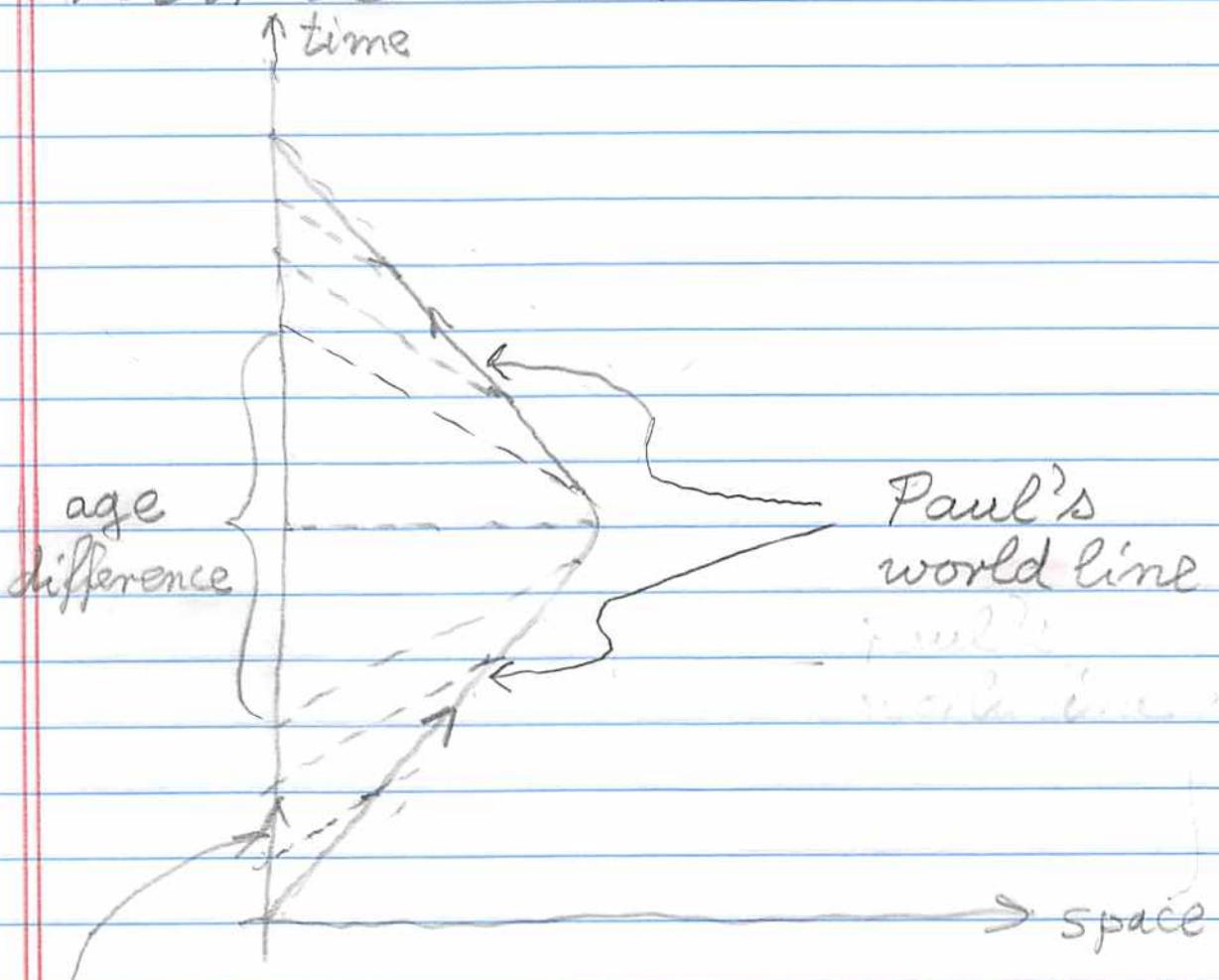
VI Constant of motion

[Read § 13.4 in MTW;]

[For a relevant review read § 13.3.]

The Twin Paradox: A Reminder. 2.0

The Twin Paradox is based on comparing two (biological or any other) clocks in relative motion



Peter's worldline | For π^\pm lifetime is 2.6×10^{-8} sec

For a π^0 meson its life-time is 8×10^{-17} sec

in its own frame. Assume $v = 0.995c$; $\gamma = \frac{1}{\sqrt{1 - (0.995)^2}} = 10$

Travel distance

$$\pi^0 \quad \left\{ \begin{array}{l} \Delta X_{\text{Newton}} = 8 \times 10^{-17} \text{ sec} \times 0.995 \times 3 \times 10^{10} = 2.4 \times 10^{-6} \text{ cm} \\ \Delta X_{\text{rel}} = 8 \times 10^{-17} \times 8 \times 0.995 \times 3 \times 10^{10} = 2.4 \times 10^{-5} \text{ cm} \end{array} \right.$$

$$\pi^\pm \quad \left\{ \begin{array}{l} \Delta X_{\text{Newton}} = 2.6 \times 10^{-8} \text{ sec} \times 0.995 \times 3 \times 10^{10} = 7.8 \times 10^2 \text{ cm} \approx 7.8 \text{ meter} \\ \Delta X_{\text{rel}} = 2.6 \times 10^{-8} \times 8 \times 0.995 \times 3 \times 10^{10} = 78 \times 10^2 \text{ cm} = 78 \text{ meters} \end{array} \right.$$

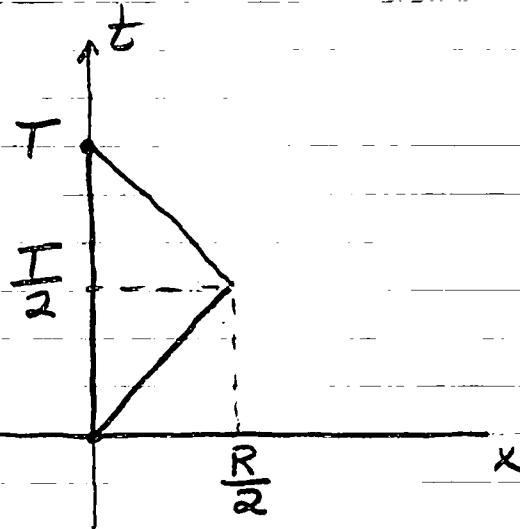
Hamilton's Principle & the Twin "Paradox."

Geodesics as worldlines of extremal length

I. THE WHY OF EXTREMAL LENGTH.

In a Lorentz frame it is easy to distinguish a straight line from one which is not.

Compare a "broken" worldline with a straight one, both starting at $(0, 0)$ and finishing at $(0, T)$



Q: What is the amount of elapsed proper time along each of these curves?

A: Along the broken worldline:

$$\tau = 2 \sqrt{\left(\frac{I}{2}\right)^2 - \left(\frac{R}{2}\right)^2} = \sqrt{T^2 - R^2}$$

Along the straight worldline:

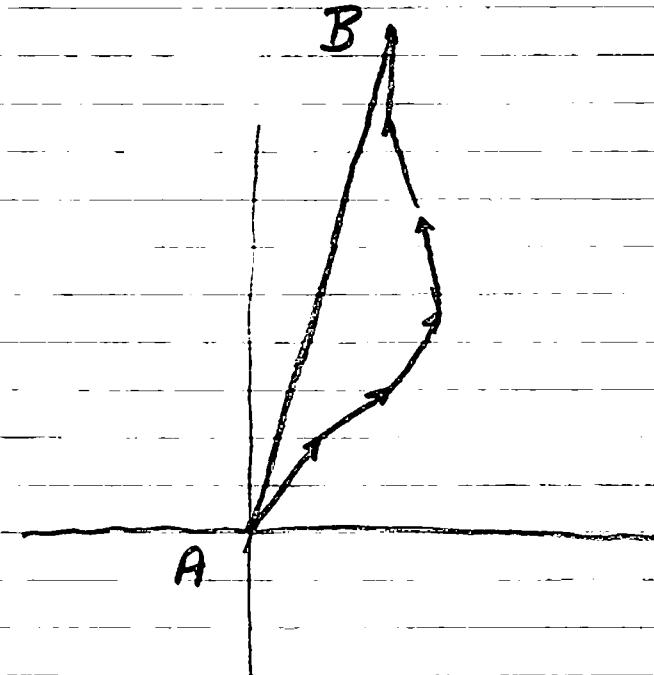
$$\tau = T$$

Comment: This illustrates the twin paradox: an inertial spacetime observer ages more than one whose world line is not straight.

Conclusion: The proper time along a straight worldline is a maximum in relation to the time along broken worldlines.

II. GENERALIZATION

This conclusion generalizes to the case where one compares multiply broken world lines with a straight line in a Lorentz frame



~ In that circumstance one has

$$(a) \tau = \int_A^B d\tau = \sqrt{\int_A^B dt^2 - dx^2 - dy^2 - dz^2}$$

$= \begin{cases} \text{maximum for a straight line} \\ \text{compared to any } \underline{\text{variant}} \\ \text{of the straight line} \end{cases}$

This maximum principle holds in any Lorentz

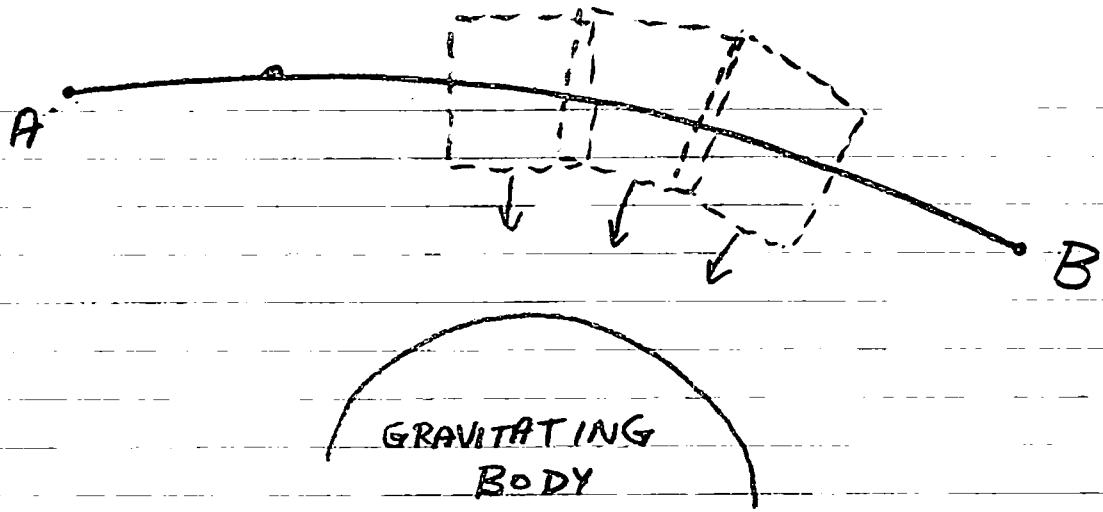
frame, even if one chooses to introduce curvilinear coordinates

$$(b) \tau = \int_A^B d\tau = \sqrt{\int_A^B -g_{\mu\nu} dx^\mu dx^\nu}$$

an extremum for time
like worldline that is
straight in each local
Lorentz frame along its
path, as compared to any nearby
variant of this worldline.

In a single Lorentz frame the introduction
of curvilinear coordinates is optional.

However if one considers a worldline
passing through a sequence of
distinct Lorentz frames, then the
use of curvilinear coordinates is
mandatory.



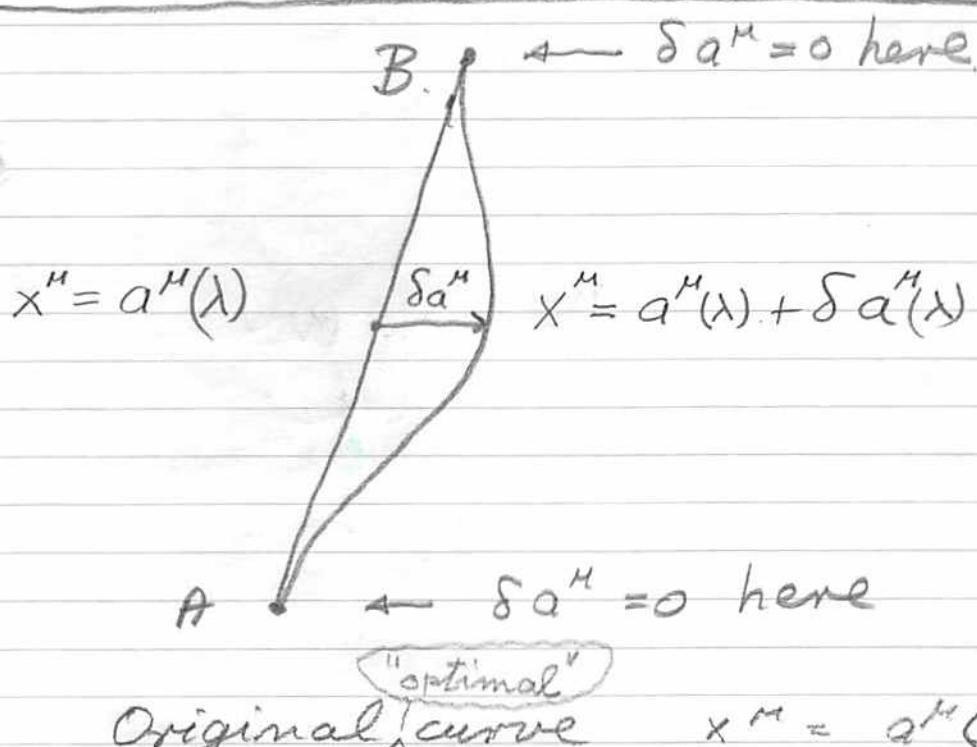
Photographic snapshot of particle about to pass through a sequence of distinct Lorentz frames

Note that in a curvilinear scenario we have replaced the "maximum" condition by an extremum condition because there may be more than one locally straight worldline connecting the two events A and B.

III. THE VARIATIONAL PRINCIPLE.

In order to determine the consequences of this extremum principle, let us compare a locally straight world line with one of its general variants.

Definition of "Variant": Different in form from others of its kind.



Variant = deformed curve: $x'' = a''(\lambda) + \delta a''(\lambda)$

" = Different in form from others of its kind

We need to calculate $\tau_A^B(a + \delta a) - \tau_A^B(a)$ 2.6
to 1st order accuracy.

a) Along either curve the proper time

is

$$\tau_A^B = \int_A^B d\tau = \int_0^1 \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

b)(i) At fixed λ the metric coefficient $g_{\mu\nu}(x^\alpha(\lambda))$ differs from one curve to another by

$$\begin{aligned}\delta g_{\mu\nu} &= g_{\mu\nu}[a^\alpha(\lambda) + \delta a^\alpha(\lambda)] - g_{\mu\nu}[a^\alpha(\lambda)] \\ &= \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta a^\alpha(\lambda) \quad (\text{the } \underline{\text{Principal linear part}} \text{ only})\end{aligned}$$

(ii) The components $\frac{dx^\mu}{d\lambda}$ of the tangent vector

differ by

$$\delta \left(\frac{dx^\mu}{d\lambda} \right) = \frac{d(a^\mu(\lambda) + \delta a^\mu(\lambda))}{d\lambda} - \frac{d a^\mu(\lambda)}{d\lambda} = \frac{d(\delta a^\mu)}{d\lambda}$$

same λ

(iii) These changes in $g_{\mu\nu}$ and $\frac{dx^\mu}{d\lambda}$, at fixed λ , produce corresponding changes in the

$$\tau_A^B[a + \delta a] - \tau_A^B[a] = \frac{d}{d\lambda} \left(g_{\mu\nu} \frac{dx^\mu}{d\lambda} \delta a^\nu + \frac{d}{d\lambda} (-g_{\mu\nu}) \delta a^\mu \right)$$

$$\delta \tau_A^B = \underbrace{\left\{ -g_{\mu\nu} \frac{da^\mu}{d\lambda} \frac{d}{d\lambda} (\delta a^\nu) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{da^\mu}{d\lambda} \frac{da^\nu}{d\lambda} \delta a^\alpha \right\}}_{\sqrt{-g_{\alpha\beta}} \frac{da^\alpha}{d\lambda} \frac{da^\beta}{d\lambda}} d\lambda$$

$$\delta \tilde{\tau}_A^B = \int_0^1 -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda$$

Integrate the first term by parts. The end point term vanishes because both paths pass through A and B:

$$\delta a^\gamma(0) = \delta a^\gamma(1) = 0$$

We therefore obtain

$$\boxed{\delta \tilde{\tau}_A^B = \int_0^1 f_\gamma(\lambda) \delta a^\gamma - g_{\alpha\beta} \frac{da^\alpha}{d\lambda} \frac{da^\beta}{d\lambda} d\lambda}$$

where

$$\boxed{f_\gamma(\lambda) = \frac{1}{\sqrt{-g}} \frac{d}{d\lambda} \left(\frac{g^{\alpha\beta} \frac{da^\mu}{d\lambda}}{\sqrt{-g}} \right) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{da^\mu}{d\lambda} \frac{da^\nu}{d\lambda} \sqrt{-g}} \quad (1)$$

with

$$\sqrt{-g} = \sqrt{-g_{\alpha\beta} \frac{da^\alpha}{d\lambda} \frac{da^\beta}{d\lambda}}$$

Comment: We shall return to this given expression for $f_\gamma(\lambda)$.

The extremum is achieved whenever $\delta \tilde{\tau}_A^B = 0$ for any first order deformation $\delta a^\gamma(\lambda)$

Thus one obtains

$$\boxed{f_\gamma(\lambda) = 0} \quad \gamma = 0, 1, 2, 3$$

IV PARAMETRIZATION

A) The equations $f_g(\lambda) = 0$ constitute mathematical overkill! There are more of them than necessary to express the extremal nature of the variational integral

$$\tau_A^B = \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda.$$

In other words, one of these equations holds for all worldlines, even those that do not extremize τ_A^B .

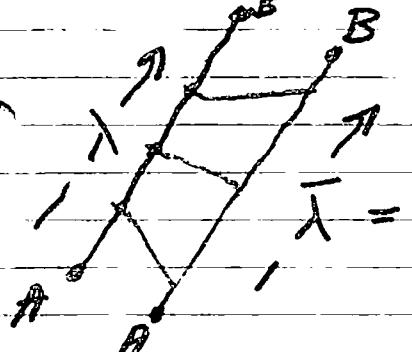
This fact follows from the parametrization independence of this integral. The reparametrization

$$\lambda \rightarrow \bar{\lambda} = \lambda + h(\lambda) \quad \frac{d\bar{\lambda}}{d\lambda} = 1 + h'(\lambda)$$

$$\left. \begin{array}{l} \bar{\lambda}(0) = 0 \\ \bar{\lambda}(1) = 1 \end{array} \right\} \Rightarrow h(0) = h(1) = 0$$

variational

does not change the value of the integral



It corresponds to a mere

"repositioning of beads along a string" (= reparametrization)

$$\bar{\lambda}(\lambda) = \lambda + h(\lambda)$$

2.9

$$\tilde{\tau}_\alpha^B = \int_0^1 -g_{\alpha\beta} \frac{da^\alpha}{d\lambda} \frac{da^\beta}{d\lambda} d\lambda = \int_0^1 -g_{\alpha\beta} \frac{d\alpha^\alpha}{d\tilde{\lambda}} \frac{d\alpha^\beta}{d\tilde{\lambda}} d\tilde{\lambda}$$

The change in $\alpha^\delta(\lambda)$ brought about by such a reparametrization is

$$a^\delta(\lambda) \rightarrow a^\delta(\tilde{\lambda}) = a^\delta(\lambda + h(\lambda)) = a^\delta(\lambda) + \delta a^\delta(\lambda)$$

where

$$\delta a^\delta(\lambda) = \frac{da^\delta}{d\lambda} h(\lambda)$$

The fact that such variations can not change the variational integral for

arbitrary $h(\lambda)$ implies $\tilde{\tau}_\alpha^B = \int_0^1 f_\delta(\lambda) \frac{da^\delta}{d\lambda} h(\lambda) d\lambda = 0$,

and hence

$$\boxed{f_\delta(\lambda) \frac{da^\delta}{d\lambda} = 0} \quad (\text{even if } f_\delta \neq 0)$$

This holds for all paths $a^\delta(\lambda)$, even those that do not extremize $\tilde{\tau}_\alpha^B$.

An equation that holds whether or not the quantities obey any differential equations is called an identity.

Ours is simply an algebraic identity.

B) The reparametrization freedom can be exploited to simplify the differential equation: Introduce the physically more relevant parameter, the proper time, by means of

$$d\tau = \sqrt{-g_{\alpha\beta}(\alpha^\lambda(\lambda)) \frac{d\alpha^\lambda}{d\lambda} \frac{d\alpha^\beta}{d\lambda}} d\lambda$$

We have

$$\frac{d\tau}{d\lambda} \neq 0 \Rightarrow \lambda = \lambda(\tau).$$

Introduce

$$x^\delta(\tau) = \alpha^\delta(\lambda(\tau)) \quad (2.a)$$

so that

$$\frac{dx^\delta(\tau)}{d\tau} = \frac{d\alpha^\delta}{d\lambda} \frac{d\lambda}{d\tau} = \frac{1}{\sqrt{-g}} \frac{d\alpha^\delta}{d\lambda} \quad (2.b)$$

or more generally

$$\frac{d}{d\tau} = \frac{1}{\sqrt{-g}} \frac{d}{d\lambda} \quad (2.c)$$

where

$$\sqrt{-g} \equiv \sqrt{-g_{\alpha\beta} \frac{d\alpha^\lambda}{d\lambda} \frac{d\alpha^\beta}{d\lambda}}$$

The introduction of the proper time τ as the world line parameter results in a non-trivial simplification in the extremum condition $\delta \Sigma_A^B = 0$ as expressed by the differential equations

$$0 = f_\gamma(\lambda) \quad \gamma = 0, 1, 2, 3$$

on page 2.7. Indeed, introducing

Eqs (2a)-(2c) on page 2.10 into Eq(1) on

page 2.7 results in

$$0 = f_\gamma(\lambda) = \frac{d}{d\tau} \left(g_{\mu\gamma} \frac{dx^\gamma}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

or

$$0 = g_{\mu\nu} \frac{d^2x^\nu}{d\tau^2} + \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}. \quad (3)$$

Following the line of reasoning in Problem 3.11b in MTW (Homework 4 in Math 5756), rewrite the middle term as

$$g_{\mu\nu, \nu} \dot{x}^\nu \dot{x}^\mu = \frac{1}{2} (g_{\mu\nu, \nu} + g_{\nu\mu, \nu}) \dot{x}^\mu \dot{x}^\nu$$

The result is that the extremal condition as expressed by the differential Eq. (3) on page 2.11 becomes simply

$$0 = g_{\mu\nu} \frac{d^2 x^\mu}{dr^2} + \frac{1}{2} (g_{\mu\gamma,\nu} + g_{\nu\gamma,\mu} - g_{\mu\nu,\gamma}) \frac{dx^\mu}{dr} \frac{dx^\nu}{dr}. \quad (4)$$

Streamline this differential equation

further by introducing the inverse

metric $g^{\nu\alpha}$:

$$g^{\nu\alpha} g_{\mu\nu} = \delta_\mu^\alpha$$

and obtain

$$0 = \frac{d^2 x^\alpha}{dr^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{dr} \frac{dx^\nu}{dr}$$

where

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\gamma} (g_{\mu\gamma,\nu} + g_{\nu\gamma,\mu} - g_{\mu\nu,\gamma}) \quad (5)$$

are the "Christoffel symbols of the 2nd kind."

II TORSIONLESS METRIC COMPATIBLE PARALLEL

By contrast:

TRANSPORT. 2.13

The Christoffel symbols of the 1st kind"

are the contents of the round parentheses in Eqs(4) and(5) before we

introduced the inverse metric,

$$\Gamma_{\gamma \mu \nu} = \frac{1}{2} (g_{\mu \delta, \nu} + g_{\nu \delta, \mu} - g_{\mu \nu, \delta})$$

These symbols are significant because

they mathematize the metric compatibility of the law of parallel transport.

Indeed, add to the above symbol the

one with ν and δ interchanged:

$$\Gamma_{\nu \mu \delta} = \frac{1}{2} (g_{\nu \mu, \delta} + g_{\nu \delta, \mu} - g_{\mu \delta, \nu})$$

One obtains

$$\frac{\partial g_{\nu \delta}}{\partial x^\mu} = \Gamma_{\delta \mu \nu} \cdot \Gamma_{\nu \mu \delta}$$

$$= \Gamma^\alpha_{\mu \nu} g_{\alpha \delta} + \Gamma^\alpha_{\mu \delta} g_{\alpha \nu}$$

namely,

$$0 = \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - g_{\alpha\sigma} \Gamma_{\nu\mu}^\alpha - g_{\nu\alpha} \Gamma_{\sigma\mu}^\alpha = g_{\nu\sigma;\mu}$$

This is the condition that the law of parallel transport as expressed by

the $\Gamma_{\mu\nu}^\alpha$ in Eq.(5) on page 2.12 is

- (a) compatible with the same metric tensor field that went into the extremization of the proper time as.

stated on page 2.3, and

- (b) has zero torsion (because $\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha$).

Two Conclusions:

- ① The principle of extremal proper time implies a unique torsionless parallel transport which compatible with the metric.

$$\boxed{\int \sqrt{-g_{\mu\nu}} dx^\mu dx^\nu = \text{extr. l}} \Rightarrow \Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma})$$

$$\Rightarrow \Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha \text{ (i.e. torsion=0)}$$

② The geodesics of curved spacetime

coincide with the world lines of extremal
proper time.

IV A GENERAL CONSTANT OF MOTION

By differentiating the squared magnitude $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ with respect to τ , one can verify that

$$f_\gamma(\lambda) \frac{d\dot{x}^\lambda}{d\tau} = 0 \Rightarrow \frac{d}{d\tau} (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = 0.$$

Thus $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \text{const} (= -1 \text{ for any timelike curve})$; $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ is always an integral of motion; it expresses the constancy of the magnitude of

the unit tangent $u = \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu}$:

$$u \cdot u = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \text{constant}$$