

LECTURE 20

I) Rotation as a tensor

II) The Hodge dual.

III) Curvature-induced rotation

Google: "Angular velocity tensor"

In MTW: Read § 15.4, again

II)

ROTATION AS A TENSOR20-1
(25.1)The Physical Origin of Rotation.

In three dimensions consider a vector \vec{v} rotating with a given angular velocity around a given axis. The vectorial change $\Delta \vec{v}$ in this vector during time Δt is



$$\Delta \vec{v} = \Delta t \vec{\omega} \times \vec{v}$$

$$= \Delta t \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega^1 & \omega^2 & \omega^3 \\ v^1 & v^2 & v^3 \end{vmatrix}$$

Figure 1

Such a vectorial determinant can be generalized to higher dimensions. But, as far as I know, it will not represent a rotation in that case.

This is because the essential (= most consequential) property of the rotation process is that it takes place in a plane.

In 4-d spacetime, unlike 3-d, a 2-d plane has no unique direction that can serve as an axis of rotation. Instead the plane of rotation is obtained mathematically by reinterpreting it as an inner product:

$$\Delta \vec{v} = \Delta t \left[-\omega^1 (\vec{e}_2 v^3 - \vec{e}_3 v^2) + \omega^2 (\vec{e}_3 v^1 - \vec{e}_1 v^3) - \omega^3 (\vec{e}_1 v^2 - \vec{e}_2 v^1) \right]$$

This vectorial determinant is basis independent.

Evaluate it relative to an o.n. basis, and obtain

$$\Delta \vec{v} = \Delta t \left[\underbrace{\omega^1 (\vec{e}_2 \otimes \vec{e}_3 - \vec{e}_3 \otimes \vec{e}_2)}_{R^{23} = R^{32}} \cdot \vec{v} + \underbrace{\omega^2 (\vec{e}_3 \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_3)}_{R^{13} = R^{31}} \cdot \vec{v} + \underbrace{\omega^3 (\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1)}_{R^{12} = R^{21}} \cdot \vec{v} \right]$$

This change is the result due to an infinitesimal rotational change

$$\vec{v} \rightarrow \vec{v} + \underbrace{\frac{\vec{e}_i \wedge \vec{e}_j R^{ij}}{2!} \cdot \vec{v}}$$

$$\vec{R} = \vec{e}_i \wedge \vec{e}_j E_R^{ijk} \omega^k = \vec{e}_i \wedge \vec{e}_j E_{gjk}^{ijk} \omega^k$$

where the basis invariant bivector

$$(20.1) \quad \boxed{\vec{e}_i \vec{e}_j \vec{R} = \frac{\vec{e}_i \wedge \vec{e}_j R^{ij}}{2!} = \vec{e}_i \wedge \vec{e}_j R^{ijk}}$$

is the rotation generator. Its basis components form the matrix

$$[R^{ij}] = [E^{ijk} \omega_k] = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

III) THE HODGE DUAL.

The angular velocity $\vec{\omega} = \vec{e}_3 \omega^k$ which gave rise to this rotation generator can be recovered from the

rotation generator by means of its Hodge dual, the result of the (linear) Hodge \star operator. For 3-d Euclidean space it is

$$\star : \Lambda^2(\mathbb{R}^3) \longrightarrow \Lambda^1(\mathbb{R}^3)$$

$$\vec{e}_i \wedge \vec{e}_j \mapsto \star(\vec{e}_i \wedge \vec{e}_j) = \epsilon_{ijk} \vec{e}_k$$

This is a linear 1-1 transformation.

Apply it to the generator \vec{R} and find that

$$\star\left(\frac{\vec{e}_i \wedge \vec{e}_j \cdot \vec{R}^{ij}}{2!}\right) = \frac{\epsilon_{ijk} \vec{e}_k \epsilon^{ijl} \omega^l}{2!}$$

$$= \delta_k^l \vec{e}_k \omega^l$$

$$= \vec{\omega}.$$

The only part of this calculation of

space

$$\star(\vec{e}_i \wedge \vec{e}_j) = \epsilon_{ijk} \vec{e}_k$$

In 3-d Euclidean space there are two equally good ways of mathematizing the rotational change of a 3-d vector: via (i) the angular velocity vector $\vec{\omega}$ and (ii) basis invariant bivector $\overset{\leftrightarrow}{R}$:

$$(i) \quad \vec{\omega} \in V : \Delta t \vec{\omega} \times (\cdot) : V \rightarrow V$$

$$\star \uparrow \downarrow \star^{-1}$$

$$\vec{v} \rightsquigarrow \Delta \vec{v} = \Delta t \vec{\omega} \times \vec{v}$$

$$(ii) \quad \overset{\leftrightarrow}{R} \in \Lambda^2 V : \Delta t \overset{\leftrightarrow}{R} : \vec{v} \rightsquigarrow \Delta \vec{v} = \Delta t \overset{\leftrightarrow}{R} \cdot \vec{v}$$

However, the mathematization via $\overset{\leftrightarrow}{R}$ is more

fundamental because its extension to

rotation in 4-d spacetime is very direct.

On the other hand, with an appropriate modification

of $\overset{\leftrightarrow}{R}$ the Hodge \star operator can be extended

to 4-d spacetime, where, as we shall see, \star plays a key role in geometrizing $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

III) CURVATURE-INDUCED ROTATION,

The concept of rotation defined by

$$\overleftrightarrow{R} = \frac{\vec{e}_i \wedge \vec{e}_j R^{ij}}{2!}$$

generalizes to four (and higher)

dimension of spaces with an inner product

(i.e. metric) structure. Indeed, recall

the curvature-induced rotational

change associated with a π -spanned

face of a cube:

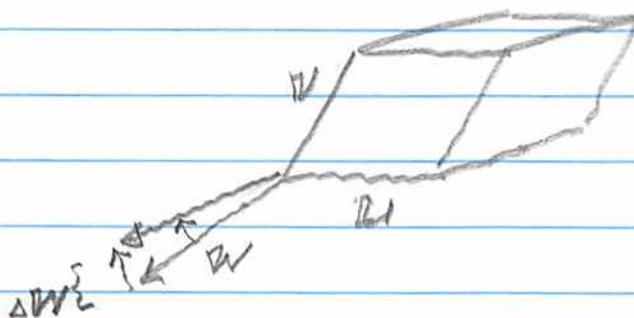


Figure 2.

One has

$$R(u, v): V \rightarrow V$$

$$\begin{aligned} w = e_p w^k \Rightarrow \Delta w &= e_p R^{\ell m} {}_{\alpha \beta} \frac{dx^\alpha \wedge dx^\beta}{2!} (u, v) w^k = \\ &= e_p R^{\ell m} {}_{\alpha \beta} g_{m k} \frac{dx^\alpha \wedge dx^\beta}{2!} (u, v) w^k \\ &= e_p \otimes e_m \cdot e_k w^k R^{\ell m} {}_{\alpha \beta} dx^\alpha \wedge dx^\beta (u, v) \end{aligned}$$

Taking advantage of metric-induced

antisymmetry $R^{\ell m} {}_{\alpha \beta} = -R^m {}^{\ell} {}_{\alpha \beta}$, one

one has

$$\begin{aligned} w \text{ and } e_p \otimes e_m \frac{R^{\ell m}}{2!} {}_{\alpha \beta} \underbrace{dx^\alpha \wedge dx^\beta}_{2!} (u, v) \cdot w \\ = \underbrace{e_p \otimes e_m \frac{R^{\ell m}}{2!} (u, v)}_{2!} \cdot w \equiv R(u, v) \cdot w \end{aligned}$$

Compare this with the bivector defined

rotation, Eq. (20,1) on page 20-3, one

arrives at

$e_p \otimes e_m \frac{R^{\ell m}}{2!} (u, v) = \text{"rotation"}$

It is induced by the curvature in the area subtended by the vectors u and v . For infinitesimal vectors u and v , $R(u, v)$'s components $\tilde{R}^m(u, v)$ are the angles by which a vector such as w gets rotated in the (e_2, e_m) -plane.

Nota bene: In the context of spacetime, rotation refers to Euclidean, rotation, Lorentz rotation or any of their combinations.

For a 3-d cube permeated by curvature, each of its faces has a the attribute of a rotation proportional to

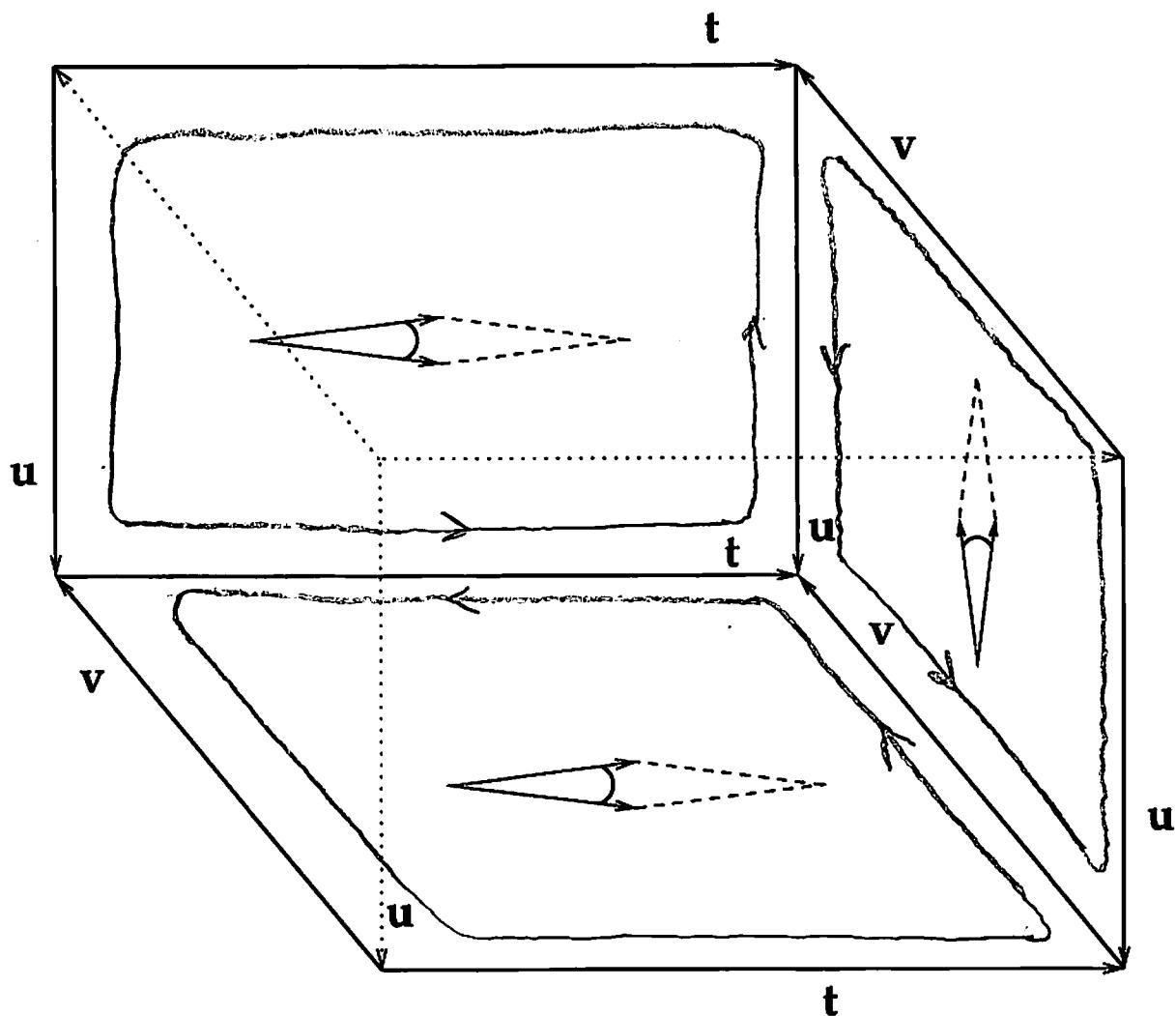


Figure
the area of the respective face.