

LECTURE 21

I) Curvature-induced rotation

applied to a 3-cube

II) Reminder about curvature

(can be skipped on the 1st reading)

III) Bianchi identity

IV) $\partial\partial C = 0$ and the Fundamental Thm
of calculus

In MTW Read § 15.2, Fig. 15.1

I CURVATURE-INDUCED ROTATION APPLIED TO A THE FACES OF A 3-CUBE

21.1

The cause of gravitation is the existence of matter in any given 3-volume.

One of the conceptually most efficient ways of geometrizing gravitation is to mathematize the curvature-induced rotation on the face of a 3-d cube in 4-d space-time.

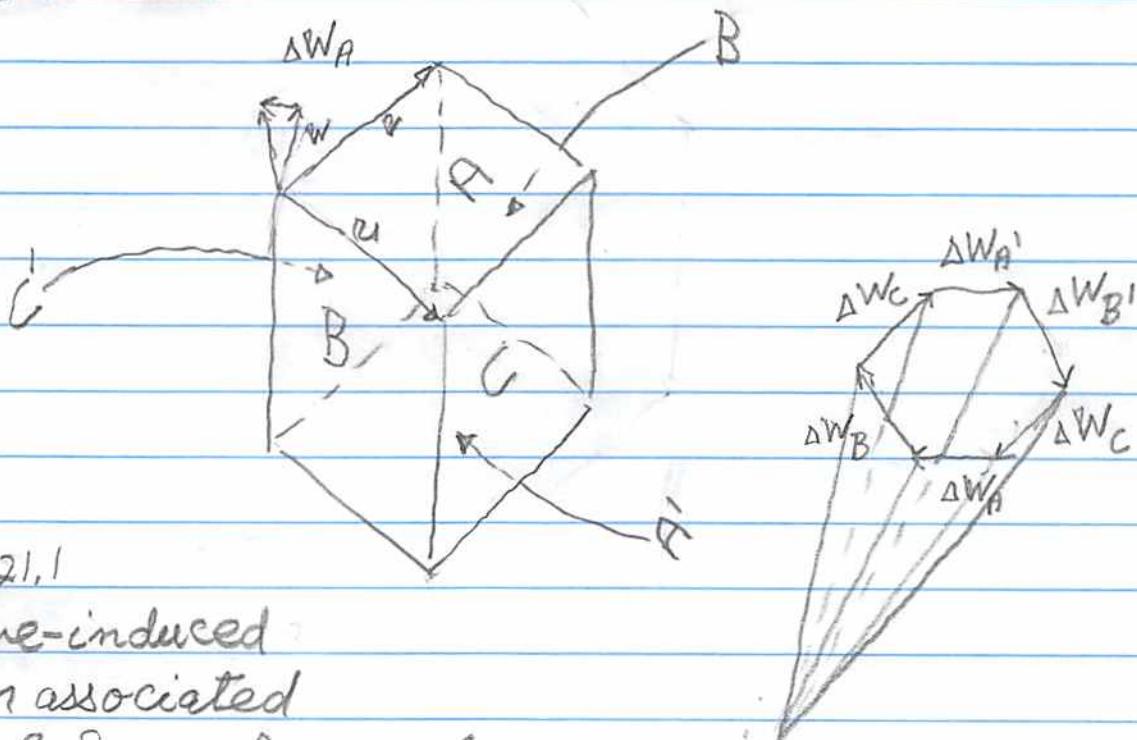


Figure 21.1
Curvature-induced
rotation associated
with each face of a 3-cube.

Transport the vector w parallel to itself around the closed square-shaped loop which bounds the face A. The result is the rotated vector $w + \Delta w_A$. The amount of this curvature-induced rotation is

$$\Delta w_A = \frac{e_2 \wedge e_m \cdot w}{2!} R_m^{lm}(u, v).$$

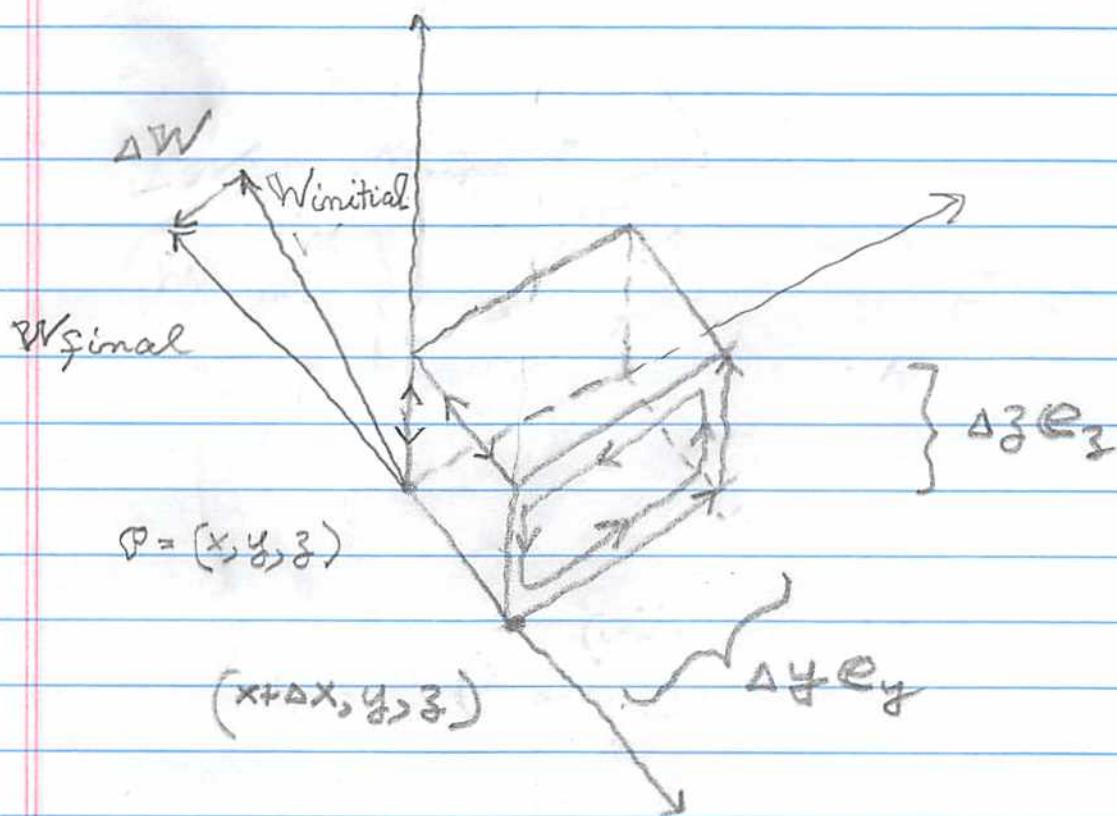
The sum total contribution from all six faces vanished:

$$\Delta w_A + \Delta w_B + \Delta w_C + \Delta w_{A'} + \Delta w_{B'} + \Delta w_{C'} = 0.$$

This is because in getting parallel transported around each of the faces w gets moved along each edge of two abutting edges twice but in opposite directions. The result, as shown

in Figure 21.1 on page 21.1, is that sum total is zero.

To mathematize this geometrical fact consider a vector field \mathbf{W} whose domain is on the surface of a 3-cube C as well as its interior



Parallel transport from a corner point along an edge to one of the 6 faces, around its boundary, and then back again to point P.

The contribution to the vectorial change from the face at $x+\Delta x$ is

$$\Delta W = e_2 w^m R^l m_{yz} (x+\Delta x, y, z) \Delta y \Delta z$$

cont'd on page 21.6

II) REMINDER ABOUT CURVATURE

using MTW's notation on p 27)
the above component representation

is related to the rotation

$$\overset{\leftrightarrow}{R} = \frac{e_i \wedge e_m}{2!} R^{lm} (u, v) = e_2 \otimes e_m R^{lm} (e_y, e_z) \Delta y \Delta z$$

as follows

$$\Delta W = R_{ieman} (\dots, W, u = \Delta y e_y, v = \Delta z e_z) \Big|_{(x+\Delta x, y, z)}$$

21.5

$$\Delta W = \{ [\nabla_u, \nabla_v] - \nabla_{[u, v]} \} W \quad (11.8, 11.9 \text{ in MTW})$$
$$= R(u, v) W \quad (x + \Delta x, y, z) \quad P 271$$

$$= e_2 \otimes w^n \underbrace{R^e_m}_m (u, v) W$$

Cartan's curvature
curvature

$$2 \text{ form} = dw^l_n + w^l_k \wedge w^k_n$$

at $(x + \Delta x, y, z)$

$$= e_2 w^n R^e_m (u, v)$$

$$= e_2 w^n g_{nm} R^e_m (u, v)$$

$$= e_2 \otimes e_m \cdot W R^e_m (u, v)$$

$$= e_2 \wedge e_m R^e_m (e_y, e_z) \Delta y \Delta z \cdot W$$

"rotation" in the (y, z) -plane,
at $(x + \Delta x, y, z)$

End of Reminder

The opposite face gives a similar contribution. The combined contribution from the pair of faces is

$$\epsilon_2 w^m \frac{\partial}{\partial x} \left(R^e m y_3 \right) \Delta x \Delta y \Delta z \quad (\text{"front-back"})$$

Here we have used a vector basis which is parallel ($\Gamma_{\alpha\beta}^M = 0$, but $\partial_\gamma \Gamma_{\alpha\beta}^M \neq 0$ at the chosen corner \mathcal{P}). Such a basis is induced by a "Riemann normal coordinate system" (See Section 11.6 and Exercise 11.9 in MTW).

Relative to such a coordinate system centered at the given point, all covariant derivatives become partial derivatives at this point. This is because the basis vectors

are parallel (to 2nd order accuracy) in its neighborhood.

III) BIANCHI IDENTITY

Other pairs of faces of that cube give similar contributions. However the contributions from common edges of abutting faces cancel. Consequently one has

$$0 = e_2 w^m \left(R^e_{m y z, x} + R^e_{m z x, y} + R^e_{m x y, z} \right)$$

front-back right-left top-bottom

More generally (because of the basis independent mathematical framework) one has

$$\boxed{0 = R^e_{m i j, k} + R^e_{m j k, i} + R^e_{m k i, j}}$$

These are the "Bianchi identities".

IV The 3-2-1 and the 2-1-0 chains
 of Stokes' theorem.

All versions of Stokes' theorem are examples of the Fundamental Theorem of Calculus. Indeed one has

$$\text{a) } \iint_C \operatorname{div} \operatorname{curl} \vec{u} \, d^3x = \iint_{\partial C} \operatorname{curl} \vec{u} \cdot d\vec{s} = \oint_{\partial C} \vec{u} \cdot d\vec{l}$$

In terms of exterior calculus one has

$$\iint_C dd\psi = \iint_{\partial C} d\psi = \int_{\partial C} \psi$$

$$\text{b) } \iint_C (\vec{\nabla} \times \vec{\nabla} \psi) \cdot d\vec{s} = \int_{\partial C} \vec{\nabla} \psi \cdot d\vec{l} = \psi \Big|_{\partial C}$$

or

$$\iint_C d\psi = \int_{\partial C} \psi = \psi \Big|_{\partial C}$$

Thus & c's one has

$$\partial \partial C = 0 \Rightarrow \left\{ \begin{array}{lcl} \operatorname{div} \operatorname{curl} \vec{u} & = 0 \\ d\psi & = 0 \\ \vec{\nabla} \times \vec{\nabla} \psi & = 0 \\ d\psi & = 0 \end{array} \right.$$