LECTURE 22

Force, rotation, and their moments mathematized as geometrical objects:

A) Mathematize rotational changes induced by curvature inside and on the boundary of a 3-cube

B) Translational equilibrium of 3-cube subjected to external forces.

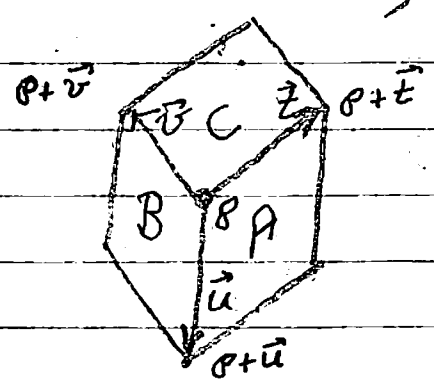
C) Moment of Force

consigned to Lecture 23

In MTW Read §15.3.

I. ROTATION vs. FORCE: Their Common Denominator.

Consider a 3-d cube a typical one of its 6 bounding faces is an element of area spanned by the infinitesimal displacement vectors $\vec{u}$, $\vec{v}$, and $\vec{t}$

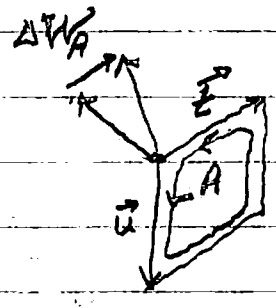


$$\vec{u} = \Delta u\, e_u \equiv \Delta u \frac{\partial}{\partial u}$$
$$\vec{v} = \Delta v\, e_v \equiv \Delta v \frac{\partial}{\partial v}$$
$$\vec{t} = \Delta t\, e_t \equiv \Delta t \frac{\partial}{\partial t}$$

First we shall compare the mathematized result of two different processes.

A) The cube is subjected to curvature.

A vector, say W at P is parallel transported around each element of area



1.) Recall that

$$\nabla_{\vec{u}} \vec{W} = \langle d(e_\beta w^\beta), \vec{u} \rangle = \int_P^{P+\vec{u}} dW$$

$$\nabla_{\vec{t}} \vec{W} = \langle d(e_\beta w^\beta, \vec{t} \rangle = \int_P^{P+\vec{t}} dW$$

Consequently, the line integral around the boundary of A, namely ∂A, is

$$\oint_{\partial A} d\vec{W} = e_\alpha \wedge e_\beta \cdot \vec{W} \, \underset{\sim}{R}^{|\alpha\beta|}(u,t)$$

$$= \left\{ \iint_A e_\alpha \wedge e_\beta \, \underset{\sim}{R}^{|\alpha\beta|}(e_u, e_t) \, du \, dt \right\} \cdot W .$$

where we used

(i) the vectorial 1-2 version of Stokes' theorem:

$$\nabla_u \underset{\sim}{\vec{\Omega}}(t) - \nabla_t \underset{\sim}{\vec{\Omega}}(u) - \underset{\sim}{\vec{\Omega}}(u,t) \overset{HW3}{=} d\vec{\Omega}(u,t)$$

with

$$\underset{\sim}{\vec{\Omega}} = d(e_\beta w^\beta) = e_\alpha \underset{\sim}{\omega}^\alpha{}_\beta w^\beta + e_\beta \underset{\sim}{d}w^\beta$$

and

(ii) Cartan's 2nd structure equation

$$d\,dW = e_\alpha\left(d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta\right) w^\beta \equiv e_\alpha R^\alpha{}_{\beta|\gamma\delta|}\, dx^\gamma \wedge dx^\delta$$

$$\equiv e_\alpha\, w^\beta\, \underset{\sim}{R}^\alpha_\beta$$

2.) We need to obtain the rotational change in W from all 3 pairs of opposing faces.

(i) From faces A' and A one has,

$$\underbrace{\oint_{\partial A'} dW - \oint_{\partial A} dW}_{} = \left\{ \iint_{A'} e_\alpha \wedge e_\beta R^{\alpha\beta}(e_u, e_t) \right|_{P+v} du\, dt \qquad (22.1)$$

$$- \iint_{A} \qquad '' \qquad \Big|_{P} \xi \cdot W$$

$$= \nabla_{\vec{v}} \Big(\underbrace{e_\alpha \wedge e_\beta R^{\alpha\beta}(u, t)}_{\overleftrightarrow{\Omega}(\vec{u}, \vec{t})} \Big)\Big|_P \cdot W$$

(ii) Apply this mathematization to the other pair of faces, B'&B and C'&C.

(iii) Take advantage of the tensorial 2-3 version of Stokes' theorem,

$$(22.2)\quad \nabla_v \overleftrightarrow{\Omega}(t,u) + \nabla_t \overleftrightarrow{\Omega}(u,v) + \nabla_u \overleftrightarrow{\Omega}(v,t) = \underset{\sim}{d}\,\overleftrightarrow{\Omega}(t,u,v)$$

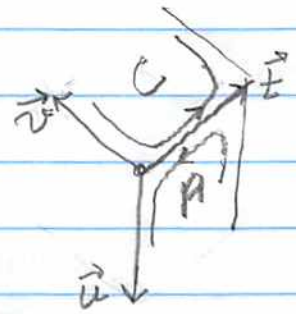
and obtain the total rotational change in $\mathbb{W}$ from all 6 faces

$$\left\{\iint_{6\,\text{faces}} \overleftrightarrow{\Omega}\right\}\cdot\mathbb{W} = \underbrace{\left[\iint_{A'} - \iint_{A} + \iint_{B'} - \iint_{B} + \iint_{C'} - \iint_{C'}\right]}\ \overleftrightarrow{\underset{\sim}{\Omega}}\cdot\mathbb{W} = \iiint_{\mathcal{D}=\text{3-cube}} d\Omega$$

$$\left[\iint_{\partial\mathcal{D}}\right] \text{ " } = \text{ "}$$

$\partial\mathcal{D}$ = oriented boundary of $\mathcal{D}$.

The sum total from all six faces vanishes because the parallel transport line integrals along each abutting edge occurs twice, but in opposite directions, namely



$$\sum_{\ell=1}^{6} \iint_{\ell^{th}\,\text{face}} \overleftrightarrow{\Omega} = \sum_{\ell=1}^{6} e_\alpha \wedge e_\beta R^{|\alpha\beta|} \Big|_{\ell^{th}\,\text{face}} = 0$$

This result holds for <u>all</u> spanning vectors. Consequently, the identity Eq. (22.2) on page 22.4 implies $d\overleftrightarrow{\Omega} \equiv d\left(e_\alpha \wedge e_\beta \underset{\sim}{R}^{|\alpha\beta|}\right) = 0$

Combining the line of reasoning in Eq. (22.1) with Eq. (22.2) one obtains

$$\int_{\partial\partial\mathcal{D}} dW = \left[\iint_{\partial\partial\mathcal{D}} e_\alpha \wedge e_\beta \underset{\sim}{R}^{|\alpha\beta|}\right]\cdot W = \left[\iiint_{\mathcal{D}} d\left(e_\alpha \wedge e_\beta R^{|\alpha\beta|}\right)\right]\cdot W$$

namely

$$\boxed{\partial\partial\mathcal{D} = 0 \Rightarrow d\left(e_\alpha \wedge e_\beta R^{|\alpha\beta|}\right) = 0}$$

The fact that one arrives at

(22.3) $$\boxed{d\left(e_\alpha \wedge e_\beta \underset{\sim}{R}^{|\alpha\beta|}\right) = 0}$$

is consistent with the fact that this is an identity also validated by exterior differention.

SUMMARY:

1. For each of the cube's 6 faces there is a curvature-induced rotation (mathematized by a bi-vector-valued 2-form).
2. Because of the 1-2 version of Stokes' theorem, and because of Cartan's 2nd structure equation, each of these rotations came from the change due to the process of parallel transporting some vector around the bounding edges of each face.
3. Because of the 2-3 version of Stokes' theorem, the sum of the rotations from all 6 faces was mathematized by a 3-d volume (interior of the cube) integral of the source of the rotations intercepted by the 6 faces. The surprising result is the fact that this source is always zero.
4. The cause of this result is the fact the process of parallel transport in Step 2 was done along each of the cube's 12 edges twice, but in opposite (!) directions: Once along the edge of a particular face, and another time along the (same) edge of the abutting face, but now moving (i.e. transporting) the vector into the opposite direction. This cancellation property is the reflection of the mathematical ("topological") principle that the boundary of a boundary is zero.