

LECTURE 23

I.) Einstein's mathematization of gravitation.

II.) How to form a concept

III.) Solid cube in translational equilibrium: its mathematization.

IV.) Rotational equilibrium:

V.) Its mathematization via the "moment of force"

VI.) Non-equilibrium.

For II.) Read Chapter 2, "Concept Formation" in Objectivist Epistemology by Ayn Rand

For III-VI.) Read § 15.3 in MTW

Read p 116-117 in A Journey into Gravity and Spacetime by J.A. Wheeler

I) EINSTEINS MATHEMATIZATION OF GRAVITATION.

Let us focus on the process of grasping the meaning of the equation

$$(23,1) \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}, \quad R = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

This equation mathematizes the fact that (i) geometry controls the motion of matter, and (ii) matter controls the geometry of spacetime.

(i.e. via Eq. (23,1))
 The r.h.s., $T_{\mu\nu}$, is well understood in terms of its physical ("momenergy") and geometric ("density-flux 4-vector") attributes.

Although Eq. (23,1) is a tensorial statement, the l.h.s., $G_{\mu\nu}$, has yet to be conceptualized

in the same way, i.e. physically and geometrically. Einstein in 1916 did not did not specify the geometrical meaning of his "Einstein tensor" $\{G_{\mu\nu}\}$, to say nothing of its physical meaning. However, Cartan, Misner and Wheeler filled this gap, at least in regard to its geometrical attributes. Their geometrization of $\{G_{\mu\nu}\}$ consisted of introducing a new geometrical concept, the "moment of curvature-induced rotation".

II.) HOW TO FORM A CONCEPT

The process of forming a concept consists of first focusing on concrete instances,

23,3

identifying their common conceptual denominator, and then, based on that, blend them into a new integrated whole identified by means of a definition. The process is completed by assigning a word or a symbol to it.

To arrive at the "moment of curvature-induced rotation", start with the concretes familiar from physics: the "moment of force" applied to a rigid 3-cube.

III) SOLID CUBE IN TRANSLATIONAL EQUILIBRIUM: ITS MATHEMATIZATION 23.4

Consider the circumstance in which that 3-cube is in a force field so that there is a flow of momentum through each face of that cube, which is spanned by the vectors $(\vec{u}, \vec{v}, \vec{E})$:

$$\frac{d}{dt}(\text{momentum}) = \vec{F}(u, v) = \vec{e}_i \cdot \vec{E}^i(u, v)$$

$$= \vec{e}_i \cdot T^{ij} \epsilon_{jlmn} dx^l dx^m (\vec{u}, \vec{v})$$

' is the rate at which momentum flows across the $\vec{u} \cdot \vec{v}$ -spanned face into the 3-cube.

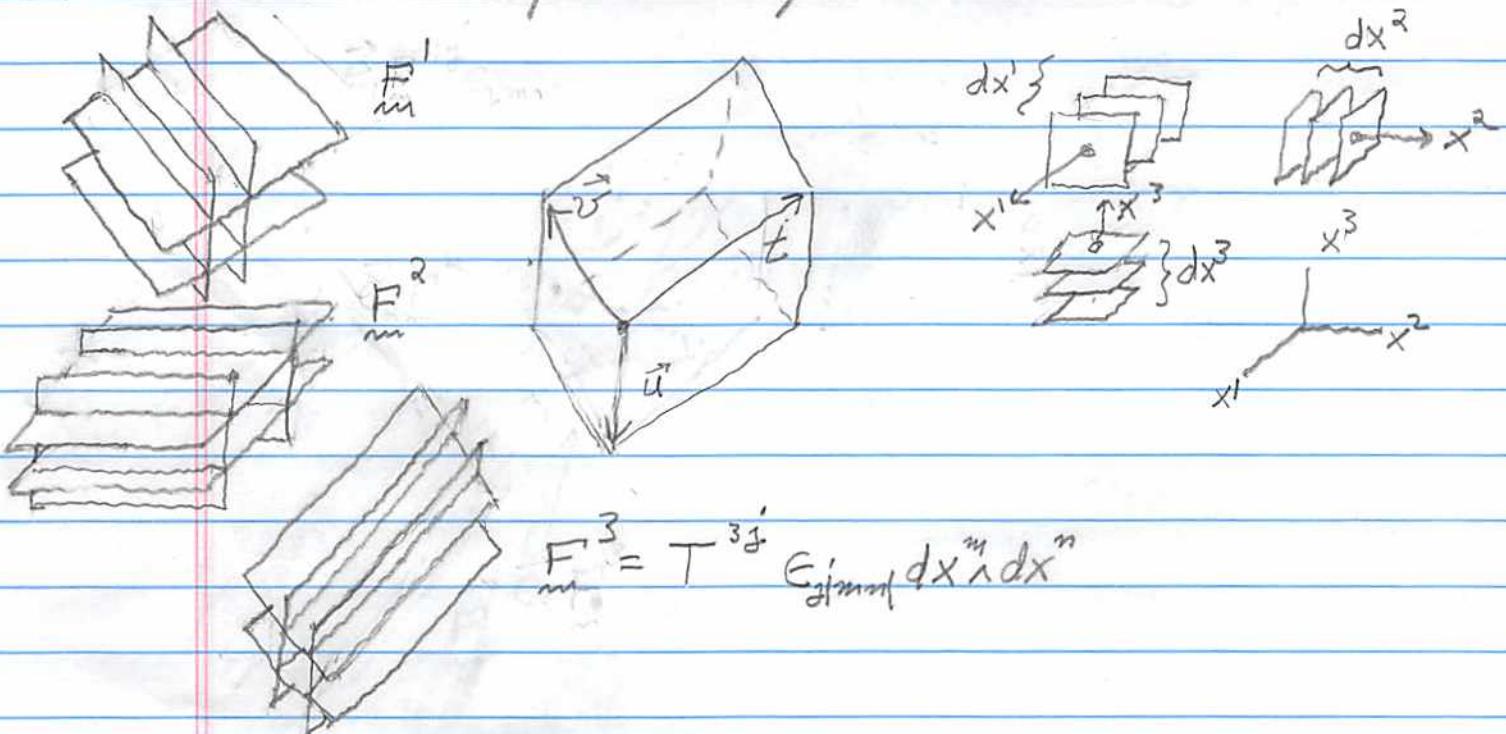


Figure 23.1

The force field $\vec{F} = \vec{e}_i \vec{F}^i$

is a three component (vector-valued) "honey-comb" structure which upon being intersected by all 6 face of the 3-cube produces a resultant vector which is the rate of momentum if it's

$$\sum_{i=1}^6 \vec{F}^i (\text{the } i^{\text{th}} \text{ face})$$

flowing into that cube, i.e., the net force on that cube.

of the interior, it has to distribute

the rigid cube's momentum on the circumstance where the 3-cube is in translational equilibrium. Consequently,

$$\sum_{\ell=1}^6 \frac{d(\text{momentum})}{dt} \Big|_{\ell^{\text{th}} \text{ face}} = \vec{0},$$

Mathematically one has

$$\sum_{\ell=1}^6 \vec{F} \Big|_{\ell^{\text{th}} \text{ face}} = \vec{0}. \quad (23.2)$$

This expresses the fact that the sum total of all forces acting on the cube

is zero; the rigid cube gains no translational momentum; there is no net amount of momentum that flows into the interior of the 3-cube.

The rigid cube is in translational equilibrium.

IV.) ROTATIONAL EQUILIBRIUM:

However, in order to be also in rotation equilibrium, the sum total of the moments of these forces

about some point P must also vanish,

$$\sum_{l=1}^6 (\vec{r}_l - \vec{r}) \wedge \vec{F}_l = \sum_{l=1}^6 (\text{torque})|_{l^{\text{th}} \text{ face}} = 0$$

If it does not vanish, then the cube will acquire angular momentum and start to spin. The fact that the cube is in translation equilibrium implies that the total moment of force is independent of P . Indeed, consider another point P' in or near the cube.

One has

$$\sum_E (\rho_E - \rho) \wedge \vec{F}_E = \sum_E [(\rho_E - \rho') + (\rho' - \rho)] \wedge \vec{F}_E$$

$$= \sum_E (\rho_E - \rho') \wedge \vec{F}_E + (\rho' - \rho) \wedge \sum_E \vec{F}_E$$

zero

The last term vanishes because

of the translation equilibrium condition,

Eq. (23,2) on page (23.6)

II). HOW TO MATHEMATIZE THE MOMENT OF FORCE.

The Archimedean concept of the moment of force ("Give me a fulcrum and I shall move the world")

$$\overleftrightarrow{\Gamma} = \sum_E (\rho_E - \rho) \wedge \vec{F}_E$$

has been mathematized by Cartan,

Wheeler, and Misner so that its

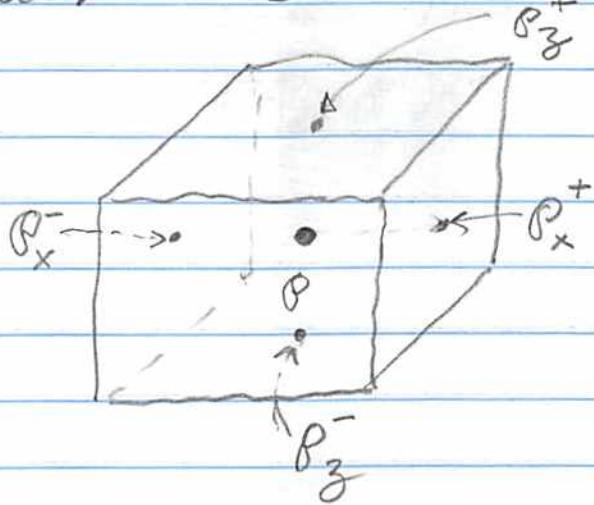
generalization yields the Einstein

tensor and hence the Einstein field

equations.

23.9

Consider a 3-cube.



with a fulcrum point inside or outside nearby.

Let

$$P_1^\pm, P_2^\pm, P_3^\pm$$

$\uparrow \quad \uparrow \quad \uparrow$
 $x \quad y \quad z$

be the six points centered on the six respective opposing faces of the cube.

The goal is to construct \vec{T} so that it is valid in all coordinate frames.

The method for doing this is not unfamiliar: Exhibit $\overset{\leftrightarrow}{T}$ explicitly relative to a particular frame.

Do this in terms of a frame

invariant geometrical object

Being frame-invariant, the expression

for $\overset{\leftrightarrow}{T}$ holds in all frames.

We start by expressing the force on the 3-cube in terms of the stress-tensor two-form

$$\vec{F} = e_i \vec{F}_m^i = e_i T^{ij} \epsilon_{jkl} dx^k dx^l$$

[in class) SKIP TO page 23.14]

stress-tensor two-form

$$\vec{F} = e_i F^i = e_i T^{ij} \epsilon_{jkl} dx^k \wedge dx^l.$$

1. In particular start with

(force on
the right
face $\Delta y \Delta z$)

$$\vec{F} (\Delta y \frac{\partial}{\partial y}, \Delta z \frac{\partial}{\partial z})$$

$$= e_i T^{ij} \epsilon_{jyz} \Delta y \Delta z$$



2.
(moment of
force associated
with right
face $\Delta y \Delta z$)

$$= (\rho_x^+ - \rho) \wedge \vec{F} (\Delta y e_y \Delta z e_z)$$

$$= (\rho_x^+ - \rho) \wedge e_i T^{ij} \epsilon_{jyz} \Delta y \Delta z$$

3.



(Net moment
of force associated
with right
and left faces $\Delta y \Delta z$)

$$= \underbrace{[(\rho_x^+ - \rho) - (\rho_x^- - \rho)]}_{e_x \Delta x} \wedge \vec{F}$$

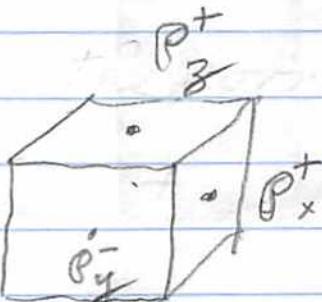
VERY GOOD, but SKIP in CLASS

23, 12

Net moment
of force
associated
with front
and back
 $\Delta y \Delta z$ face

$$= e_x \wedge e_i T^{ij} E_{jyz} \Delta x \Delta y \Delta z$$

4.



$$\oint = \left(\begin{array}{l} \text{Net moment of force} \\ \text{associated} \\ \text{with all 3} \\ \text{pairs of faces} \end{array} \right) = e_x \wedge e_i T^{ij} E_{jyz} \Delta x \Delta y \Delta z \quad (\text{left-right})$$
$$+ e_y \wedge e_i T^{ij} E_{jzx} \Delta y \Delta z \Delta x \quad (\text{front-back})$$
$$+ e_z \wedge e_i T^{ij} E_{xy} \Delta z \Delta x \Delta y \quad (\text{top-bottom})$$

$$= e_x \wedge e_i T^{2x} E_{xyz} \Delta x \Delta y \Delta z \quad (\Delta x e_x, \Delta y e_y, \Delta z e_z)$$

$$+ e_y \wedge e_i T^{iy} E_{yzx} \Delta y \Delta z \Delta x \quad (" -)$$

$$+ e_z \wedge e_i T^{iz} E_{xy} \Delta z \Delta x \Delta y \quad (" -)$$

VERY GOOD, but SKIP in CLASS,

23.13

5. Taking advantage of the antisymmetry of the Levi-Civita ϵ_{jkl} and the exterior product, $\overset{\leftrightarrow}{\mathcal{T}}$ condenses into

$$\overset{\leftrightarrow}{\mathcal{T}} = \overset{\leftrightarrow}{e}_m \wedge \overset{\leftrightarrow}{e}_i T^{ij} \frac{1}{2!} \epsilon_{jkl} dx^m \wedge dx^k \wedge dx^l (\quad)$$

Thus the total moment of force to which the cube has been subjected is a bi-vectorial 3-form, a $\binom{2}{3}$ tensor, evaluated on the volume $\Delta x \wedge \Delta y \wedge \Delta z$ of the 3-cube spanned by the ordered basis vectors

$$\{e_x \Delta x, e_y \Delta y, e_z \Delta z\} = \{e_1 \Delta x^1, e_2 \Delta x^2, e_3 \Delta x^3\}$$

This expression for $\overset{\leftrightarrow}{\mathcal{T}}$ is a geometrical object, i.e. it is invariant under any change of basis.

(In class cont'd from 23.10)

23.14 13

The result is that the moment of force, the torque, associated with all 3 pairs of faces is

$$\overleftrightarrow{\tau}_m(e_1, e_2, e_3) =$$

$$= \vec{E}_m \wedge \vec{E}_i T^i \frac{1}{2!} E_{jkl} dx^m \wedge dx^k \wedge dx^l (e_1, e_2, e_3)$$

where $\{e_1, e_2, e_3\}$ spans the volume of the particular cube.

$\overleftrightarrow{\tau}_m$ is a bivectorial 3-form, a tensor of rank $\binom{2}{3}$.

The line of reasoning leading to this conclusion is on pages 23.11 - 23.13.

The condition for any solid 3-cube in a

force field \vec{F} be in rotational equilibrium

is $\boxed{\overleftrightarrow{\tau}_m = 0}$

6. Introduce Cartan's unit tensor

$$d\rho = e_m \otimes dx^m$$

and use it to write $\overset{\leftrightarrow}{T}$ in the form

$$\begin{aligned} \overset{\leftrightarrow}{T}_m &= d\rho \wedge e_i T^{ij} \frac{1}{2!} \underbrace{e_{jkl}}_{\vec{F}_m} dx^k \wedge dx^l \quad (\quad) \\ &= d\rho \wedge \vec{F}_m \quad (\quad) \end{aligned}$$

7. Summary

The bi-vector valued 3-form

$$\overset{\leftrightarrow}{T}_m = d\rho \wedge \vec{F}_m$$

$$= e_m \wedge e_i T^{ij} \frac{1}{2!} \underbrace{e_{jkl}}_{dx^m \wedge dx^k \wedge dx^l} dx^m \wedge dx^k \wedge dx^l$$

a $(\frac{2}{3})$ tensor, is Cartan's mathematization

of the moment of force acting on an as-yet-unspecified element of volume.

VI ROTATIONAL EQUILIBRIUM and NON-EQUILIBRIUM

23, 16

The condition for any solid 3-cube
in a given force field \vec{F} to be in rotational
equilibrium is

$$\stackrel{\leftrightarrow}{\int_m} = 0$$

On the other hand, the condition for
non-equilibrium is

$$\stackrel{\leftrightarrow}{\int_m} \neq 0$$

Consider a homogeneous electric field
 \vec{E} . An uncharged or neutrally charged
dielectric 3-cube will be in trans-
lation equilibrium. However, if that

3-cube has a net dipole moment $d = q\vec{r}$
then this dielectric cube will be
subjected to a torque, namely,

23/7

$$\vec{\tau} = \vec{d} \times \vec{E}$$

The corresponding torque two-form field.

ω

$$\overset{\leftrightarrow}{\tau} = \dots TBJ$$

23.18

CURVATURE-INDUCED

MOMENT OF ROTATION

The above 6 step Cartan-Wheeler

- Misner method of mathematizing

the moment of a 2-form applies

not only to the force 2-form

$$\tilde{F} = e_i \tilde{\tau}^{ij} \frac{1}{2} \epsilon_{jkl} dx^k \wedge dx^l$$

but also to the rotation form

$$\overset{\leftrightarrow}{R} = e_\mu \wedge e_\nu R^{\mu\nu}$$

$$= e_\mu \wedge e_\nu R^{\mu\nu} \int_{\partial\Omega} dx^\alpha \wedge dx^\beta$$

In that case one obtains with
 $dP = e_\lambda \otimes dx^\lambda$

(§15.4 in MTW) the moment of rotation

$$dP \wedge \overset{\leftrightarrow}{R} = e_\lambda \wedge e_\mu \wedge e_\nu R^{\mu\nu} \int_{\partial\Omega} dx^\lambda \wedge dx^\alpha \wedge dx^\beta$$

in an as-yet-unspecified element of volume,