

## LECTURE 24 & 25

LECTURE 24

I) The moment of force

A) 3-cube with an electric dipole moment

B) Translational response to a uniform electric field.

C) Moment of force method vs. the cross product method.

D) Arbitrarily located fulcrum

E) Cartan's "displacement" ( $dP$ ) conceptualized.

F) The total moment of force as a generator of rotation

LECTURE

II) Comparison with Einstein's field equation

III) The  $\star$  isomorphism

IV) conclusion torque generates rotation

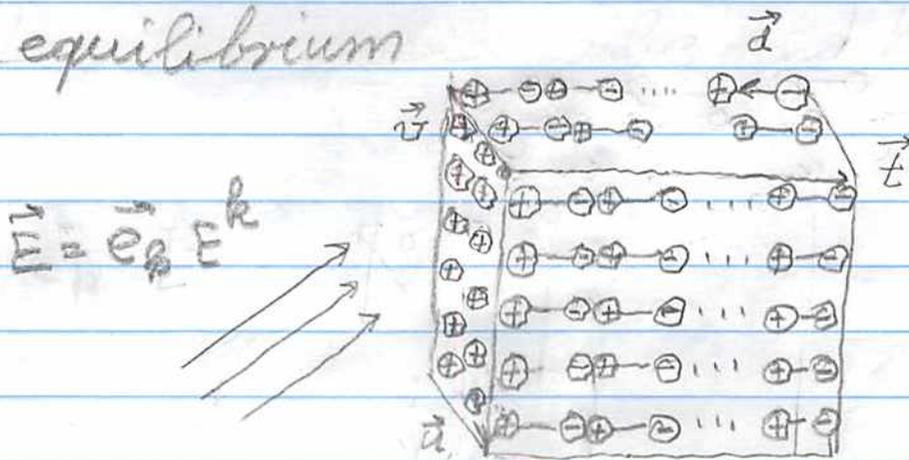
The line of reasoning leading to the geometrization of the l.h.s. of

Einstein's

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

consists of extending to 4-d spacetime the 3-d concept of "moment of force."

I.) The prerequisite for this concept is the mathematization of "force" on a 3-d rigid parallelepiped in translational equilibrium



It is spanned by

$$\begin{aligned}\vec{u} &= \Delta x^1 \vec{e}_1 = \Delta x^1 \frac{\partial}{\partial x^1} \\ \vec{v} &= \Delta x^2 \vec{e}_2 = \Delta x^2 \frac{\partial}{\partial x^2} \\ \vec{z} &= \Delta x^3 \vec{e}_3 = \Delta x^3 \frac{\partial}{\partial x^3}\end{aligned}$$

FIGURE 24.1

## Caption to Figure 24,1

An electret parallelepiped subjected to an external electric field  $\vec{E}$ . The rigid (From now on called a "cube") parallelepiped is spanned by the three infinitesimal vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{E}$ . The

## A) ELECTRIC DIPOLE MOMENT

The cube consists of an array of alligned molecular dipole moments  $q\vec{d}$ .

$$q\vec{d} : -q \xrightarrow{\vec{d}} +q$$

The charges inside the cube cancel. On the surface of the mini-cube there is a net charge on one side and an opposite charge on the other side. But the total charge of the cube is zero. By contrast, the

total dipole moment is  $(\# \text{ of molecules}) \times q\vec{d} \equiv \vec{D} = \left( \begin{array}{l} \text{"total"} \\ \text{dipole} \\ \text{moment"} \end{array} \right)$

Being exposed to the  $\vec{E}$  field, the torque experienced by the cube is  $\boxed{\vec{D} \times \vec{E} = \vec{T}}$ .

The cube has a permanent dipole moment. It comes about follows

$$\underbrace{\Delta x^1 \Delta x^2 \Delta x^3 N}_{\text{\# of molecules}} \times \underbrace{q d}_{\substack{\text{(dipole} \\ \text{moment)} \\ \text{(molecule)}}} = \underbrace{\quad}_{\text{(total dipole moment)}}$$

("surface density of molecules") =  $\sigma^3$

### B) TRANSLATIONAL RESPONSE TO AN ELECTRIC FIELD

This cube is subjected to an electric field

$$\vec{E} = \vec{e}_k E^k$$

and hence to the force field

$$(24.2) \quad \vec{F}_m = \vec{e}_k E^k \underbrace{q \sigma^m}_{\substack{\text{"surface density"} \\ \text{of molecules}}} \underbrace{\epsilon_{mij}}_{\equiv F^k_{ij}} \frac{dx^i dx^j}{2!}$$

Thus the forces on the two opposing oriented faces spanned by  $\{\vec{u}, \vec{v}\}$  are

$$\vec{F}(\vec{u}, \vec{v}) \Big|_{x^3+\Delta x^3} = \vec{F}(\vec{e}_1, \vec{e}_2) \Delta x^1 \Delta x^2 \Big|_{x^3+\Delta x^3} = e_R F_{12}^R \Delta x^1 \Delta x^2 \Big|_{x^3+\Delta x^3}$$

= force on  $u, v$  area at  $x^3+\Delta x^3$

$$\vec{F}(\vec{v}, \vec{u}) \Big|_{x^3} = \vec{e}_R \leftarrow F_{12}^R \Delta x^1 \Delta x^2 \Big|_{x^3}$$

= force on  $v, u$  area at  $x^3$

The forces on the other faces spanned by

$\{\vec{v}, \vec{e}_1\}$  and  $\{\vec{e}_1, \vec{u}\}$  are given by similar expressions.

The fact that the parallelepiped is in

translational equilibrium is

mathematized by the statement that

there is no net force acting on it,

i. e., that the sum total of the forces on the parallelepiped's 6 faces vanishes

(24,3)

$$\sum_{\ell=1}^6 \vec{F}_m(\ell^{\text{th}} \text{ face}) = \vec{0}$$

c) MOMENT OF FORCE VERSUS CROSS PRODUCT  
There also is rotational equilibrium and non-equilibrium, 24.5

In mechanics one learns the concepts "rotational motion", "axis", "force", "fulcrum", "moment of force", "torque", etc. They are mathematized in terms of the cross product in 3-d Euclidean space in order to put the observation "torque generates rotation" into quantitative form.

Even though the cross product method is a valid method for mathematizing this observation, this method is restricted to 3-d Euclidean where it serves admirably. However, we need geometrical methods that extend into 4-d spacetime, the environ-

ment for Einstein's gravitational field equation.

The "moment of force" method fulfill this need. It is an alternative way of mathematizing "torque generates rotation" and it extends to 4-d space-time where it <sup>easily</sup> geometrizes the left hand side of Einstein's equation <sup>4-version</sup>

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

## D) ARBITRARILY LOCATED FULCRUM.

24.7

The "moment of force" method is a geometrical approach which starts with a fulcrum, say  $P'$ , either inside or outside the parallelepiped in Figure 24.1. Then there are the levers to the centers of each of the 6 faces,  $\{P_i^{\pm}, i=1,2,3\}$

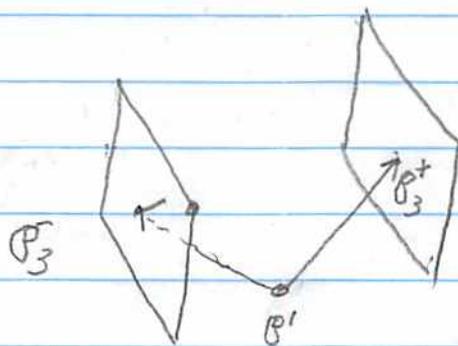


Figure 24.2

$$\overrightarrow{P_3^+ - P_3} = \Delta x^3 \vec{e}_3$$

are the six respective displacement vectors

$$\overrightarrow{(P_i^{\pm} - P')}, i=1,2,3$$

The 3-cube is set into rotational motion by the sum of the six corresponding moments of force.

The sum of

24.8

the moments of force ("torque") involving  
oriented  
the two faces at  $x^3$  and  $x^3 + \Delta x^3$  is

$$(24.4) \quad (\rho_3^+ - \rho_3^-) \wedge \vec{F}_m \Big|_{x^3}^{x^3 + \Delta x^3} (u, v) + (\rho_3^- - \rho_3^+) \wedge \vec{F}_m \Big|_{x^3}^{x^3 + \Delta x^3} (v, u),$$

with similar expressions for the other two pairs of faces.

The fact that the points  $\rho_1^\pm, \rho_2^\pm, \rho_3^\pm$  are centered at the 6 opposing faces as in

Figure 24.2 on page 24.5 implies

the existence of the three lever arms

$$\begin{aligned} \rho_1^+ - \rho_1^- &= \vec{e}_1 \Delta x^1 \\ \rho_2^+ - \rho_2^- &= \vec{e}_2 \Delta x^2 \\ \rho_3^+ - \rho_3^- &= \vec{e}_3 \Delta x^3 \end{aligned}$$

Following Cartan, one conceptually integrates them (See Ch. 2 & 3 in Ayn Rand's "Introduction to Objectivist Epistemology") into a wider mathematical concept

$$\vec{e}_1 dx^1 + \vec{e}_2 dx^2 + \vec{e}_3 dx^3 \equiv d\rho \in (1),$$

which Cartan call the "displacement vector,"

and which MTW call "Cartan's unit tensor". It has the property that

$$d\rho(\vec{u}) = \vec{e}_1 \Delta x^1$$

$$d\rho(\vec{v}) = \vec{e}_2 \Delta x^2$$

$$d\rho(\vec{t}) = \vec{e}_3 \Delta x^3$$

[E]

START: The conceptualization Cartan's "displacement"

If a vector such as

$$\vec{w} = w^1 \frac{\partial}{\partial x^1} + w^2 \frac{\partial}{\partial x^2} + w^3 \frac{\partial}{\partial x^3} = \vec{e}_i w^i$$

conceptualizes the change of an as-yet-unspecified scalar, say  $\psi$ , into the direction specified by the observed/measured numbers  $w^1, w^2, w^3$ , then

$d\rho$  is the change of this as-yet-unspecified scalar due to the as-yet-unspecified direction

$$\begin{aligned} d\rho(\vec{w}, \psi) &= dx^i(\vec{w}) \delta_j^i \frac{\partial(\psi)}{\partial x^j} \\ &= w^j \delta_j^i \frac{\partial \psi}{\partial x^i} \left( = w^j \frac{\partial \psi}{\partial x^j} \right) \end{aligned}$$

The verbal of definition of Cartan's  
"displacement"  $dP$  is therefore

$$dP = dx^i \otimes \frac{\partial}{\partial x^i} = \left( \begin{array}{l} \text{rate of change of an} \\ \text{as-yet-unspecified} \\ \text{scalar into an} \\ \text{as-yet-unspecified} \\ \text{direction} \end{array} \right) \quad (24.5)$$

where  $\{dx^i \equiv \omega^i\}$  is the basis dual to

$$\left\{ \frac{\partial}{\partial x^i} \equiv \vec{e}_i \right\}!$$

$$\underbrace{\omega^i(\vec{e}_j)}_{\text{mathematicians' notation}} = \delta^i_j = \underbrace{\langle dx^i, \frac{\partial}{\partial x^j} \rangle}_{\text{MTW's (physicists') notation}} = \underbrace{\left( \frac{\partial}{\partial x^j} x^i = \frac{\partial x^i}{\partial x^j} \right)}_{\text{calculus notation}}$$

In terms of the process of concept formation  
(as spelled out in "Introduction to Objectivist Epistemology"), Cartan performed with his  $dP$  a

process of double abstraction:

Starting with a scalar field, which  
a familiar abstraction that starts  
with concretes (directly observable  
or perceivable), he abstracted from  
that a higher order abstraction:

a vector as an abstraction of change.  
From that he went to a still higher  
level of abstraction by introducing his  
infinitesimal "displacement"  $dp$  as  
defined verbally and mathematically  
on page 24.10

FINISH

F) THE TOTAL MOMENT OF FORCE AS A ROTATION GENERATOR 24/12  
 on page 24.8

Augment Eq. (24.4) to obtain the total

moment of force experienced by the

cube. Referring to Figure 24.1, one has

(24.6 a,b,c)

$$\vec{J} = \underbrace{\vec{e}_3 \Delta X^3 \wedge e_R}_{dx^3(\vec{t})} F^R_{[2j]} dx^i \wedge dx^j (u, v) \quad (a)$$

$$+ \underbrace{\vec{e}_1 \Delta X^1 \wedge e_R}_{dx^1(u)} F^R_{[ij]} dx^i \wedge dx^j (v, t) \quad (b)$$

$$+ \underbrace{\vec{e}_2 \Delta X^2 \wedge e_R}_{dx^2(v)} F^R_{[ij]} dx^i \wedge dx^j (t, u) \quad (c)$$

Note that in Eq. (24.6a)  $\Delta X^3 = dx^3(\vec{t})$  is the extent of the

$\vec{e}_3$  contribution, namely  $dP(\vec{t})$ , to the lever

arm  $dP$ . This contribution generates rotation in the  $\vec{e}_3 - \vec{e}_R$  plane. But it is non-zero only if the lever is pointing into the  $\vec{e}_3$  (i.e. the  $\vec{t}$ )-direction.

Consequently, one has

$$\vec{J} = \vec{e}_3 \wedge e_R F^R_{[2j]} dx^3 \wedge dx^i \wedge dx^j (t, u, v)$$

$$+ \vec{e}_1 \wedge e_R F^R_{[ij]} dx^1 \wedge dx^i \wedge dx^j (u, v, t)$$

$$+ \vec{e}_2 \wedge e_R F^R_{[ij]} dx^2 \wedge dx^i \wedge dx^j (v, t, u)$$

For the moment-of-force-induced infinitesimal

rotation. Leaving cube vectors  $u, v, t$  as-yet-unspecified, obtain the generator of spatial rotations

$$(24,7) \quad \boxed{\vec{J}_m = \vec{e}_L \wedge \vec{e}_R F^R_{[L]} \underbrace{dx^2 \wedge dx^3 \wedge dx^3}_{= dP}} \quad (24,5)$$

or in Cartan's and MTW's frame

-invariant notation

$$(24,8) \quad \boxed{\vec{J}_m = dP \wedge \vec{F}_m}$$

II.)

COMPARISON WITH EINSTEIN FIELD EQ'N

This (infinitesimal) rotation generator

is the Euclidean version of what in

spacetime is MTW's "moment of rotation"

(their Eq. (15.12)), which in essence (via

their special  $\star$  transformation) is the

left hand side of the Einstein field

equation.

$$(24.9) \quad \star(dP \wedge \vec{R}) = \frac{8\pi G}{c^4} \star T$$

If one recognizes <sup>that</sup> energy and momentum of a system are generators of time and space translation, then one must say

this: MTW's "moment of rotation" on <sup>the above</sup> the l.h.s. of Eq. (24.9) mathematizes

(modulo their one-to-one  $\star$  transformation) <sup>both</sup> in a geometrical way, time and space translations of a gravitating system in its 4-d spacetime environment.

⊛ Recalling that Eq. (24.9) stipulates that (in appropriate units)

$$\star(dP \wedge \vec{R}) = \frac{\text{(amount of momentum energy)}}{\text{(as-yet-unspecified volume)}}$$

This refers to a physical attribute of moving matter expressed in geometrical terms ("moment of curvature-induced rotation".)

Q: What is its Euclidean version, <sup>?</sup> i.e., what, if any, is the physical attribute corresponding to geometrical concept expressed by Eqs. (24.7) or (24.8)

("moment of electrostatically-induced force")

A: To that end introduce the Euclidean version of MTW's  $\star$  transformation. Noticing that for 3-d Euclidean space  $E^3$

$$\dim \Lambda^2(E^3) = 3 = \dim E^3,$$

### III) THE $\star$ ISOMORPHISM

24.16

Introduce the 1-1 correspondence

$$\star: \Lambda^2(E^3) \rightarrow E^3$$

$$\vec{e}_2 \wedge \vec{e}_3 \mapsto \star(\vec{e}_2 \wedge \vec{e}_3) = \vec{e}_m \epsilon^m{}_{23}$$

Here

$\epsilon^m{}_{23} = g^{mm} \epsilon_{n23} = g^{mm} |g|^{-1/2} [m23]$   
are the components of the Levi-Civita tensor in (2).

Consequently,

$$\begin{aligned} \star(\vec{S}) &= \star(\vec{e}_2 \wedge \vec{e}_3 F^k{}_{ij} dx^2 \wedge dx^3) \\ &= \vec{e}_m \epsilon^m{}_{23} F^k{}_{ij} dx^2 \wedge dx^3 \end{aligned}$$

Simplify by referring to Eq. (24.2), namely,

$$F^k{}_{ij} = E^k g \sigma^n \epsilon_{nij} \equiv F^k \sigma^n \epsilon_{nij}$$

Here  $F^k = E^k g$  the vectorial force components of force

surface density ("shear", "pressure") 2-form

They are  
 $\vec{F}_m$ , Eq. (24.2), due to the electric field  $\vec{E}$  acting  
on the surface charge density 2-form

$$q\sigma = q\sigma^n \epsilon_{nij} \frac{dx^i dx^j}{2!}$$

24,17

In terms of these force component

$$\begin{aligned} F^k_{[kij]} dx^l dx^i dx^j &= F^k \sigma^n \epsilon_{nij} dx^l dx^i dx^j \\ &= F^k \sigma^n \sqrt{g} \underbrace{[nlij]}_{\delta^n_l} [lij] dx^1 dx^2 dx^3 \\ &= F^k \sigma^l \sqrt{g} dx^1 dx^2 dx^3 \end{aligned}$$

Consequently,

$$\begin{aligned} \star(\vec{F}_m) &= \vec{e}_m \epsilon^m_{lk} F^k \sigma^l \sqrt{g} dx^1 dx^2 dx^3 \\ &= \vec{e}_m \underbrace{\epsilon^m_{lk}}_{\frac{1}{\sqrt{g}} [mlk]} F^k \sigma^l \sqrt{g} dx^1 dx^2 dx^3 \end{aligned}$$

Here  $\{F_k\}$  and  $\{\sigma_l\}$  are the metric-related components corresponding to  $\{F^k\}$  and  $\{\sigma^l\}$ .

Thus one obtain the result that

$$\star(\vec{F}_m) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ F_1 & F_2 & F_3 \end{vmatrix} dx^1 dx^2 dx^3$$

Relative an orthonormal basis one has

$$(24.10) \star \left( \vec{\mathcal{J}}_{\vec{m}} \right) = (\vec{\sigma} \times \vec{F}) \cdot dx^1 \wedge dx^2 \wedge dx^3 = (\vec{D} \times \vec{E}) dx^1 dx^2 dx^3$$

From mechanics one uses the "vector cross product" of the lever arm vector  $\vec{\sigma}$  and the force  $\vec{F}$  to mathematize the torque applied to a body. Here that body is a 3-cube subtended by the vectors  $\vec{u}, \vec{v}, \vec{t}$ :

$$\Delta(\text{volume}) \equiv dx^1 dx^2 dx^3 (\vec{u}, \vec{v}, \vec{t}) = \Delta x^1 \Delta x^2 \Delta x^3$$

Thus one has

$$\star \left( \vec{\mathcal{J}}_{\vec{m}} \right) (u, v, t) = (\text{torque}) \Delta(\text{volume})$$

IV) TORQUE GENERATES ROTATION

The conclusions are

$$(24.11) \quad \textcircled{1} \quad \boxed{\star \left( \vec{\mathcal{J}}_{\vec{m}} \right) = \frac{(\text{torque})}{(\text{volume})}}$$

and from Eqs (25, 7) - (25, 8)

②

(24.12)

$$\vec{\tau} = \frac{\text{(rotation generator)}}{\text{(volume)}} (= dP \wedge \vec{F})$$

The electric-field-induced torque  $\vec{\tau} = \vec{D} \times \vec{E}$

$$\vec{D} \times \vec{F} = \vec{D} \times \vec{F}$$

in a 3-cube of a given volume equals

(modulo  $\star$ ) the generated rotation, or

in familiar language

(24.13)

③

"torque generates rotation"