Lecture 24

(Purpose: Attain mastery of using modern multivariable calculus methods for mathematizing a constellation of key concepts from electrostatics to be exteded to gravitation physics)

I.) EINSTEIN'S EQUATIONS: WHAT THEY MATHEMATIZE

The Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

mathematizes two facts:

(i) geometry controls the motion of matter (via $T_{\mu}{}^{\nu}{}_{;\nu} = 0$), and

(ii) matter controls the geometry of spacetime.

Einstein's line of reasoning for arriving at his tensorial equation was guided primarily by the physical and geometrical properties of the right hand side. The l.h.s. came out as a deductive consequence of his inductive line of reasoning applied to the right hand side. Although the l.h.s was a tensorial consequence, Einstein never identified its physical or its geometrical meaning and origin. This gap was filled later by Cartan and Wheeler. They filled it with the geometrical concept of "moment of rotation". Among other things, this resulted in the conservation of momenergy principle to be mathematized by them in terms of the topological principle that "the boundary of a boundary is zero".

The concept "moment of rotation" is an extension of the one familiar from mechanics in 3-d Euclidean space: torque, the moment of force. Both force and torque cause motion of bodies, translation and rotation. But in order to bring out its relevance to the Einstein field equations, both force and the moment of force need to be geometrized in the form surface and volume densities.

II.) DIELECTRIC IN A STATIC FORCE FIELD)

To this end consider a rigid parallelopiped (which for shorthand we will call a "cube", a "3-cube", or a "3-d cube") composed of an array of uniformly distributed and rigidly aligned permanent molecular dipole moments,

$$\vec{p} = q\,\vec{d} = q\,\vec{e}_m d^m$$

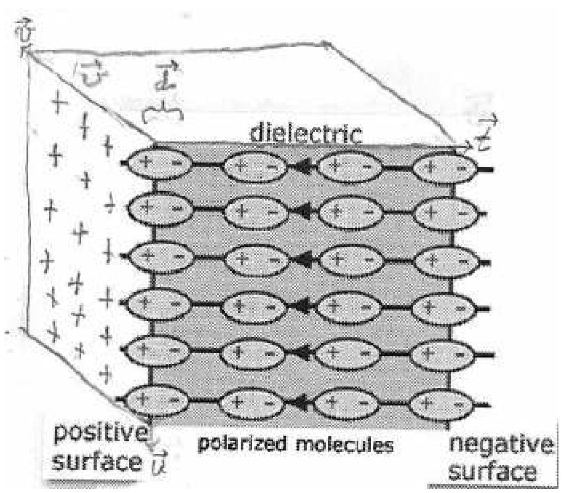

Figure 1: A dielectric parallelopiped of volume $\Delta x^1 \Delta x^2 \delta x^3$ subjected to an external electrostatic field.

The volume of this cube is spanned by the triad of vectors

$$\vec{u} = \Delta x^1 \frac{\partial}{\partial x^1} \equiv \Delta x^1\, \vec{e}_1, \tag{2}$$

$$\vec{v} = \Delta x^2 \frac{\partial}{\partial x^1} \equiv \Delta x^2\, \vec{e}_3, \tag{3}$$

$$\vec{t} = \Delta x^3 \frac{\partial}{\partial x^1} \equiv \Delta x^2\, \vec{e}_3. \tag{4}$$

The electrostatic polarization in this 3-d cube is

$$\overrightarrow{P} = N\,\vec{p} = qN\vec{d}\ \left(= \frac{\textbf{(dipole moment)}}{\textbf{(volume)}}\right) \tag{5}$$

$$\equiv \vec{e}_m d^m\, qN \tag{6}$$

Here

$$N = \frac{\textbf{(\# of molecules)}}{\textbf{(volume)}} \tag{7}$$

is the density of molecules in this cube. The total polarization is

$$\overrightarrow{P} \times \textbf{(volume)} = \begin{pmatrix} \textbf{total} \\ \textbf{dipole} \\ \textbf{moment} \end{pmatrix} \tag{8}$$

The molecular dipoles in their uniform alignment yield surface charge densities on each of the six oriented faces of the cube, namely

$$\begin{pmatrix} \textbf{surface density} \\ \textbf{of molecules} \end{pmatrix} \equiv \sigma^3$$

$$\begin{aligned} q\ \overbrace{Nd^3}\ \ \epsilon_{312}\,\Delta x^1 \Delta x^2|_{x^3+\Delta x^3} &= qNd^m \epsilon_{m|ij|} dx^i \wedge dx^j (\vec{u},\vec{v})|_{x^3+\Delta x^3}, \\ q\ Nd^3\ \ \epsilon_{312}\,\Delta x^1 \Delta x^2|_{x^3} &= qNd^m \epsilon_{m|ij|} dx^i \wedge dx^j (\vec{u},\vec{v})|_{x^3}, \end{aligned}$$

and similarly for the other two pairs of faces.

III.) THE FORCE FIELD

Upon subjecting the cube to an electrostatic field

$$\overrightarrow{E} = \vec{e}_k E^k, \tag{9}$$

the force field acting on the charged surfaces is mathematized by

$$\underline{\overrightarrow{F}} = \vec{e}_k E^k\ q\ \ \overbrace{\sigma^m \epsilon_{m|ij|} dx^i \wedge dx^j}^{\begin{pmatrix} \textbf{surface density} \\ \textbf{of molecules on an} \\ \textbf{as-yet-unspecified} \\ \textbf{area} \end{pmatrix}} \tag{10}$$

$$= \vec{e}_k E^k\ \underbrace{q\ \ Nd^m \epsilon_{m|ij|} dx^i \wedge dx^j}_{\begin{pmatrix} \textbf{surface charge} \\ \textbf{density on an} \\ \textbf{as-yet-unspecified} \\ \textbf{area} \end{pmatrix}} \tag{11}$$

or more economically by

$$\boxed{\underline{\vec{F}} \equiv \vec{F}_{ij}\ dx^i \wedge dx^j,} \tag{12}$$

the vectorial force field (surface density) acting on the faces of a 3-cube. This is the momentum flux 2-form, a vectorial flux across an as-yet-unspecified element of area[1]. It is a vector-valued 2-form, a tensor of rank $\binom{1}{2}$.
The forces on the opposing oriented faces spanned by $\{\vec{u}, \vec{v}\}$ are

$$\begin{aligned} \underline{\vec{F}}\ (\vec{u}, \vec{v})|_{x^3+\Delta x^3} &= \vec{F}_{12}\ |_{x^3+\Delta x^3} \Delta x^1 \Delta x^2 \\ &= \textbf{force on } (\vec{u}, \vec{v})\textbf{-area at } x^3 + \Delta x^3 \end{aligned} \tag{16}$$

and

$$\begin{aligned} \underline{\vec{F}}\ (\vec{v}, \vec{u})|_{x^3} &= (-)\ \vec{F}_{12}\ |_{x^3} \Delta x^1 \Delta x^2 \\ &= \textbf{force on } (\vec{v}, \vec{u})\textbf{-area at } x^3, \end{aligned} \tag{17}$$

with similar expressions for the other faces spanned by $\{\vec{v}, \vec{t}\}$ and $\{\vec{t}, \vec{u}\}$.

IV.) SURFACE FORCES VS. VOLUME FORCE

The total force on these opposing faces, all six of them, is

$$\begin{aligned} \vec{F}_{total} =& \underline{\vec{F}}\ (\vec{u}, \vec{v})|_{x^3+\Delta x^3} + \underline{\vec{F}}\ (\vec{v}, \vec{u})|_{x^3} \\ &+ \underline{\vec{F}}\ (\vec{v}, \vec{t})|_{x^1+\Delta x^1} + \underline{\vec{F}}\ (\vec{t}, \vec{v})|_{x^1} \\ &+ \underline{\vec{F}}\ (\vec{t}, \vec{u})|_{x^2+\Delta x^2} + \underline{\vec{F}}\ (\vec{u}, \vec{t})|_{x^2} \end{aligned} \tag{18}$$

which in light of Eq.(12) and Eqs.(2)-(4) becomes

$$\vec{F}_{total} = \nabla_{\vec{t}} (\vec{F}_{12}\ \Delta x^1 \Delta x^2) \tag{19}$$

$$+ \nabla_{\vec{u}} (\vec{F}_{23}\ \Delta x^2 \Delta x^3) \tag{20}$$

$$+ \nabla_{\vec{v}} (\vec{F}_{31}\ \Delta x^3 \Delta x^1). \tag{21}$$

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[1] The force field exerts stresses on the faces of the electretized cube in Figure 1. These stresses are mathematized in terms of the stress tensor familiar from continuum mechanics. Let

$$d^2\Sigma_m = \epsilon_{m|ij|} dx^i \wedge dx^j \quad m = 1, 2, 3 \tag{13}$$

be the 2-form of an element of area spanned by an as-yet-unspecified pair of vectors. The stress tensor is a vector (force) valued surface-density 2-form

$$\vec{e}_k T^{km} d^2\Sigma_m \equiv \vec{F}_{|ij|}\ dx^i \wedge dx^j. \tag{14}$$

Its components, in light of Eq.(11) are

$$T^{km} = E^k N q d^m = E^k \times (\text{dipole moment})^m \tag{15}$$

This stress-tensor is not symmetric: $T^{km} \neq T^{mk}$. This happens when the dipole vector density is not collinear with the applied electrostatic field. Consequently, the stresses acting on the cube exert a non-zero torque on it. This, as we know, imparts angular momentum to the cube.

By introducing the vector valued three-form

$$
\begin{aligned}
d\,\underline{\vec{F}} &= d\,(\vec{F}_{|ij|}\;dx^i\wedge dx^j) && (22)\\
&= d\,(\vec{F}_{12}\;dx^1\wedge dx^2) && (23)\\
&+ d\,(\vec{F}_{23}\;dx^2\wedge dx^3) && (24)\\
&+ d\,(\vec{F}_{31}\;dx^3\wedge dx^1), && (25)
\end{aligned}
$$

one recognizes that the total force vector, Eqs.(19)-(21), condenses into the value of that three-form evaluated on the vectors that span the volume of the cube,

$$\boxed{\vec{F}_{total}= d\,\underline{\vec{F}}\,(\vec{u},\vec{v},\vec{t}).} \tag{26}$$

Physically this is the total *volume force* acting on and averaged over the cube's interior domain, which is spanned by the three vectors $\vec{u},\vec{v}$ and $\vec{t}$. Mathematically $d\,\underline{\vec{F}}$ is the familiar *exterior derivative* of $\underline{\vec{F}}$. Next substitute Eq.(12) into Eq.(26), use the triad of vectors Eqs.(2)-(4) and thus obtain

$$
\begin{aligned}
\vec{F}_{total} &= \nabla_{\vec{e}_n}\;\vec{F}_{|ij|}\;dx^n\wedge dx^i\wedge dx^j\,(\vec{u},\vec{v},\vec{t}) && (27)\\
&= \left(\nabla_{\vec{e}_1}\;\vec{F}_{23} + \nabla_{\vec{e}_2}\;\vec{F}_{31} + \nabla_{\vec{e}_3}\;\vec{F}_{12}\right)\times \Delta x^1\Delta x^2\Delta x^3 && (28)\\
&= \left(\frac{\textbf{(force)}}{\textbf{(volume)}}\right)\times \Delta x^1\Delta x^2)\Delta x^3, && (29)
\end{aligned}
$$

the *volume force* experienced by the cube in its interior. Here

$$\vec{F}_{ij}= \vec{e}_k E^k\,qNd^m\epsilon_{mij} \tag{30}$$

are the *surface force densities* acting on the ij-labeled faces of the cube.

By equating Eq.(18) to Eq.(28) one obtains

$$
\begin{aligned}
\begin{pmatrix}\textbf{total force on}\\ \textbf{all 6 faces}\end{pmatrix} &\equiv \sum_{\ell=1}^{6}\underline{\vec{F}}\,(\ell^{th}\textbf{face}) && (31)\\
&= \left(\nabla_{\vec{e}_1}\;\vec{F}_{23} + \nabla_{\vec{e}_2}\;\vec{F}_{31} + \nabla_{\vec{e}_3}\;\vec{F}_{12}\right)\times \Delta x^1\Delta x^2\Delta x^3 && (32)
\end{aligned}
$$

The l.h.s. of this equation refers to the totality of the *surface forces* acting on the (6-faced) boundary of the cube. The r.h.s. of Eq.(32) refers to the *volume force* on the interior of the cube. Thus the 6 conditions on the surface boundary of the cube are sufficient for inferring the mean condition inside:

$$\frac{\sum_{\ell=1}^{6}\underline{\vec{F}}\,(\ell^{th}\textbf{face})}{\Delta x^1\Delta x^2)\Delta x^3} = \nabla_{\vec{e}_1}\;\vec{F}_{23} + \nabla_{\vec{e}_2}\;\vec{F}_{31} + \nabla_{\vec{e}_3}\;\vec{F}_{12}\;. \tag{33}$$

When the electrostatic field $\vec{E}$, Eq.(9), is homogeneous, i.e.

$$\nabla_{\vec{e}_i}\left(\vec{e}_k E^k\right) = 0, \quad (i = 1,2,3), \tag{34}$$

Eq.(33) becomes

$$\begin{aligned}\sum_{\ell=1}^{6} \underline{\vec{F}}\,(\ell^{th}\textbf{face}) &= \vec{e}_k E^k [\,\nabla_{\vec{e}_1}(Nqd^m\epsilon_{m23}) \\ &\qquad +\nabla_{\vec{e}_2}(Nqd^m\epsilon_{m31}) \\ &\qquad +\nabla_{\vec{e}_3}(Nqd^m\epsilon_{m12})]\,\Delta x^1\Delta x^2\Delta x^3 \\ &= \vec{e}_k E^k \quad \sum_{n=1}^{3}\frac{\partial(\sqrt{g}\,Nqd^n)}{\partial x^n}\Delta x^1\Delta x^2\Delta x^3 \\ &= \vec{e}_k E^k\,(N\,q\,d^n)_{;n}\underbrace{\sqrt{g}\,\Delta x^1\Delta x^2\Delta x^3}_{\begin{pmatrix}\textbf{invariant}\\ \textbf{volume}\end{pmatrix}} \end{aligned} \tag{35}$$

Here $\vec{e}_k E^k\,(N\,q\,d^n)_{;n}$ is the force density relative to an physical/orthonormal frame.

V.) ENERGY INJECTED INTO THE CUBE

If the cube undergoes displacement, say $\vec{w}= \vec{e}_\ell\, w^\ell$, then each of its 6 faces receives mechanical energy from the force field. The amount of that energy is

$$\vec{w}\cdot\underline{\vec{F}}\,(\ell^{th}\textbf{face}),\qquad \ell=1,\cdots,6, \tag{36}$$

the work done by each of the respective forces listed in Eq.(18). The union of the 6 oriented faces is the boundary $\partial\mathcal{D}$ of the cube's interior domain $\mathcal{D}$:

$$\bigcup_{\ell=1}^{6}(\ell^{th}face)=\partial\mathcal{D}.$$

The function

$$\underline{\vec{F}}\,(\cdots) = \cdot\,\vec{F}_{i,j}\;dx^i\wedge dx^j(\cdots) \tag{37}$$

$$= \vec{E}\quad N\,q\,d^m\epsilon_{mij}dx^i\wedge dx^j(\cdots) \tag{38}$$

is a linear vector-valued function on its components:

$$\vec{F}_{total}\left(\bigcup_{\ell=1}^{6}(\ell^{th}face)\right)=\sum_{\ell=1}^{6}\underline{\vec{F}}\,(\ell^{th}\textbf{face})$$

From this line of reasoning one arrives from Eq.(36) that

$$\boxed{\vec{w}\cdot\,\underline{\vec{F}}=\vec{E}\,\cdot\vec{w}\,N q\,d^m\epsilon_{mij}dx^i\wedge dx^j} \tag{39}$$

is the translational energy injected into the cube's interior through one of its as-yet-unspecified faces of its boundary.

The total change in mechanical energy of the dielectric cube is therefore

$$\vec{w}\cdot\vec{F}_{total} = \vec{w}\cdot\left[\sum_{\ell=1}^{6}\underline{\vec{F}}\,(\ell^{th}\textbf{face})\right], \tag{40}$$

In light of Eqs.(32) this total is

$$\vec{w}\cdot\vec{F}_{total} = \vec{w}\cdot\left(\nabla_{\vec{e}_1}\vec{F}_{23}+\nabla_{\vec{e}_2}\vec{F}_{31}+\nabla_{\vec{e}_3}\vec{F}_{12}\right)\times\Delta x^1\Delta x^2\Delta x^3, \tag{41}$$

or, equivalently, because of Eq.(26),

$$\vec{w}\cdot\vec{F}_{total}=\vec{w}\cdot d\,\underline{\vec{F}}\,(\vec{u},\vec{v},\vec{t}). \tag{42}$$

VI.) TRANSLATIONAL EQUILIBRIUM

Even though the opposing faces carry non-zero charges, the total charge on the 3-d cube is zero. Such a cube, when subjected to an $\vec{E}$**-field, which we take to be homogeneous** ($\nabla_{\vec{u}}\,\vec{E}=\nabla_{\vec{v}}\,\vec{E}=\nabla_{\vec{t}}\,\vec{E}=0$)**, exerts no net force on the cube. The sum total of the forces on the cube's 6 faces vanishes:**

$$\vec{F}_{total}=\sum_{\ell=1}^{6}\underline{\vec{F}}\,(\ell^{th}\textbf{face})=\vec{0}\ . \tag{43}$$

Thus the cube is in a state of *translational equilibrium.* It gains no translational energy. In light of the fact that Eq.(43) holds for any set of spanning vectors in Eq.(26), one concludes that

$$\boxed{d\,\underline{\vec{F}}=0} \tag{44}$$

mathematizes that condition for translational equilibrium. In light of Eq.(33) this is equivalent to

$$\nabla_{\vec{e}_1}\,\vec{F}_{23}+\nabla_{\vec{e}_2}\,\vec{F}_{31}+\nabla_{\vec{e}_3}\,\vec{F}_{12}=0 \tag{45}$$

or more compactly

$$\boxed{\nabla_i\,\vec{F}_{jk}+\nabla_j\,\vec{F}_{ki}+\nabla_k\,\vec{F}_{ij}=0} \tag{46}$$

VI.) DIVERGENCELESS VECTOR FIELD

The internal charge structure of the cube consists of dipoles distributed uniformly throughout its interior. If $\vec{e}_m d^m(\vec{E})q$ **is the molecular dipole moment**[2]**, then**

$$\vec{P}=\vec{e}_n d^n q\,N\equiv\vec{e}_n P^n$$

in Eq.(6) is the macroscopic polarization vector field. Compare its components with those in the volume force, Eq.(35), experienced by the cube under the condition of translational equilibrium, Eq.(43). Based on this, the conclusion is that the divergence of the polarization vector field vanishes:

$$0=(N\,q\,d^n)_{;n}\equiv P^n_{\ ;n}\equiv\nabla\cdot\vec{P}\equiv\textbf{div (polarization)} \tag{47}$$

[2]The molecular charge separation vector $\{d^m(\vec{E})\}$ is typically a linear, but not necessarily a coliniear, function of the externally applied electrostatic field.

VII.) CONCLUSION

Mathematically the Einstein field equations (EFE), Eq.(1), is a geometrical extension[3] of the interaction between a dielectric body and the electrostatic forces acting on it. The forces in Euclidean space have two types of causal attributes:

1. **those that result in translational motion and**
2. **those that result in rotational motion.**

For Einstein's field equations (EFE) both types need to be extended to the 4-d spacetime. Moreover, both of them require the conservation laws stated in the form of the generalized vectorial and tensorial Stokes' theorem, the relation between vectorial, as well as tensorial, physical attributes inside a given 3-d domain to those on its 2-d boundary.

	Electrostatics	**Gravitation**
Eq.(#)	**Electrostatic-induced force field:**	**Curvature-induced rotation field:**
Eq.(12)	$\underline{\vec{F}}=\vec{F}_{i,j}\ dx^i \wedge dx^j$	$\underline{\overleftrightarrow{\mathcal{R}}}=\overleftrightarrow{R}_{i,j}\ dx^i \wedge dx^j$
Eq.(11)	$=\vec{E}\quad N\ q\, d^m \epsilon_{m\vert ij\vert} dx^i \wedge dx^j$	$= \vec{e}_\mu \wedge \vec{e}_\nu R^{\mu\nu}{}_{\alpha\beta} dx^\alpha dx^\beta$
	Volume force:	
Eq.(26)	$\vec{F}_{total}= d\ \underline{\vec{F}}\ (\vec{u},\vec{v},\vec{t})$	$d\ \underline{\overleftrightarrow{\mathcal{R}}}\ (\mathbf{u},\mathbf{v},\mathbf{t})$
	Change in energy due to displacement shift $\vec{w}$:	**Rotational change in movement due to displacement shift $\mathbf{w} = \mathbf{e}_\nu w^\nu$:**
Eq.(39)	$\underline{\vec{F}}\cdot\vec{w} =\vec{E}\cdot\vec{w}\, N q\, d^m \epsilon_{mij} dx^i \wedge dx^j$	$\underline{\overleftrightarrow{\mathcal{R}}}\cdot(\mathbf{e}_\nu w^\nu) = \vec{e}_\mu w^\nu R^\mu{}_{\nu\alpha\beta} dx^\alpha dx^\beta$
	Translational Equilibrium:	**Bianchi Identity:**
Eq.(44)	$d\ \underline{\vec{F}}= 0$	$d\ \underline{\overleftrightarrow{\mathcal{R}}}= 0$
Eq.(46)	$\nabla_i\ \vec{F}_{jk} +\nabla_j\ \vec{F}_{ki} +\nabla_k\ \vec{F}_{ij}= 0$	$\nabla_\gamma\ \overleftrightarrow{R}_{\alpha\beta} +\nabla_\alpha\ \overleftrightarrow{R}_{\beta\gamma} +\nabla_\beta\ \overleftrightarrow{R}_{\gamma\alpha}= 0$

Table 1 above highlights the extension of vectorial concepts from electrostatics in a 3-d Euclidean environment to tensorial concepts in 4-d spacetime necessary for the EFE.

[3]The extension is one from vectors in Euclidean space to tensors in 4-d spacetime.