

LECTURE 27

I Rotational kinematics

II Lever arms

III Moment of force

IV Torque: two mathematizations

V The \star ("Hodge dual") isomorphism

Reading assignment

1. Typeset Lecture 24

2. MTW §15.3

I) ROTATIONAL KINEMATICS

However, translational equilibrium does not imply rotational equilibrium.

Consider the forces acting on a pair of opposing faces in relation to a fulcrum, say P' , either inside or outside a cube:

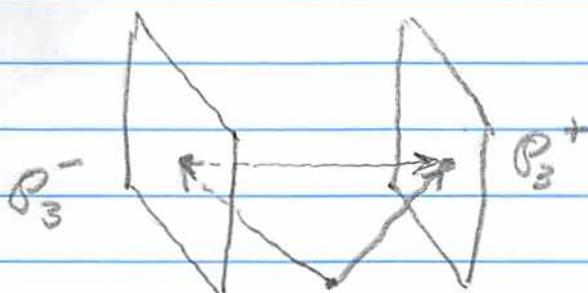


Figure 27.1

Fulcrum P' and
two of its levers

$$\overrightarrow{P_3^+} - \overrightarrow{P_3^-} = \Delta x \hat{\vec{e}}_3$$

The fulcrum P' gives rise to 6 displacement vectors

$$\overrightarrow{P_i^\pm - P'} ; \quad i = 1, 2, 3$$

These vectors give rise to moments of force applied to each pair of oppositely oriented faces

$$(27.1) \quad (\overleftrightarrow{\tau})_3 = (\rho_3^+ - \rho_1^-) \wedge \vec{F}(u, v) + (\rho_3^- - \rho_1^+) \wedge \vec{F}(v, u)$$

$x^3 + \Delta x^3$ "opposite orientation"

with similar expression for the other faces.

II.) LEVER ARMS

The points ρ_i^\pm , $i=1,2,3$, are centered on the opposite faces of the opposing faces. Consequently, one has three connecting levers

$$\overrightarrow{\rho_i^+ - \rho_i^-} = \vec{e}_i \Delta x^i \text{ (nosum)}; i=1,2,3$$

Following Cartan, one integrates them conceptually (see Ch. 2 & 3 in Ayn Rand's "Introduction to Epistemology") into

into the wider (more abstract) mathematical concept

$$\vec{e}_1 dx^1 + \vec{e}_2 dx^2 + \vec{e}_3 dx^3 = d\rho \quad \in (1).$$

Cartan call it the "displacement vector", which MTW call "Cartan's unit" tensor. Its dictionary definition would be

$$(27.2) \quad d\rho = dx^i \otimes \frac{\partial}{\partial x^i} = \left[\begin{array}{l} \text{rate of change of an} \\ \text{as-yet-unspecified} \\ \text{scalar into an} \\ \text{as-yet-unspecified} \\ \text{direction} \end{array} \right] \in (1)$$

For a specified scalar, say ψ , and a specified direction, say \vec{w} , that rate of change is

$$d\rho(\vec{w}, \psi) = \langle dx^i, \vec{w} \rangle \frac{\partial \psi}{\partial x^i} = D_{\vec{w}} \psi$$

27.4

27.4

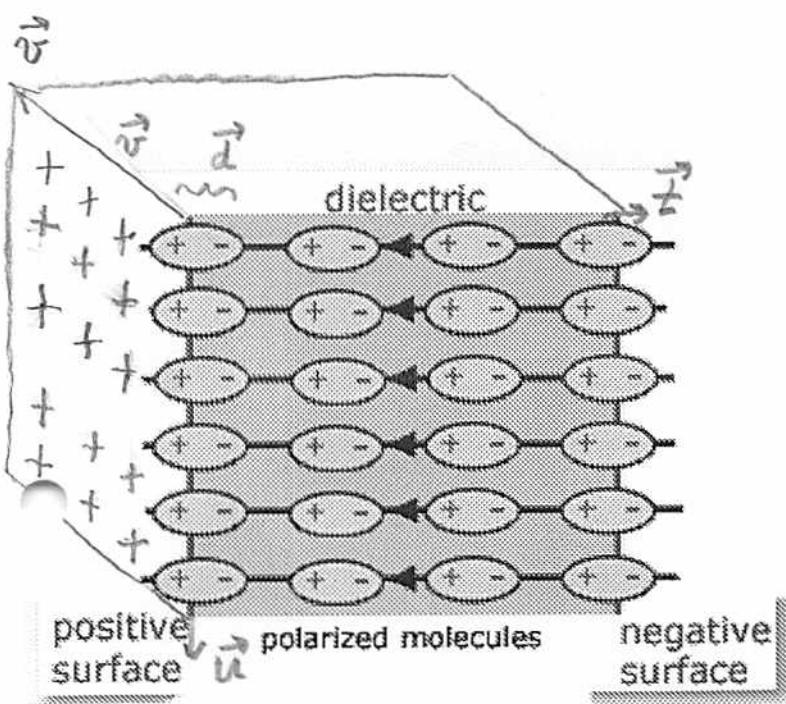


Figure 27.2
Polarized dielectric cube subjected to electric field \vec{E} .

A diagram showing three parallel horizontal arrows pointing to the left, representing an electric field \vec{E} .

$$\vec{E} = \epsilon_0 \vec{E}_0$$

III) MOMENT OF FORCE

26.10
27.5

Referring to Figure 27.2, augment Eq.(27.1)

on page 27.2 in order to obtain the

total moment of force (= "torque")

$$\tilde{\tau}(u, v, t) = \overrightarrow{e}_3 \Delta x^3 \wedge \overrightarrow{e}_k F_{12j1}^k dx^i \wedge dx^j (u, v)$$

$$+ \overrightarrow{e}_1 \Delta x^1 \wedge \overrightarrow{e}_k F_{12j1}^k dx^i \wedge dx^j (v, t)$$

$$+ \overrightarrow{e}_2 \Delta x^2 \wedge \overrightarrow{e}_k F_{12j1}^k dx^i \wedge dx^j (t, u)$$

$$- \rho' \sum_{l=1}^6 \overrightarrow{F} \quad (\text{l}^{\text{th}} \text{ face})$$

xational \Rightarrow zero.
equilibrium

Consequently, $\tilde{\tau}(u, v, t)$ is independent of the location of ρ' , the chosen fulcrum point. It a torque.

Thus, for the volume spanned by (u, v, t) , the total

$$\begin{aligned} \text{is } \overleftrightarrow{\Gamma}(u, v, t) &= \vec{e}_3 \wedge \vec{e}_k F_{12j_1}^k dx^3 \wedge dx^1 \wedge dx^2(t, u, v) \\ &+ \vec{e}_1 \wedge \vec{e}_k F_{12j_1}^k dx^1 \wedge dx^2 \wedge dx^3(u, v, t) \\ &+ \vec{e}_2 \wedge \vec{e}_k F_{12j_1}^k dx^2 \wedge dx^3 \wedge dx^1(v, t, u) \end{aligned}$$

Leaving the cubes spanning vectors

$\{u, v, t\}$ as-yet-unspecified, one obtains

the generator of spatial rotations

$$(27.3) \quad \boxed{\overleftrightarrow{\Gamma} = \vec{e}_2 \wedge \vec{e}_k F_{12j_1}^k dx^2 \wedge dx^1 \wedge dx^3} \quad \stackrel{\uparrow}{=} dP$$

Eq. (27.2) on page 27.3

Using Cartan's and MTW frame

invariant notation one has therefore

$$(27.4) \quad \boxed{\overleftrightarrow{\Gamma} = dP \wedge \vec{F}}$$

which is the amount of moment of force ("torque", "generator of rotation") per as-yet-unspecified volume of a dielectric medium subjected to the electrostatic field $\vec{E} = \hat{e}_k E^k$.

II) TWO MATHEMATIZATIONS OF "TORQUE"

There is a big difference between the two ways of mathematizing the concept "torque,"

(i) via the familiar cross product,
the vector $\vec{T} = \vec{R} \times \vec{F}$, as compared to

(ii) via the bivector-valued three

form, Eqs. (27.3) and (27.4) on page (27.6)

First of all, the ^{boxed} expressions on page

(27.6), by virtue of

$$\vec{F} = \vec{e}_k F_{(i,j)}^k dx^i \wedge dx^j,$$

contain explicit reference to the 6

boundary faces of the 3-cube; while

$\vec{T} = \vec{R} \times \vec{F}$ does not.

Secondly, the displacement vector

$$d\vec{p} = \vec{e}_k \otimes dx^k \quad \text{in } \vec{F}$$

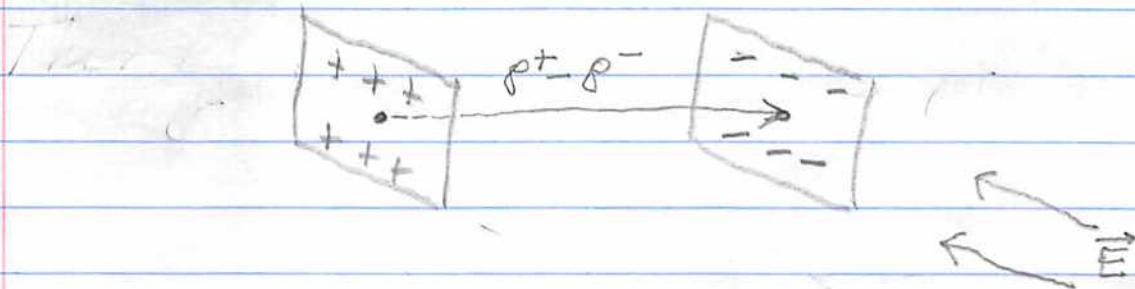
Subsumes the three connecting levers

$$\rho_i^+ - \rho_i^-; i=1,2,3$$

$$\left. \begin{array}{l} \rho_1^+ - \rho_1^- = \vec{e}_1 \Delta x^1 = u \\ \rho_2^+ - \rho_2^- = \vec{e}_2 \Delta x^2 = v \\ \rho_3^+ - \rho_3^- = \vec{e}_3 \Delta x^3 = t \end{array} \right\} \subset d\rho = \vec{e}_k \otimes dx^k$$

which conceptualizes

to all 3 connecting levers between
all 3 pairs of
opposing faces that are subjected to
the action of a force couple.



Thirdly, the bivector-valued 3-form

$$\tilde{\omega} = d\rho \wedge \vec{F} = \vec{e}_2 \wedge \vec{F}_{23} \delta_n^l dx_1^n dx_2^i dx_3^j$$

$$= (\vec{e}_1 \wedge \vec{F}_{23} + \vec{e}_2 \wedge \vec{F}_{31} + \vec{e}_3 \wedge \vec{F}_{12}) \times dx^1 \wedge dx^2 \wedge dx^3$$

is an attribute of the dielectric immersed in
an \vec{E} -field. Being the product $(\vec{e}_1 \wedge \vec{F}_{23}) \times (dx^1 \wedge dx^2 \wedge dx^3)$,

Janus - the Roman god of the before and after,

27.10

like Janus, \overleftrightarrow{J} has two faces, one before
and one after it has been evaluated
on a triad of vectors (u, v, t) .

Before evaluation \overleftrightarrow{J} is a function

which refers to an intensive property,
namely the density of moment of
force ("torque per unit volume")
of "stuff"
permeating the dielectric medium

After evaluation on, say, the

triad of spanning vectors (u, v, t)

$\overleftrightarrow{J}(u, v, t)$ is the value of this function,

which refers to an extensive property,
namely the amount of moment
of force, of "stuff," contained

in the spanned cube. Thus doubling u , v , or t will double the spanned volume and hence double the amount of measurable moment of force in that cube.

In brief, $\overset{\leftrightarrow}{T}$ is an intensive property, while $\overset{\leftrightarrow}{T}(u, v, t)$ is an extensive property of the dielectric medium in an \vec{E} -field.