LECTURE 28

I. Einstein's line of reasoning

II. Moment of force

III. Translational equilibrium

IV. Moment of force as torque

V. Equivalence via Hodge dual.

The equations of geometrodynamics, i.e. Einstein's gravitational field equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu},$$

which he arrived at in 1916, are the result of his following line of reasoning

I.) EINSTEINS LINE OF REASONING.

a) They must be in the framework of Special Relativity within any local spacetime neighborhood via Einstein's (1907) instantaneous Lorentz ("inertial," "free float") frames of reference.

b) The equations of motion for freely moving (or "falling") bodies must be independent of their composition, and hence geometrical. This is because of the Eotvos experiment and its restate- by Einstein (1907) as the Equivalence Principle, "inertial mass = grav'l mass".

c) The Newtonian scalar gravitational field equation for the gravitational potential must generalized to a system of 2^{nd} order P.D.E.'s in the metric coefficients.

d) The field equations must

(i) satisfy momenergy conservation and

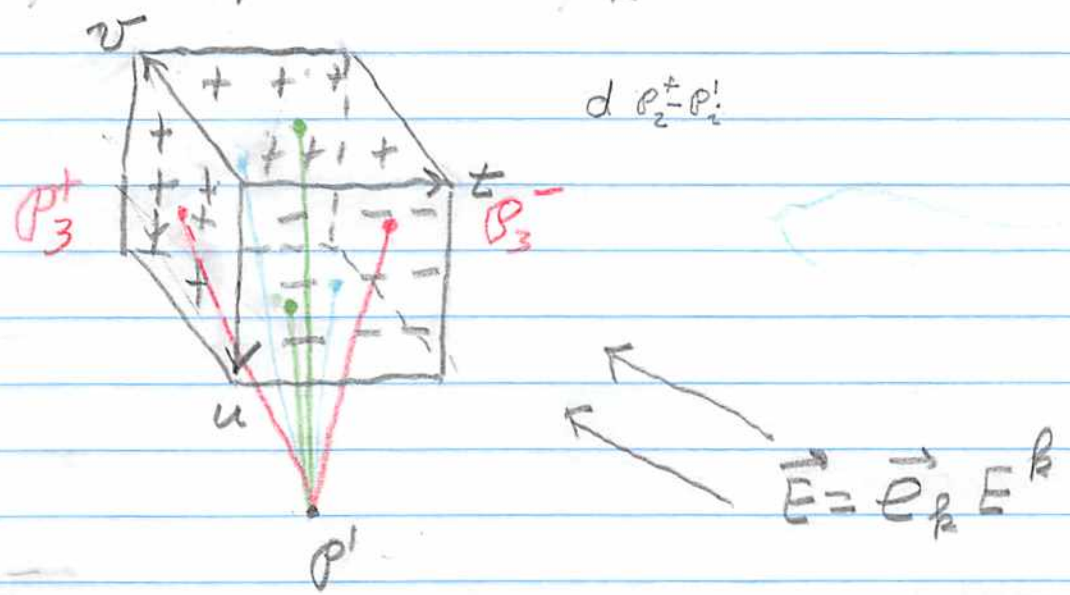
(ii) be tensorial equations.

The r.h.s. of Einstein's 1916 tensor equation is well understood both physically and geometrically. However, such understanding did not extend to the l.h.s.

This gap was filled subsequently by E. Cartan and Wheeler. They
They rematHematized the Einstein field equations in terms of Cartan's exterior calculus and introduced the new geometrical concept of the moment relative to a fulcrum point.
This concept occurs non-trivially in a 3-d environment and generalizes to four and higher dimensions.

II) MOMENT OF FORCE

In three dimensions consider a dielectric cube subjected to an electric field $\vec{E} = \vec{e}_k E^k$



The total moment of force acting on this cube is

$\vec{\tau}(u,v,t) =$

$$(28.1)\quad \underbrace{\overleftrightarrow{\underset{\sim}{\tau}}(u,v,t)}_{\text{total moment of force}} = \underbrace{d\rho \wedge \vec{F}(uvt)}_{\text{"torque"}} - \underbrace{\rho' \underbrace{\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\ \text{face})}_{\text{force on all 6 faces}}}_{\text{moment of force around } \rho'}$$

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The total moment of force acting on the cube is

$$\vec{\mathcal{T}}(u,v,t) = (\mathcal{P}_3^+ - \mathcal{P}^-) \wedge \underset{\sim}{\vec{F}}(u,v)\Big|_{\mathcal{P}_3^+}^{(1^{st}\text{ face})} + (\mathcal{P}_3^- - \mathcal{P}^-) \wedge F(v,u)\Big|_{\mathcal{P}_3^-}^{(2^{nd}\text{ face})} + \text{etc.}$$

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}}(u,v,t) = \underbrace{\vec{e}_3\, \Delta x^3}_{\mathcal{P}_3^+ - \mathcal{P}_3^- = t} \wedge \vec{e}_k\, F^k_{|ij|}\, dx^i \wedge dx^j (u,v)$$

$$\underbrace{\vec{e}_1\, \Delta x^1}_{\mathcal{P}_1^+ - \mathcal{P}_1^- = u} \wedge \vec{e}_k\, F^k_{ij}\, dx^i \wedge dx^j (v,t)$$

$$\underbrace{\vec{e}_2\, \Delta x^2}_{\mathcal{P}_2^+ - \mathcal{P}_2^- = v} \wedge \vec{e}_k\, F^k_{ij}\, dx^i \wedge dx^j (t,u)$$

$$-\mathcal{P}'\left\{\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{ face})\right)$$

$$\begin{aligned}
\underset{\sim}{\vec{\mathcal{T}}}(u,v,t) = \;& \vec{e}_3 \wedge \vec{e}_k\, F^k_{|ij|}\, dx^3(t)\, dx^i \wedge dx^j(u,v) \\
& \vec{e}_1 \wedge \vec{e}_k\, F^k_{|ij|}\, dx^1(u)\, dx^i \wedge dx^j(v,t) \\
& \vec{e}_2 \wedge \vec{e}_k\, F^k_{|ij|}\, dx^2(v)\, dx^i \wedge dx^j(t,u) \;-\mathcal{P}' \sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{ face}) \\
= \;& \vec{e}_\ell \wedge \vec{e}_k\, F^k_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j (u,v,t) -
\end{aligned}$$

$$\underbrace{\underset{\sim}{\overleftrightarrow{\mathcal{T}}}(u,v,t)}_{\text{(total moment of force)}} \equiv \underbrace{d\mathcal{P} \wedge \underset{\sim}{\vec{F}}(u,v)}_{\text{("torque")}} \underbrace{- \mathcal{P}' \underbrace{\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{ face})}_{\text{(Force on all 6 faces)}}}_{\text{(moment of force around } \mathcal{P}'\text{)}}$$

Here $\underset{\sim}{\vec{F}}$ is the given force field acting on the uncharged dielectric body

$$\underset{\sim}{\vec{F}} = \vec{e}_k E^k q N d^m \epsilon_{m|ij|} dx^i \wedge dx^j$$

with

$q\vec{d} = q\vec{e}_\ell d^\ell$ = elementary dipole moment

N = density of molecules each carrying depole moment $q\vec{d}$.

If the electric field is non-homogeneous, i.e $d\vec{E} \neq 0$, then the total force on the dielectric body will not vanish

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{ face}) \neq 0,$$

and will generate <u>linear</u> momentum, and hence

$$\mathcal{P}' \sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{ face}) \neq 0,$$

generates "orbital ∡ mom. around $\mathcal{P}'$"

which is orbital angular momentum.

around/relative to the fulcrum $\mathcal{P}$!

By contrast

$$d\mathcal{P}\wedge \underset{\sim}{\vec{E}}(u,v;t) \equiv \text{"torque"} \neq 0$$

generates "intrinsic ∡ mom. ("spin")"

will generate spinning motion, i.e. "spin", which is angular momentum intrinsic to the dielectric body.

Thus the virtue of Cartan's moment concept is this: Starting with the force field $\vec{E}$ it is the means for mathematizing (in a frame invariant way) the generators of total angular momentum, both spin and orbital. In the context of 4-d spacetime that starting point is the curvature-induced rotation field, and it leads to the Einstein tensor.

III.) TRANSLATIONAL EQUILIBRIUM
(IN A HOMOGENEOUS ELECTROSTATIC FIELD)

The force field $\underset{\sim}{\vec{F}}$ acts on each of the 6 faces with a force which is

$$\underset{\sim}{\vec{F}}(\ell^{th}\text{ face}), \quad \ell = 1, \ldots, 6$$

Denote the interior domain of the dielectric cube by $\mathcal{D}$ and it boundary by $\partial\mathcal{D}$. It is the union

$$\bigcup_{\ell=1}^{6} \ell^{th}\text{ face} = \partial\mathcal{D},$$

of the cube's six faces.

The total force, which is distributed additively over these faces, is

$$\underset{\sim}{\vec{F}}(\partial\mathcal{D}) = \underset{\sim}{\vec{F}}\Big(\bigcup_{\ell=1}^{6} \ell^{th}\text{ face}\Big) = \sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\text{ face})$$

There is no charge inside the cube, However, there are charges on each of the faces of the cube, but they add up to zero:

Consequently, all the forces due to the electostatic field $\vec{E}$, which we take to be homogeneous, also add to zero. The cube is in tranlational equilibrium:

$$\vec{\underset{\sim}{F}}(\partial\mathcal{D}) = \sum_{\ell=1}^{6} \vec{\underset{\sim}{F}}(\ell^{th}\text{face}) = 0$$

Apply this condition to the total moment of force, Eq. (28.1) on page (28.4):

(28.2) $$\underbrace{\overleftrightarrow{\underset{\sim}{\mathcal{T}}}(u,v,t)}_{\substack{\text{moment}\\ \text{of force.}}} = \underbrace{d\mathcal{P}\wedge \vec{\underset{\sim}{F}}(u,v,t)}_{\text{"torque"}}$$

One arrives a the more abstract mathematical object

(28.3) $$\boxed{\overleftrightarrow{\underset{\sim}{\mathcal{T}}} = d\mathcal{P}\wedge \vec{\underset{\sim}{F}}} = \frac{\begin{pmatrix}\text{moment}\\ \text{of force}\end{pmatrix}}{(\text{volume})},$$

the "moment of force density", by following the principle that $\overleftrightarrow{\underset{\sim}{\mathcal{T}}}$ must be evaluate for some triad of vectors, but may be evaluated for any triad.

IV.) MOMENT OF FORCE AS TORQUE: ITS VALIDATION.

The moment of force density,

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}} = d\mathcal{P} \wedge \underset{\sim\sim}{\vec{F}}$$

$$= \vec{e}_\ell \, dx^\ell \wedge \vec{F}_{|ij|} \, dx^i \wedge dx^j$$

(28.4) $$= \vec{e}_\ell \wedge \vec{e}_k \, E^k q N d^m \epsilon_{m|ij|} \, dx^\ell \wedge dx^i \wedge dx^j ,$$

evaluated on the cube spanned by the triad of vectors (u, v, t), Eq. (28.2) represents torque as a bivector. By contrast, the familiar cross product

$$\vec{T} = \vec{R} \times \vec{F}$$

represents the same torque as a vector. To

To validate this claim, we show that these two representations are isomorphic.

V THE ★ ("HODGE DUAL") ISOMORPHISM

From the observation the bases

$\{ \vec{e}_\ell \wedge \vec{e}_k : {}^{\ell}_{k} = 1,2,3 \}$ for $\Lambda^2(E^3)$

and

$\{ \vec{e}_m : m = 1,2,3 \}$ for E^3

have the same dimension

$$\dim \Lambda^2(E^3) = \dim E^3 \; (=3)$$

one constructs the linear xformation

★ according to the following

<u>Definition</u> ("Hodge dual")

$$\star : \quad \Lambda^2(E^3) \longrightarrow E^3$$

$$\vec{e}_\ell \wedge \vec{e}_k \rightsquigarrow \star(\vec{e}_\ell \wedge \vec{e}_k) = \vec{e}_m \epsilon^m{}_{\ell k}$$

where

$$\epsilon^m{}_{\ell k} = g^{mn} \epsilon_{n\ell k}$$

$$= g^{mn} \sqrt{g}\, [n \ell k]$$

are the $\binom{1}{2}$ tensor components of $\epsilon \in \Lambda^3(E^3)$.

Comment: Q: Where does this definition come from?

$\boxed{\text{SKIP to P28.12}}$

A: It comes from the metric-induced fact that

$$\epsilon_{n\ell k} = \sqrt{g}\,[n\ell k].$$

and the observation that for

$$\begin{aligned} u &= \vec{e}_n u^n \\ v &= \vec{e}_\ell v^\ell \\ t &= \vec{e}_k t^k \end{aligned}$$

one has the basis invariant inner product

$$\epsilon_{n\ell k} u^n v^\ell t^k = u^n e_n \cdot e_m \epsilon^m{}_{\ell k} v^\ell t^k$$

$$\sqrt{g} \begin{vmatrix} u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \\ t^1 & t^2 & t^3 \end{vmatrix} \equiv u \cdot \star(e_\ell \wedge e_k v^\ell t^k) \quad \forall\, u, v, t \in E^3$$

$$= u \cdot \star(v \wedge t)$$

Apply the $\star$ transformation to Eq. (28.4) on page 28.9, a bivector-valued 3-form. The result is the vector-valued 3-form

$$\star\left(\underset{\sim}{\overleftrightarrow{T}}\right) = \star\left(\vec{e}_\ell \wedge \vec{e}_k E^k q N d^m \epsilon_{m|ij|}\, dx^\ell \wedge dx^i \wedge dx^j\right)$$

$$= \vec{e}_n \epsilon^n{}_{\ell k} E^k q N d^m \epsilon_{m|ij|} [\ell i j]\, dx^1 \wedge dx^2 dx^3$$

Take advantage of the fact that

$$\epsilon_{mij}[\ell i j] = \sqrt{g}\,[m i j][\ell i j] = \sqrt{g}\,\delta^\ell_m$$

to obtain

$$\star\left(\underset{\sim}{\overleftrightarrow{T}}\right) = \vec{e}_n \epsilon^{n\ell k} E_k q N d_\ell \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

Use $\epsilon^{n\ell k} = g^{n\bar{n}} g^{\ell\bar{\ell}} g^{k\bar{k}} \epsilon_{\bar{n}\bar{\ell}\bar{k}} = \frac{1}{g}\sqrt{g}\,[n\ell k]$

$$= \frac{1}{\sqrt{g}}[n\ell k]$$

$$\star\left(\underset{\sim}{\overleftrightarrow{T}}\right) = \vec{e}_n [n\ell k] E_k q N d_\ell\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= \frac{1}{\sqrt{g}} \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ d_1 & d_2 & d_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix} N\sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$\star\left(\overleftrightarrow{\underset{\sim}{S}}\right) = \underbrace{\frac{1}{\sqrt{g}}\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ d_1 & d_2 & d_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix}}_{\text{(relative to a generically given bases)}} \underbrace{N\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3}_{\substack{N\times\text{ invariant volume} \\ = \#\text{ of molecules}}}$$

Evaluate this vectorial 3-form on the triad of spanning vectors $(u, v; t)$

$$\star\left(\overleftrightarrow{\underset{\sim}{S}}\right)(u, v, t) = \left(q\vec{d}\times\vec{E}\right)\cdot\left(\#\text{ of molecules}\right)$$

$$= \left(\overrightarrow{\text{dipole moment}}\right)\times\vec{E}$$

$$= \vec{R}\times\vec{F}$$

where $\vec{R} = \vec{d}\cdot(\#\text{ of molecules})$

$\vec{F} = q\vec{E}$

OR

$\vec{F} = \vec{E}\cdot(\text{total }\#\text{ of surface charges } q)$

$\vec{R} = \vec{d}$