

APPENDIX TO LECTURE 29

29,5

5.) Hodge dual
on Euclidean
Space E^3

Hodge dual
on Minkowski
Spacetime R^4

$$a) \star : \Lambda^2(E^3) \rightarrow E^3 \quad a) \star : \Lambda^3(R^4) \rightarrow R^4$$

$$\begin{aligned} e_{\lambda} \wedge e_{\mu} \mapsto \star(e_{\lambda} \wedge e_{\mu}) &= e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \mapsto \star(e_{\nu} \wedge e_{\lambda} \wedge e_{\mu}) = \\ &= e_n \epsilon^n_{\lambda \mu} \end{aligned}$$

$$= e_{\nu \lambda \mu} \delta^{\nu}_{\sigma}$$

$$\star(d\rho \wedge \vec{F}) = \quad \star(d\rho \wedge \vec{R}) =$$

$$\begin{aligned} &= \vec{e}_n \epsilon^n_{\lambda \mu} F^{\lambda \mu}_{ij} dx^i \wedge dx^j \\ &= e_{\nu \lambda \mu} \delta^{\nu}_{\sigma} R^{\lambda \mu}_{\nu \rho} dx^{\rho} \wedge dx^{\sigma} \end{aligned}$$

b) Inverse Hodge dual

$$\begin{aligned} \tilde{\star}^{-1} : e_m \mapsto \tilde{\star}^{-1}(e_m) &= \\ &= \frac{1}{2!} e_{\lambda} \wedge e_{\mu} \epsilon^{\lambda \mu}_m \end{aligned}$$

$$\begin{aligned} \tilde{\star}^{-1} : e_p \mapsto \tilde{\star}^{-1}(e_p) &= \\ &= \frac{1}{3!} e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \epsilon^{\nu \lambda \mu}_p \end{aligned}$$

$$\begin{aligned} c) \tilde{\star} \tilde{\star}^{-1}(e_m) &= \frac{1}{2!} e_n \epsilon^n_{\lambda \mu} \epsilon^{\lambda \mu}_m \\ &= e_n \delta^n_m \quad (\text{identity!}) \end{aligned}$$

$$\begin{aligned} \tilde{\star} \tilde{\star}^{-1}(e_p) &= \frac{1}{3!} e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \epsilon^{\nu \lambda \mu}_p \\ &= (\pm) \delta^{\nu}_{\rho} e_{\nu} \quad (\text{identity!}) \end{aligned}$$

$$\begin{aligned} d) \tilde{\star} \star(e_{\lambda} \wedge e_{\mu}) &= \tilde{\star}^{-1}(e_n \epsilon^n_{\lambda \mu}) \\ &= \frac{1}{2!} e_{\lambda} \wedge e_{\mu} \epsilon^{ij}_n \epsilon^n_{\lambda \mu} \\ &= \frac{1}{2!} e_{\lambda} \wedge e_{\mu} \delta^{ij}_{\lambda \mu} \\ &= e_{\lambda} \wedge e_{\mu} \quad (\text{identity!}) \end{aligned}$$

$$\begin{aligned} \star^{-1} \star(e_{\nu} \wedge e_{\lambda} \wedge e_{\mu}) &= \tilde{\star}^{-1}(e_{\nu \lambda \mu} \delta^{\nu}_{\sigma}) \\ &= e_{\nu \lambda \mu} \frac{1}{3!} e_{\alpha} \wedge e_{\beta} \wedge e_{\gamma} \epsilon^{\alpha \beta \gamma}_{\sigma} \\ &= \frac{1}{3!} e_{\alpha} \wedge e_{\beta} \wedge e_{\gamma} \delta^{\alpha \beta \gamma}_{\nu \lambda \mu} \\ &= e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \quad (\text{identity!}) \end{aligned}$$