

LECTURE 29

I) The method of moments to arrive at bulk properties from environmentally induced surface properties.

A. Moment of force vs

B. moment of rotation

II,) Einstein's field eq'n's

III,) Two equivalent momenergy representation.

The geometrical meaning of the L.h.s. of Einstein's field equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

was not part of Einstein's 1916 formulation of geometrodynamics.

That gap was filled via Cartan's method of "momentizing" geometrical objects.

I. THE MOMENT METHOD

This method applies to infinitesimal

3-d bodies immersed in an environment that acts on the 2-d boundary

$\partial\mathcal{D}$ and thereby produces a kinematic bulk effect on its interior \mathcal{D} .

$$\left(\begin{matrix} \text{Surface, } \partial\mathcal{D} \\ \text{property} \end{matrix} \right) \rightsquigarrow \left(\begin{matrix} \text{Bulk, } \mathcal{D} \\ \text{property} \end{matrix} \right) = dP_A \left(\begin{matrix} \text{surface} \\ \text{property} \end{matrix} \right)$$

$$dP^A_{\lambda} \begin{pmatrix} \text{surface} \\ \text{property} \end{pmatrix} = \begin{pmatrix} \text{Bulk} \\ \text{property} \end{pmatrix} : \quad 292$$

A) The novelty of Cartan's method

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is illustrated by his mathematization

a particular bulk property, the moment of

force volume density inside a 3-cube in E^3 ,
 $\overset{(1)}{\vec{F}} = dP^A_{\lambda} \vec{F}_{ij} dx^i dx^j$

$$\overset{(m)}{\vec{F}} = dP^A_{\lambda} \vec{F}_{ij} dx^i dx^j = e_P^A \vec{F}_{ij} dx^i dx^j$$

Hence

$$\overset{(m)}{\vec{F}} = \vec{F}_{ij} dx^i dx^j, \text{ a } \underline{\text{surface ppty.}}$$

is the environmental force field acting

on ∂D of the 3-cube, and

$$dP = \vec{e}_P dx^P$$

is Cartan's displacement vector.

emanating from some point P and

applied to a point on some face of the
of the cube.



Upon evaluating this bivector volume density, a bulk property, one obtains

$$\begin{aligned}\tilde{\mathcal{J}}(u, v, t) &= \vec{e}_E \wedge \vec{e}_R F^k d^m N E_{m|123} dx^2 \wedge dx^1 \wedge dx^3 (u, v, t) \\ &= \vec{e}_E \wedge \vec{e}_R F^k d^m N \underbrace{\sqrt{g} [m|i;j] [\ell^i, j]}_{\delta_m^k} dx^1 \wedge dx^2 \wedge dx^3 (u, v, t)\end{aligned}$$

$\left(\begin{array}{c} \text{mom.} \\ \text{of} \\ \text{force} \end{array}\right)$

$$= \vec{e}_E \wedge \vec{e}_R F^k d^2 N \cdot (\text{volume}) \quad \left(\begin{array}{l} \text{"Hodge} \\ \text{dual} \\ \text{represent'n"} \end{array}\right)$$

$R^2 = d^2 \#$

$\star \left(\begin{array}{c} \text{mom.} \\ \text{of} \\ \text{force} \end{array}\right) = \frac{1}{\sqrt{g}} \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ R_1 & R_2 & R_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{R} \times \vec{F} = (\overrightarrow{\text{Torque}})$

("Standard representation")

I. B) 4-d SPACETIME

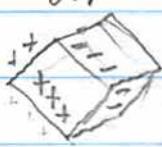
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The generalization to 4-d space is now straight and proceeds as follows:

1. 3-cube in E^3

1. 3-cube in 4-d spacetime

2. Vector-valued
force field
density



2. Bivector-valued,
curvature-induced
rotation field
density

$$\vec{F} = \vec{e}_k F_{[ij]}^k dx^i dx^j$$

$$\overleftrightarrow{R} = e_\lambda \wedge e_\mu R^{[\lambda\mu]}_{[\alpha\beta]} dx^\alpha \wedge dx^\beta$$

3. Translational
equilibrium

$$a) \sum_{k=1}^6 \vec{F} (\text{k-th face}) = 0$$

$$b) d\vec{F} = 0$$

$$c) F^k_{[i;j;n]} = 0$$

3. Bianchi identity

$$a) \sum_{k=1}^6 \overleftrightarrow{R} (\text{k-th face}) = 0$$

$$b) d\overleftrightarrow{R} = 0$$

$$c) R^{\lambda\mu}_{[\alpha\beta;\gamma]} = 0$$

4. Moment
of Force/volume

Moment of
Rotation/volume

$$\overleftrightarrow{T} = d\theta \wedge \vec{F}$$

$$d\theta \wedge \overleftrightarrow{R} =$$

$$= e_\ell \wedge e_k F_{[ij]}^k dx^\ell \wedge dx^i \wedge dx^j$$

$$= e_\nu \wedge e_\lambda \wedge e_\mu R^{[\nu\mu]}_{[\alpha\beta]} dx^\alpha \wedge dx^\beta$$

II. EINSTEIN'S FIELD EQN

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A.) The geometrical statement of the Einstein field equations is

$$\boxed{\frac{(\text{Moment of})}{\substack{\text{rotation} \\ \text{(Spacetime)}}}} = \frac{8\pi G}{c^4} \frac{(\text{Momenenergy})}{\substack{\text{(Spacetime)} \\ \text{3-volume}}}$$

The mathematized version of this statement is

$$\boxed{dP \wedge \tilde{R} = \frac{8\pi G}{c^4} \star (\star T)}$$

or equivalently

$$\boxed{\star (dP \wedge \tilde{R}) = \frac{8\pi}{c^4} \star T}$$

The momenergy-valued 3-form is a bulk property.

III. Momentum/volume has two equivalent representation

1.) As a vector-valued 3-volume density

$$\star T = e_p T^{\rho\sigma} \epsilon_{\sigma\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

$d^3\Sigma_0 = (\text{inv vol})_0$

and

2.) as a tri-vector valued 3-volume density

$$\star^{-1}(\star T) = e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu} T^{\rho\sigma} d^3\Sigma_0$$

This equivalence is based on the isomorphism between two 4-dimensional linear spaces

$$\begin{bmatrix} \star \\ \star^{-1} \end{bmatrix}: \mathbb{R}^4 \wedge \mathbb{R}^4 \wedge \mathbb{R}^4 \xrightleftharpoons[\star^{-1}]{\star} \mathbb{R}^4$$

$$e_\nu \wedge e_\lambda \wedge e_\mu \mapsto \star(e_\nu \wedge e_\lambda \wedge e_\mu) = \epsilon_{\nu\lambda\mu}{}^\rho e_\rho$$

$$e_p \mapsto \star^{-1}(e_p) = \frac{1}{3!} e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu} {}_\rho$$

$$\star \star^{-1}(e_p) = e_p$$

B) Component formulation of the E.F.E.:

$$\begin{aligned} dP\Lambda \overset{\leftrightarrow}{R} &= e_\nu e_\lambda e_\mu R^{\lambda\mu}_{\nu\beta} dx^\nu \wedge dx^\lambda \wedge dx^\beta = \frac{8\pi G}{c^4} \star' (*T) \\ &= \frac{8\pi G}{c^4} e_\nu e_\lambda e_\mu \frac{\epsilon^{\nu\lambda\mu}}{3!} g T^{\rho\sigma} d\Sigma_\rho \end{aligned}$$

OR equivalently

$$\begin{aligned} \star(dP\Lambda \overset{\leftrightarrow}{R}) &= \epsilon_{\nu\lambda\mu}{}^\rho e_\rho R^{\lambda\mu}_{\nu\beta} dx^\nu \wedge dx^\lambda \wedge dx^\beta \\ &= \frac{8\pi G}{c^4} e_\rho T^{\rho\sigma} d\Sigma_\sigma \end{aligned}$$

Introduce

$$R^{\delta^M}_{\nu\nu} = R^M_\nu \quad (\text{Ricci})$$

$$R^{\delta^S}_S = R \quad (\text{Curvature inbt})$$

and obtain

$$\boxed{R^M_\nu - \frac{1}{2} \delta^M_\nu R = \frac{8\pi G}{c^4} T^M_\nu}$$

APPENDIX TO LECTURE 29

5.) Hodge dual
on Euclidean
Space E^3

Hodge dual
on Minkowski
Spacetime R^4

$$a) \star : \Lambda^2(E^3) \rightarrow E^3 \quad a) \star : \Lambda^3(R^4) \rightarrow R^4$$

$$\begin{aligned} e_{\lambda} \wedge e_{\mu} \mapsto \star(e_{\lambda} \wedge e_{\mu}) &= e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \mapsto \star(e_{\nu} \wedge e_{\lambda} \wedge e_{\mu}) = \\ &= e_n \epsilon^n_{\nu \lambda \mu} \end{aligned}$$

$$= e_{\nu \lambda \mu} \delta^{\nu}_{\sigma}$$

$$\star(d\rho \wedge \vec{F}) = \quad \star(d\rho \wedge \vec{R}) =$$

$$\begin{aligned} &= \vec{e}_n \epsilon^n_{\nu \lambda \mu} F^{\nu \lambda}_{ij} dx^i \wedge dx^j \\ &= e_{\nu \lambda \mu} \delta^{\nu}_{\sigma} R^{\lambda \mu}_{\nu \lambda \rho} dx^{\lambda} \wedge dx^{\mu} \wedge dx^{\rho} \end{aligned}$$

b) Inverse Hodge dual

$$\begin{aligned} \tilde{\star}^{-1} : e_m \mapsto \tilde{\star}^{-1}(e_m) &= \\ &= \frac{1}{2!} e_{\nu} \wedge e_{\mu} \epsilon^{\nu \mu}_m \end{aligned}$$

$$\begin{aligned} \tilde{\star}^{-1} : e_p \mapsto \tilde{\star}^{-1}(e_p) &= \\ &= \frac{-1}{3!} e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \epsilon^{\nu \lambda \mu}_p \end{aligned}$$

$$\begin{aligned} c) \tilde{\star} \tilde{\star}^{-1}(e_m) &= \frac{1}{2!} e_n \epsilon^n_{\nu \mu} \epsilon^{\nu \mu}_m \\ &= e_n \delta^n_m \text{ (identity!) } \end{aligned}$$

$$\begin{aligned} \tilde{\star} \tilde{\star}^{-1}(e_p) &= \frac{-1}{3!} e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \epsilon^{\nu \lambda \mu}_p \\ &= (-)\delta^{\nu}_{\rho} e_{\nu} \text{ (identity!) } \end{aligned}$$

$$\begin{aligned} d) \tilde{\star} \tilde{\star}^{-1}(e_{\nu} \wedge e_{\mu}) &= \tilde{\star}^{-1}(e_n \epsilon^n_{\nu \mu}) \\ &= \frac{1}{2!} e_{\nu} \wedge e_{\mu} \epsilon^{\nu \mu}_n \epsilon^n_{\rho \sigma} \\ &= \frac{1}{2!} e_{\nu} \wedge e_{\mu} \delta^{\nu \mu}_{\rho \sigma} \\ &= e_{\nu} \wedge e_{\mu} \text{ (identity!) } \end{aligned}$$

$$\begin{aligned} \tilde{\star} \tilde{\star}^{-1}(e_{\nu} \wedge e_{\lambda} \wedge e_{\mu}) &= \tilde{\star}^{-1}(e_{\nu} \wedge e_{\lambda} \wedge e_{\mu}) \\ &= e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \frac{1}{3!} e_{\rho} \wedge e_{\sigma} \wedge e_{\tau} \epsilon^{\rho \sigma \tau} \\ &= \frac{1}{3!} e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \delta^{\nu \lambda \mu}_{\rho \sigma \tau} \\ &= e_{\nu} \wedge e_{\lambda} \wedge e_{\mu} \text{ (identity!) } \end{aligned}$$