

# LECTURE 3

Continuation  
from LECTURE 2

IV. Reparametrization

V. Torsionless, metric-compatible,  
parallel transport

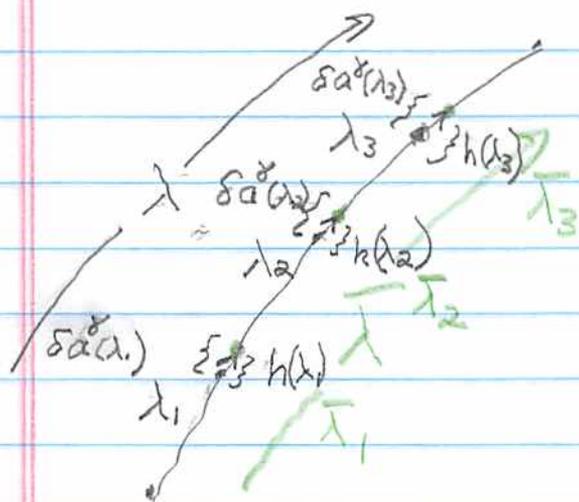
VI. Constant of motion.

ACCELERATED FRAMES:

For LECTURE 4

1. Vectorial change in the static (fixed  
star) frame vs

vectorial change in the rotating  
("body") frame,



$$\bar{\lambda} = \lambda + h(\lambda)$$

Reminder

$$f_{\delta}(\lambda) = \frac{1}{\sqrt{\dots}} \left\{ \frac{g_{\delta\mu} \frac{da^{\mu}}{d\lambda}}{\sqrt{\dots}} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^{\delta}} \frac{da^{\mu}}{d\lambda} \frac{da^{\nu}}{d\lambda} \frac{1}{\sqrt{\dots}} \right\}$$

where

$$\sqrt{\dots} = \sqrt{-g_{\alpha\beta} \frac{da^{\alpha}}{d\lambda} \frac{da^{\beta}}{d\lambda}}$$

## IV PARAMETRIZATION

A) The equations  $f_\gamma(\lambda) = 0$  constitutes mathematical overkill (i.e. over generalization) There are more of them than necessary to express the extremal nature of the variational integral

$$\tau_A^B = \int_A^B \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda.$$

In other words, one of these equations holds for all worldlines, even those that do not extremize  $\tau_A^B$ .

This fact follows from the parametrization independence of this integral. The reparametrization

$$\lambda \rightarrow \bar{\lambda} = \lambda + h(\lambda) \quad \frac{d\bar{\lambda}}{d\lambda} = 1 + h'(\lambda)$$

$$\left. \begin{array}{l} \bar{\lambda}(0) = 0 \\ \bar{\lambda}(1) = 1 \end{array} \right\} \Rightarrow h(0) = h(1) = 0$$

does not change the value of the <sup>variational</sup> integral

It corresponds to a mere

"repositioning of beads along a string" (= reparametrization)

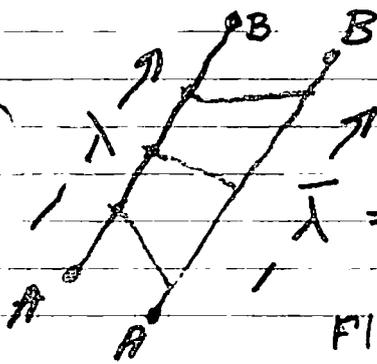


FIGURE 3.1

$$\bar{\lambda}(\lambda) = \lambda + h(\lambda)$$

3.2

$$\tau_A^B = \int_0^1 \sqrt{-g_{\alpha\beta} \frac{da^\alpha(\lambda)}{d\lambda} \frac{da^\beta(\lambda)}{d\lambda}} d\lambda = \int_0^1 \sqrt{-g_{\alpha\beta} \frac{da^\alpha(\bar{\lambda})}{d\bar{\lambda}} \frac{da^\beta(\bar{\lambda})}{d\bar{\lambda}}} d\bar{\lambda}$$

The change in  $a^\delta(\lambda)$  brought about by such a reparametrization is

$$a^\delta(\lambda) \rightarrow a^\delta(\bar{\lambda}) = a^\delta(\lambda + h(\lambda)) = a^\delta(\lambda) + \delta a^\delta(\lambda)$$

where 
$$\delta a^\delta(\lambda) = \frac{da^\delta}{d\lambda} h(\lambda)$$

The fact that such variations can not change the variational integral for

arbitrary  $h(\lambda)$  implies  $\delta \tau_A^B = \int_0^1 f_\delta(\lambda) \frac{da^\delta}{d\lambda} h(\lambda) d\lambda = 0$ ,  
and hence

$$\boxed{f_\delta(\lambda) \frac{da^\delta}{d\lambda} = 0} \quad (\text{even if } f_\delta \neq 0)$$

This holds for all paths  $a^\delta(\lambda)$ , even those that do not extremize  $\tau_A^B$ !

An equation that holds whether or not the quantities obey any differential equation is called an identity.

Ours is simply an algebraic identity.

B) The reparametrization freedom can be exploited to simplify the differential equation: Introduce the physically more relevant parameter, the proper time, by means of

$$d\tau = \sqrt{-g_{\alpha\beta}(x^\alpha(\lambda)) \frac{dx^\alpha(\lambda)}{d\lambda} \frac{dx^\beta(\lambda)}{d\lambda}} d\lambda$$

We have

$$\frac{d\tau}{d\lambda} \neq 0 \Rightarrow \tau \text{ is a } \underline{\text{monotonic}} \text{ function of } \lambda, \therefore \lambda = \lambda(\tau) \text{ is well-defined.}$$

Introduce

$$x^\alpha(\tau) \equiv x^\alpha(\lambda(\tau)) \quad (2a)$$

so that

$$\frac{dx^\alpha(\tau)}{d\tau} = \frac{dx^\alpha(\lambda)}{d\lambda} \frac{d\lambda}{d\tau} = \frac{1}{\sqrt{\dots}} \frac{dx^\alpha}{d\lambda} \quad (2b)$$

or more generally

$$\frac{d}{d\tau} = \frac{1}{\sqrt{\dots}} \frac{d}{d\lambda} \quad (2c)$$

where

$$\sqrt{\dots} \equiv \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}$$

The introduction of the proper time  $\tau$  as the world line parameter results in a non-trivial simplification in the extremum condition  $\delta \tau_A^B = 0$  as expressed by the differential equations

$$0 = f_\gamma(\lambda) \quad \gamma = 0, 1, 2, 3$$

on page 2.7. Indeed, introducing Eqs (2a)-(2c) on page 2.10 into Eq. (1) on page 2.7 results in

$$0 = f_\gamma(\lambda) = \frac{d}{d\tau} \left( g_{\mu\gamma} \frac{dx^\mu}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

or

$$0 = g_{\mu\gamma} \frac{d^2 x^\mu}{d\tau^2} + \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\gamma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad (3)$$

Following the line of reasoning in Problem 3.11b in MTW (Homework 4 in Math 5756), rewrite the middle term as

$$g_{\mu\gamma, \nu} \dot{x}^\nu \dot{x}^\mu = \frac{1}{2} (g_{\mu\gamma, \nu} + g_{\nu\gamma, \mu}) \dot{x}^\mu \dot{x}^\nu$$

The result is that the external condition as expressed by the differential Eq. (3) on page 2,11 becomes simply

$$0 = g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \underbrace{\frac{1}{2} (g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta})}_{\equiv \Gamma_{\mu\nu}^\delta} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad (4)$$

Streamline this differential equation

further by introducing the inverse

metric  $g^{\nu\alpha}$ :

$$g^{\nu\alpha} g_{\mu\nu} = \delta_\mu^\alpha$$

and obtain

$$0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad (3,5)$$

where

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\delta} (g_{\delta\mu,\nu} + g_{\delta\nu,\mu} - g_{\mu\nu,\delta}) \quad (5)$$

are the "Christoffel symbols of the

2<sup>nd</sup> kind"

V TORSIONLESS METRIC COMPATIBLE PARALLEL TRANSPORT. 3,6

By contrast;  
The Christoffel symbols of the first kind are the contents of the round parentheses in Eqs (4) and (5) before we introduced the inverse metric,

$$\Gamma_{\gamma \mu \nu} = \frac{1}{2} (g_{\mu \delta, \nu} + g_{\nu \delta, \mu} - g_{\mu \nu, \delta})$$

These symbols are significant because they mathematize the metric compatibility of the law of parallel transport.

Indeed, add to the above symbol the one with  $\nu$  and  $\delta$  interchanged:

$$\Gamma_{\nu \mu \delta} = \frac{1}{2} (g_{\nu \mu, \delta} + g_{\nu \delta, \mu} - g_{\mu \delta, \nu})$$

One obtains

$$\begin{aligned} \frac{\partial g_{\nu \delta}}{\partial x^{\mu}} &= \Gamma_{\gamma \mu \nu} + \Gamma_{\nu \mu \delta} \\ &= \Gamma^{\alpha}_{\mu \nu} g_{\alpha \delta} + \Gamma^{\alpha}_{\mu \delta} g_{\alpha \nu}, \end{aligned}$$

namely,

$$0 = \frac{\partial g_{\nu\mu}}{\partial x^\alpha} - g_{\alpha\gamma} \Gamma_{\nu\mu}^\alpha - g_{\nu\alpha} \Gamma_{\gamma\mu}^\alpha = g_{\nu\gamma;\mu}$$

This is the condition that the law of parallel transport as expressed by the  $\Gamma_{\mu\nu}^\alpha$  in Eq. (5) on page 2.12 is

(a) compatible with the same metric

tensor field that went into the extremization of the proper time as

stated on page 2.3, and

(b) has zero torsion (because  $\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha$ ).

Two Conclusions:

- ① The principle of extremal proper time implies a unique torsionless parallel transport which compatible with the metric.

$$\int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} = \text{extr.} \left\{ \begin{array}{l} \Rightarrow \Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\gamma} (g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma}) \\ \Rightarrow \Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha \text{ (i.e. torsion=0)} \end{array} \right.$$

relative

② The geodesics of curved spacetime

coincide with the world lines of extremal proper time.

#### IV A GENERAL CONSTANT OF MOTION

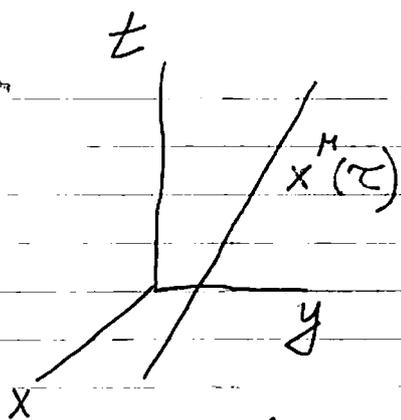
By differentiating the squared magnitude  $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$  with respect to  $\tau$ , one can verify that

$$f_{\gamma}(\lambda) \frac{dx^{\gamma}}{d\tau} = 0 \Rightarrow \frac{d}{d\tau} (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = 0.$$

Thus  $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \text{const} (= -1 \text{ for } \underline{\text{any}} \text{ timelike curve; } g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \text{ is } \underline{\text{always}} \text{ an integral of motion; it expresses the constancy of the magnitude of the unit tangent } u = \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu} ;$

$$u \cdot u = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \text{constant}$$

# WHY A GEOMETRICAL INTERPRETATION OF FREE PARTICLE MECHANICS?



These equations for the geodesic worldline comprise the geometric formulation of the following physical aspects of a free particle:

$$(\text{mass}) \times \frac{d^2 x^\mu}{d\tau^2} = -(\text{mass}) \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \quad \mu = 0, 1, 2, 3$$

Here the right hand side may refer to

- Coriolis force
- Centrifugal force
- Gravitational force

In other words,

-  $(\text{mass}) \Gamma_{\alpha\beta}^\mu$  are the "force" components  
 so that  $\Gamma_{\alpha\beta}^\mu$  are the "potentials" from which these forces are derived by taking appropriate partial derivatives.

This farreaching observation is illustrated by the following examples.

During the time interval  $dt$ ,  
the total change in this vector is

$$\boxed{(d\vec{G})_p = (d\vec{G})_{rot} + dt \vec{\omega} \times \vec{G}}$$

relative to the inertial frame static w.r.t. the fixed stars,

$$\left(\frac{d}{dt}\right)_{static} = \left(\frac{d}{dt}\right)_{rot} + \vec{\omega} \times$$

to the position vector  $\vec{R}(t)$

$$\left(\frac{d\vec{R}}{dt}\right)_p = \left(\frac{d\vec{R}}{dt}\right)_{rot} + \vec{\omega} \times \vec{R}$$

and to the velocity vector  $\vec{v}_p = \left(\frac{d\vec{R}}{dt}\right)_p$ ,  
one obtains that the eq'n of motion

for a free particle is

$$0 = \frac{d}{dt} \left[ \left(\frac{d\vec{R}}{dt}\right)_p \right] = \frac{d}{dt} \left[ \left(\frac{d\vec{R}}{dt}\right)_{rot} + \vec{\omega} \times \vec{R} \right] + \vec{\omega} \times \left[ \left(\frac{d\vec{R}}{dt}\right)_{rot} + \vec{\omega} \times \vec{R} \right]$$

$$0 = m \left[ \vec{a}_{rot} + 2\vec{\omega} \times \vec{v}_{rot} + \vec{\omega} \times (\vec{\omega} \times \vec{R}) \right]$$

$$m \frac{d^2 x^i}{dt^2} = -2m \left[ \vec{\omega} \times \vec{v}_{rot} \right]^i - m \left[ \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right]^i$$

relative to the rotating coordinate frame.

Point of clarification

The equation

$$d\vec{G})_s = d\vec{G})_{rot} + dt \vec{\omega} \times \vec{G}$$

is a relation between the change  $d\vec{G})_{rot}$  of the vector  $\vec{G}$  as determined and measured relative to rotating frame and the change  $d\vec{G})_s$  as measured relative to the static frame which is fixed relative to the fixed stars.

Both frames have the same origin, but they are characterized by different bases of orthonormal vectors.

(i)  $\{\vec{e}_i\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ , the "static" basis, is taken (for the static frame = static vect. sp) to be fixed relative to the fixed stars.

(ii)  $\{\vec{e}_i^*\} = \{\vec{e}_1^*(t), \vec{e}_2^*(t), \vec{e}_3^*(t)\}$ , the "rotating" basis, is one which is taken to rotate (for the rotating frame = rotating vector space)

with angular velocity  $\vec{\omega}$  relative to the fixed stars, and hence relative to the static basis. (We take  $\vec{\omega}$  to constant, although this is not essential for our purposes)

A time parametrized family of vectors, i.e. a moving vector, say  $\vec{G}(t)$ , lends itself to being expanded relative to either orthonormal basis

$$G_1 \vec{e}_1 + G_2 \vec{e}_2 + G_3 \vec{e}_3 = G_i \vec{e}_i = \vec{G} = G_i^* \vec{e}_i^* = G_1^* \vec{e}_1^* + G_2^* \vec{e}_2^* + G_3^* \vec{e}_3^*$$

One says that  $\vec{G}$  is constant and unchanging in the static frame if its expansion coefficient relative to the fixed star basis are constant. In that case one writes its time derivative as

$$\left. \frac{d\vec{G}}{dt} \right|_s = \frac{d(G_i \vec{e}_i)}{dt} = 0 \quad \left\{ \begin{array}{l} \text{assumes a} \\ \text{static basis} \end{array} \right.$$

On the other hand, if  $\vec{G}$  is changing

relative to the static frame, then non-surprisingly one has

$$\left. \frac{d\vec{G}}{dt} \right|_s = \frac{d}{dt} (G_i \vec{e}_i) = \frac{d G_i(t)}{dt} \cdot \vec{e}_i \quad (*)$$

A ← assumes a static basis

One says that  $\vec{G}$  is constant and unchanging

in the rotating frame if its coefficients

relative to the rotating basis  $\{\vec{e}_i^*\}$  are

independent of time

$$\vec{G} = G_i^* \vec{e}_i^* = G_1^* \vec{e}_1^*(t) + G_2^* \vec{e}_2^*(t) + G_3^* \vec{e}_3^*(t)$$

where  $G_i^*$  are constants, even though

$\{\vec{e}_i^*(t)\}$  changes relative to the static

(i.e. fixed star) frame. In that case

one writes

$$\left. \frac{d\vec{G}}{dt} \right|_r = \frac{d}{dt} (G_i^* \vec{e}_i^*(t)) = 0$$

because

$$\frac{dG_i^*}{dt} = 0$$

$$i = 1, 2, 3$$

+ ← assumes a rotating basis,

In particular, one has

$$\left. \frac{d\vec{e}_1^*(t)}{dt} \right|_r = \left. \frac{d\vec{e}_2^*(t)}{dt} \right|_r = \left. \frac{d\vec{e}_3^*(t)}{dt} \right|_r = 0$$

because the rotating basis vectors have constant expansion coefficients relative to the rotating  $\{\vec{e}_i^*(t)\}$  basis,

By contrast

$$\left. \frac{d\vec{G}}{dt} \right|_A \neq 0$$

for obvious reasons. In fact, for such a vector one has

$$\left. \frac{d\vec{G}}{dt} \right|_A = \vec{\omega} \times \vec{G},$$

as is evident from the Figure on page 3.2,

In particular,

$$\left. \frac{d\vec{e}_i^*(t)}{dt} \right|_A = \vec{\omega} \times \vec{e}_i^*(t) \quad i = 1, 2, 3$$

On the other hand, if  $\vec{G}(t)$  is changing relative to the rotating frame, then this is solely due to the coefficients in the rotating frame representation changing as a function of time. In that case one writes

$$\left. \frac{d\vec{G}}{dt} \right|_r = \left. \frac{d(G_i^*(t) \vec{e}_i^*)}{dt} \right|_r = \frac{dG_1^*}{dt} \vec{e}_1^* + \frac{dG_2^*}{dt} \vec{e}_2^* + \frac{dG_3^*}{dt} \vec{e}_3^*$$

This is the most general case for which we need to ask: how does this change relate to the change in the static frame as expressed by the Eq. (\*) near the top of page 3,6? To this end one notes that

$$\left. \frac{d\vec{G}}{dt} \right|_s = \left. \frac{d(G_i^* \vec{e}_i^*)}{dt} \right|_s = \frac{dG_i^*}{dt} \vec{e}_i^* + G_i^* \left. \frac{d\vec{e}_i^*}{dt} \right|_s$$

$$\begin{aligned} \left. \frac{d\vec{G}}{dt} \right|_D &= \underbrace{\frac{dG_1^*}{dt} \vec{e}_1^* + \frac{dG_2^*}{dt} \vec{e}_2^* + \frac{dG_3^*}{dt} \vec{e}_3^*}_{\left. \frac{d\vec{G}}{dt} \right|_r} + G_i^* \left. \frac{d\vec{e}_i^*}{dt} \right|_D \\ &= \left. \frac{d\vec{G}}{dt} \right|_r + G_i^* \left. \frac{d\vec{e}_i^*}{dt} \right|_D \end{aligned}$$

Each of the vectors  $\vec{e}_i^*$  is constant in the rotating frame. Consequently, with the help of the Eq. at the bottom of p 3.7

one has

$$\begin{aligned} \left. \frac{d\vec{G}}{dt} \right|_D &= \left. \frac{d\vec{G}}{dt} \right|_r + G_1^* \vec{\omega} \times \vec{e}_1^* + G_2^* \vec{\omega} \times \vec{e}_2^* + G_3^* \vec{\omega} \times \vec{e}_3^* \\ &= \left. \frac{d\vec{G}}{dt} \right|_r + \vec{\omega} \times \underbrace{G_i^* \vec{e}_i^*}_{\vec{G}} \end{aligned}$$

so that

$$\boxed{\left. \frac{d\vec{G}}{dt} \right|_D = \left. \frac{d\vec{G}}{dt} \right|_r + \vec{\omega} \times \vec{G}}$$

which is the Eq. at the top of p 3.4