

LECTURE 30

I, Moment of rotation

II, Einstein's field equations stated
geometrically

III, Scholium on linear algebra of
two mappings

IV, Transition from Cartan-Wheeler's
to Einstein's representation of
his field equation

I MOMENT OF ROTATION

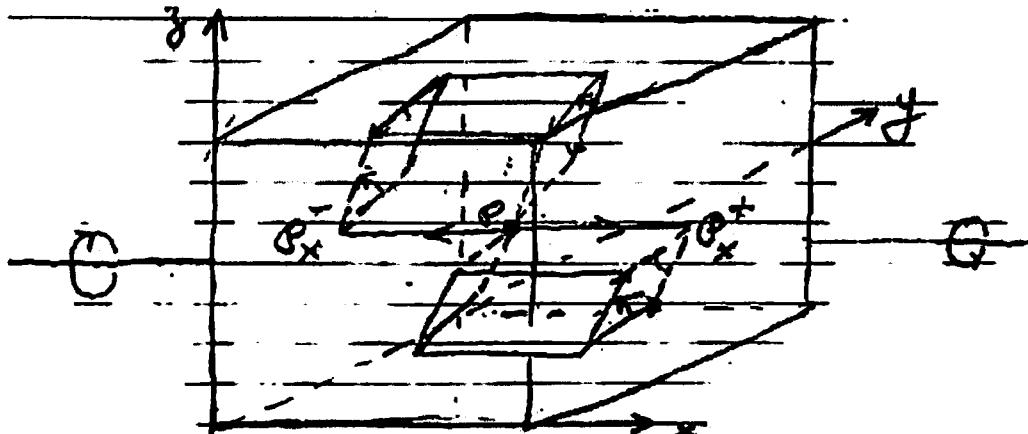
30.0

- A pictorial rendition of a moment of rotation consists of the total P-independent trielectorial parallelopiped

$$dP \mathbf{R} (\Delta x e_x, \Delta y e_y, \Delta z e_z)$$

inside the cubical volume spanned by $(\Delta x e_x, \Delta y e_y, \Delta z e_z)$ in the figure below.

$$\begin{aligned} & \text{(rotation associated with face } \Delta z) = e_x \wedge e_y R^{LM} \\ & = e_x \wedge e_y \underbrace{R^{LM}}_{(\Delta y e_y, \Delta z e_z)} \end{aligned}$$



Moment of rotation associated with a pair of opposing faces. P = "fulcrum"; an arbitrarily located point in or near the cube $\Delta x \Delta y \Delta z$.

II. GEOMETRICAL STATEMENT OF
EINSTEIN'S FIELD EQNS.

30,1

The two equivalent (i.e. Hodge dual related) geometrical statements of the Einstein field equation

$$\frac{(\text{Moment of rotation})}{(\text{spacetime 3-volume})} = \frac{8\pi G}{c^4} \frac{(\text{Momenenergy})}{(\text{spacetime 3-volume})}$$

are

$$(30,1a) \quad d\varrho_1 \overset{\leftrightarrow}{R} = \frac{8\pi G}{c^4} \star^{-1} (\star T)$$

and

$$(30,1b) \quad \star(d\varrho_1 \overset{\leftrightarrow}{R}) = \frac{8\pi G}{c^4} \star T$$

Here $\star T$ is a bulk property (i.e. how much "stuff"/3-volume), the momenergy density

$$\star T = e_\rho T^\rho{}^\sigma \underbrace{\epsilon_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma}_{III}$$

$$d^3 \Sigma_\rho = (\text{inst volume})_\rho$$

III. SCHOLIUM on LINEAR ALGEBRA
of TWO MAPPINGS

30.2

a) The mapping.

$$\star : \mathbb{R}^4 \setminus \mathbb{R}^4 \setminus \mathbb{R}^4 \xrightleftharpoons[\star^{-1}]{\star} \mathbb{R}^4$$

$$(30.2) \quad e_{\nu \lambda \mu} \mapsto \star(e_{\nu \lambda \mu} e_{\mu}) = e_{\nu \lambda \mu}{}^\rho e_{\mu}$$

$$e_\rho \mapsto \star^{-1}(e_\rho) = \frac{1}{3!} e_{\nu \lambda \mu} e_{\lambda} e_{\mu} e_{\nu}{}^\rho$$

is the linear transformation between
the trivector and the vector repre-
sentation of momenergy and moment
of rotation.

The mapping

("volume 3-form")

$$(30.3) \quad \star : e_0 \mapsto \star(e_0) = E_0 \epsilon_{\alpha \beta \gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma = d\Sigma_0$$

$$\star^{-1} : dx^\nu \wedge dx^\alpha \wedge dx^\beta \mapsto \star^{-1}(dx^\nu \wedge dx^\alpha \wedge dx^\beta) = E^\nu{}^{\alpha \beta \gamma} e_\gamma$$

$$J = J^\alpha e_\alpha \mapsto \star J = J^\alpha \epsilon_{\alpha \beta \gamma} dx^\beta \wedge dx^\gamma \wedge dx^\alpha$$

$$= J^\alpha d^3 \Sigma_\alpha$$

(current)
4-vector

(density-flux)
3-form

30,3

$$T = e_\beta T^{\alpha\beta} e_\alpha \rightsquigarrow *T = e_\beta T^{\alpha\beta} \epsilon_{\alpha\beta\gamma\delta} dx^\alpha dx^\beta dx^\gamma dx^\delta \\ = e_\beta T^{\alpha\beta} d^3 \Sigma_\alpha$$

(stress-energy tensor) \rightsquigarrow (Momentum-density-flux 3-form)

transforms vectors and tensors into their corresponding density-flux 3-forms.

TRANSITING FROM CARTAN-WHEELER'S
TO EINSTEIN'S REPRESENTATION OF HIS FIELD EQNS. 30.4

In order to show that the Cartan-Wheeler form of the gravitational field eqns are in fact the same as Einstein's

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu},$$

a) start with Eq(30.1b),

$$\begin{aligned} \epsilon_p \epsilon_{\nu\lambda\mu\rho} R^{(\lambda\mu)}_{(\alpha\beta)} dx^\nu dx^\alpha dx^\beta &= \frac{8\pi G}{c^4} * T \\ &= \frac{8\pi G}{c^4} \epsilon_p T^{\rho\sigma} \underbrace{\epsilon_{\alpha\beta\gamma\delta} dx^\nu dx^\alpha dx^\beta}_{d^3\Sigma_\sigma} \end{aligned}$$

b) isolate the components $T^{\rho\sigma}$ of the stress-energy tensor T by using

$$dx^\nu dx^\alpha dx^\beta = \epsilon^{\nu\alpha\beta\sigma} \underbrace{\epsilon_{\sigma\tau\bar{\nu}\bar{\alpha}\bar{\beta}} dx^\tau dx^{\bar{\alpha}} dx^{\bar{\beta}}}_{d^3\Sigma_\sigma}$$

c) use the linear independence $d^3\Sigma_\sigma$ of $\{d^3\Sigma_\sigma : \sigma=0,1,2,3\}$ and $\{\epsilon_p : p=0,1,2,3\}$

and thus obtain

$$E_{\lambda\mu\beta} \frac{1}{2!} R^{\lambda\mu}_{\alpha\beta} \frac{1}{2} \epsilon^{\nu\alpha\beta\sigma} = \frac{8\pi G}{c^4} T_\nu^\sigma,$$

$\ell, h, s = r, h, s$

Evaluate the sum over ν by noting

that $E_{\nu\lambda\mu\beta} = \sqrt{-g} [E_{\lambda\mu\beta}]$

$$\epsilon^{\nu\alpha\beta\sigma} = \frac{\sqrt{-g}}{g} [\nu\alpha\beta\sigma] = -\frac{1}{\sqrt{g}} [\nu\alpha\beta\sigma]$$

Consequently

$$E_{\nu\lambda\mu\beta} \epsilon^{\nu\alpha\beta\sigma} = -\delta^{\alpha\beta\sigma}_{\lambda\mu\beta} = - \begin{vmatrix} \delta^\alpha_\lambda & \delta^\alpha_\mu & \delta^\alpha_\beta \\ \delta^\alpha_\lambda & \delta^\alpha_\mu & \delta^\alpha_\beta \\ \delta^\alpha_\lambda & \delta^\alpha_\mu & \delta^\alpha_\beta \end{vmatrix} =$$

$$= -[\delta^\alpha_\lambda \delta^\beta_\mu \delta^\sigma_\beta + \delta^\alpha_\mu \delta^\beta_\beta \delta^\sigma_\lambda + \delta^\alpha_\beta \delta^\sigma_\lambda \delta^\beta_\mu]$$

$$- \delta^\alpha_\lambda \delta^\sigma_\mu \delta^\beta_\beta - \delta^\alpha_\mu \delta^\beta_\lambda \delta^\sigma_\beta - \delta^\alpha_\beta \delta^\beta_\mu \delta^\sigma_\lambda]$$

The left hand side becomes

$$L.H.S = \frac{-1}{4} R^{\lambda\mu}_{\alpha\beta} [\delta^\alpha_\lambda \delta^\beta_\mu \delta^\sigma_\beta + \delta^\alpha_\mu \delta^\beta_\beta \delta^\sigma_\lambda + \delta^\alpha_\beta \delta^\sigma_\lambda \delta^\beta_\mu] \times 2$$

$$= \frac{-1}{4} R^{\lambda\mu}_{\alpha\beta} \delta^\beta_\mu \delta^\sigma_\beta + R^{\lambda\mu}_{\alpha\beta} \delta^\beta_\sigma \delta^\sigma_\lambda + R^{\lambda\mu}_{\alpha\beta} \delta^\sigma_\mu \delta^\beta_\lambda \times 2$$

$$= \frac{-1}{4} [R^\mu_\beta \delta^\beta_\mu \delta^\sigma_\beta - R^\lambda_\beta \delta^\beta_\sigma \delta^\sigma_\lambda + R^{\beta\sigma}_{\beta\beta} \delta^\sigma_\mu \delta^\beta_\lambda] \times 2$$

$$= \frac{-1}{4} [R \delta^\sigma_\beta - R^\sigma_\beta - R^\sigma_\beta] \times 2$$

$L.H.S = [R^\sigma_\beta - \frac{1}{2} \delta^\sigma_\beta R] = r.h.s \equiv \frac{8\pi G}{c^4} T_\beta^\sigma]$ Q.E.D.

30.6

Comment

The Cartan-Wheeler representation

$$\star d\bar{P} \wedge \overset{\leftrightarrow}{R} = \star (e_{\nu} e_{\lambda} e_{\mu} R^{(\lambda\mu)}{}_{|\alpha\beta|} dx^{\nu} dx^{\alpha} dx^{\beta}) = \frac{8\pi G}{c^4} * \bar{T}$$

$$= e_{\rho} E_{\nu\lambda\mu}{}^{\sigma} R^{(\lambda\mu)}{}_{|\alpha\beta|} E^{\nu\alpha\beta\sigma} d^3\Sigma_{\sigma} = \frac{8\pi G}{c^4} e_{\rho} T^{\rho\sigma} d^3\Sigma_{\sigma}$$

of the Einstein field equations is expressed
in terms of the contracted (over ν) "double
dual" of the Riemann curvature tensor,

$$\overset{\leftrightarrow}{G} = e_{\rho} G^{\rho\sigma} d^3\Sigma_{\sigma} = e_{\rho} (\underset{\sim}{\star \bar{R} \star})_{,\nu}{}^{\rho\nu} d^3\Sigma_{\sigma} = \frac{8\pi G}{c^4} e_{\rho} T^{\rho\sigma} d^3\Sigma_{\sigma}$$

$$E_{\nu\lambda\mu}{}^{\sigma} R^{(\lambda\mu)}{}_{|\alpha\beta|} E^{\nu\alpha\beta\sigma} = R^{\rho\sigma} -$$

$$= R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R \equiv G^{\rho\sigma}$$

The most important feature of the Cartan-Wheeler formulation

$$\star d\eta \tilde{\Omega} = \frac{8\pi G}{c^4} * T$$

of the Einstein field equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}$$

is that it mathematizes two universal aspects of the world:

1. Conservation of momenergy

$$d*T = 0 \quad (T^{\mu\nu}_{;\nu} = 0)$$

and

2. the topological fact that
the boundary of the boundary of
a domain \mathcal{D} is zero:

$$\partial\partial\mathcal{D} = 0$$

In the context of a 4-d spacetime domain

\mathcal{D} which is permeated by a curvature

field $\overset{\leftrightarrow}{R} = e_\lambda \wedge e_\mu R^{(\lambda\mu)}_{(\alpha\beta)} dx^\alpha \wedge dx^\beta$

one has

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

$$\boxed{G^{\mu\nu};_i = 0 \Leftrightarrow T^{\mu\nu};_i = 0}$$

$$\boxed{G^{\mu\nu} = 0 \Leftrightarrow T^{\mu\nu} = 0}$$

$$\partial\partial\vartheta = 0 \quad \xleftarrow{\text{mathematized by}} \quad d(\star d\vartheta \overset{\leftrightarrow}{R}) = 0 \Leftrightarrow d\star T = 0$$

YD

(2-3 Stokes' thm)
(3-4 Stokes' thm)

("Fund'l thm
of calculus")

$$\star d\vartheta \overset{\leftrightarrow}{R} = \frac{8\pi G}{c^4} \star T$$

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Nota bene:

" \Rightarrow " means "implies" (via deduction)

" \Leftarrow " means "is mathematized by"
(via induction)

30.9

The Cartan-Wheeler formulation of gravitation (in term of the moment of rotation) puts its mathematization under the purview of freshman and sophomore calculus (MATH 1152 and 2153), namely the fundamental theorem of calculus.

It manifests itself in the various versions

of Stokes' theorem.

[2-3 version]



[3-4 version]

$(= 0 \star)$

$\star d\varphi \wedge \tilde{R}$

$$\star \sum_{k=1}^8 \sum_{l=1}^6 \iint \varphi \wedge \tilde{R} = \sum_{k=1}^8 \iiint \star d\varphi \wedge \tilde{R} = \iint \star d(\varphi \wedge \tilde{R})$$

(2th face)
of kth cube

cube

D

$\partial \partial \partial =$
$= 8 \cdot 6$
faces

(zero)
 $\forall \partial$

$\partial \partial =$
$\oplus 3\text{-cubes}$

$D = 4 - d$
block
of spacetime