

LECTURE 31

Einstein Field Equations for
Spherically Symmetric Systems

1. 2+2 decomposition of M^4

$$\text{into } M^4 = M^2 \times S^2$$

2. The decomposed Einstein
field equations

II. THE EINSTEIN FIELD EQUATIONS

An ubiquitously observed manifestation of q.m. is the vacuum (zero-point) fluctuations of a wave field permeating any given spatial domain[8]. Given this fact, (i) what is the effect of gravitation on the stresses and energies of these fluctuations in the collapsing body? and (ii) conversely, what is their effect on the evolution and ultimate fate of that body? We shall answer the first question while giving a heuristic answer to the second.

A. The 2+2 Decomposition

The prototypical collapse scenario is one which is spherically symmetric. The spacetime geometry for such an environment is

$$ds^2 = g_{AB}(x^C)dx^A dx^B + R^2(x^C)[d\theta^2 + \sin^2\theta d\phi^2]. \quad (1)$$

Here

$$g_{AB}(x^C)dx^A dx^B \quad (2)$$

and

$$R(x^C) \quad (3)$$

are to-be-determined metric and scalar field on the 2-d Lorentzian manifold

$$M^2 = M^4/S^2. \quad (4)$$

This is the radial-time plane. Its 2-d events are labeled by coordinate functions whose indeces are capital Latin letters near the beginning of the alphabet

$$\left. \begin{array}{c} A \\ B \\ C \\ D \\ \vdots \end{array} \right\} = 0, 1.$$

The kinematics and dynamics of all spherically symmetric systems are reducible to statements about geometrical objects (scalar, vector, tensor fields ...) on the 2-d spacetime M^2 .

1. This includes the scalar wave equation

$$\square\Phi = 0, \quad (5)$$

which, for each field harmonic proportional to the spherical harmonic $Y_\ell^m(\theta, \phi)$, reduces to the 2-d wave equation

$$(R^2 g^{AB}\Phi_{;B})_{;A} - \ell(\ell+1)\Phi = 0 \quad (6)$$

on M^2 . The vertical bar " $|$ " refers to the metric induced covariant derivative on M^2 ,

$| \leftarrow$ covariant derivative.

2. It also includes the Einstein field equations

$$G_{\mu\nu} = 8\pi t_{\mu\nu} \quad (7)$$

which reduce to a tensor and a scalar equation[9-11]

$$\begin{aligned} R^2 G_{AB} &\equiv -2RR_{,A|B} + g_{AB} \left(2RR_{,C}^{;C} + R_{,C}R^{;C} - 1 \right) \\ &= 8\pi R^2 t_{AB} \end{aligned} \quad (8)$$

$$\frac{1}{2}G_a^a \equiv \frac{R_{,C}^{;C}}{R} - \mathcal{R} = 4\pi t_a^a \quad (9)$$

on M^2 . Here t_{AB} and t_a^a are the components of the M^4 stress-energy tensor

$$t_{\mu\nu} dx^\mu dx^\nu$$

reduced to the geometrical objects

$$t_{AB} dx^A dx^B \quad (10)$$

and

$$t_a^a \equiv t_\theta^\theta + t_\phi^\phi \quad (11)$$

on M^2 , while \mathcal{R} is M^2 's Gaussian curvature defined by

$$(2) R^{AB}_{CD} \left(= {}^4R^{AB}_{CD} \right) = \mathcal{R} (\delta_C^A \delta_D^B - \delta_D^A \delta_C^B). \quad .$$

The divergence in M^4 of the Einstein tensor reduces to the vectorial identity

$$(R^2 G_A^B)_{;B} - RR_{,A}G_a^a = 0 \quad (12)$$

$$(13)$$

on M^2 . That this is an identity can also be verified directly by using the expressions for $R^2 G_A^B$ and G_a^a defined by Eqs.(8) and (9). The corresponding M^2 Euler equations implied by this are

$$(R^2 t_A^B)_{;B} - RR_{,A}t_a^a = 0 \quad (14)$$

B. Integrating the Reduced Field Equations

In spite of their non-linearity, the Einstein field Eqs.(8)-(9) can always be integrated analytically, at least in part. This is because they contain a conservation law in the form of a conservative vector field.

The vectorial identity Eq.(12) implies that the divergence of the M^2 vector

$$J^A = -\frac{1}{2}R_{,C}\epsilon^{CD}R^2 G_D^A \quad (15)$$

vanishes:

$$J^A_{;A} = 0. \quad (16)$$

SOLUTIONS TO THE EINSTEIN FIELD

EQUATIONS

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

One of the most important applications of the Einstein field equations is for spherically symmetric systems. Examples of such systems include a star, a black hole, the universe, a star collapsing towards a black hole, a star exploding and leaving behind a wildly pulsating neutron star.

A space-time with such spherically symmetric spatial symmetry is of additional interest because it serves as an arena for the generation and propagation of gravitational and hydrodynamical asymmetric disturbances: gravitational waves, acoustic waves, non-spherical oscillations and other perturbations away from spherical symmetry.

I. A four dimensional spacetime M^4 which is spatially spherically symmetric reveals this symmetry by the fact that the metric can be written in block diagonal form.

$$M^4: g_{\mu\nu} dx^\mu dx^\nu = g_{AB} dx^A dx^B + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\text{E.g. } = g_{AB}(x^c) dx^A dx^B + r^2(x^c) g_{ab} dx^a dx^b$$

Here $\begin{cases} A \\ B \\ C \end{cases} = 0, 1, 2, 3$ $\begin{cases} a \\ b \end{cases} = \theta, \phi$ $\text{on } S^2 = \text{transverse manifold}$

$\text{on } M^2 = \text{longitudinal manifold}$

Thus one can associate a sphere S of surface area $4\pi r^2(x^c)$ with every event (x^0, x') on the two dimensional spacetime M^2 spanned by x^0 and x' .

S^2 : spanned by $x^a = (\theta, \phi)$ (Transverse manifold)

M^2 : spanned by $x^c = (x^0, x')$ (Longitudinal manifold)

The sphere S^2 at x^c is an equivalence class induced by rotational symmetry. The set M^2 can be viewed as the set of equivalence classes under rotation

$$M^2 = M^4 / S^2$$

and the block diagonal nature of the metric implies that

$$M^4 = M^2 \times S^2$$



is the product of two orthogonal submanifolds.

Note $g_{\mu\nu} dx^\mu dx^\nu$: $\begin{bmatrix} \delta_{AB}(x^c) & 0 & 0 \\ 0 & r^2(x^c) & 0 \\ 0 & 0 & r^2(x^c) \sin^2 \theta \end{bmatrix}$ 31.3

a) $\delta_{AB}(x^c) dx^A dx^B$ is a tensor field on M^2

$r^2(x^c)$ is a scalar field on M^2

while

$$g_{ab} dx^a dx^b = d\theta^2 + \sin^2 \theta d\varphi^2$$

is a tensor field on S^2 .

b) A general symmetric tensor field, for example the momentum stress tensor, having spherical symmetry has the form

$$t_{\mu\nu} dx^\mu dx^\nu : \begin{bmatrix} t_{AB} & 0 & 0 \\ 0 & t_a & 0 \\ 0 & 0 & \frac{1}{2} t_a \end{bmatrix} = \begin{bmatrix} t_{AB} & 0 \\ 0 & \frac{1}{2} t_a & 0 \\ 0 & 0 & \frac{1}{2} t_a \sin^2 \theta \end{bmatrix}$$

It follows that such a tensor field can be viewed as the two fields

$$t_{AB}(x^c) dx^A dx^B \text{ and } t_a(x^c) = t_\theta^\theta + t_\varphi^\varphi$$

on M^2 .

c) A general spherically symmetric (co)vector field $v_\mu dx^\mu : \begin{pmatrix} v_A(x^c) \\ 0 \\ 0 \end{pmatrix}$

has no angular components. (The "fuzzy tennisball" theorem forbids it).

A spherically symmetric covector field on M^4 is thus characterized by a covector field field on M^2 .

It follows that spherical symmetry establishes the following scheme between spherical tensor fields on M^4 and tensor fields on M^2 :

M^4

M^2

$$g_{\mu\nu} \longleftrightarrow \begin{cases} g_{AB} & \text{tensor} \\ \gamma^2 & \text{scalar} \end{cases}$$

$$t_{\mu\nu} = t_{\nu\mu} \longleftrightarrow \begin{cases} t_{AB} & \text{tensor} \\ t^a = t^{\theta}_\theta + t^\varphi_\varphi & \text{scalar} \end{cases}$$

$$\omega_A \longleftrightarrow \omega_B \quad \text{covector}$$

$$\omega^A \longleftrightarrow \omega^B \quad \text{vector.}$$

I. Results of the 2+2 Decomposition Process

A) Applied to the Einstein tensor of his field equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \quad (\text{10 equations on } M^4)$$

one finds

$$G_{AB} = 8\pi T_{AB} \quad \text{tensor: 3 equations}$$

$$G_a{}^a = 8\pi T_a{}^a \quad \text{scalar: 1 equation}$$

} after some calculations (5.756Final or MTW Ex 14.16)
and rewriting the answer in terms of $g_{AB}(x^4)$,
 $r(x^4)$ and their derivatives
where

$$a) r^2 G_{AB} = -2 r_{;AB} + (2r r_c{}^{1c} + r_c r^{1c} - 1) g_{AB}$$

or with $v_c = \frac{r_c}{r}$

$$a) G_{AB} = -2 (v_{AIB} + v_A v_B) + (2 v_c v_c{}^{1c} + 3 v_c v^c - \frac{1}{r^2}) g_{AB}$$

b) $G_{Aa} = 0$

c) $G_{ab} = \frac{1}{2} G_d{}^d g_{ab} = \left(\frac{r_c{}^{1c}}{r} - R \right) g_{ab}; R_{cd}^{AB} = R \left(\delta_c^A \delta_d^B - \delta_c^B \delta_d^A \right)$
 $= (v_c v_c{}^{1c} + v_c v^c - R) g_{ab}$

or

$$c) \frac{1}{2} G^d_d = G^0_0 = G^{\theta}_{\phi}$$

$$= \frac{\gamma_c^{1c}}{r} - R$$

Thus, the 2+2 decomposition of the Einstein's field equations on M^4 reduce to the tensor equations

$$r^2 G_{AB} = -2r \gamma_{AB} + g_{AB} (2r \gamma_c^{1c} + \gamma_c \gamma_D g^{CD} - 1) = 8\pi r^2 t_{AB}$$

and the scalar equation

$$\frac{1}{2} G_a^a = \frac{\gamma_c \gamma_D g^{CD}}{r} - R = \frac{1}{2} 8\pi t_a^a$$

both on M^2 .

B) Applied to the conservation equations

$$G_\mu^{\nu} ; \nu = 0 = t_\mu^{\nu} ; \nu, \quad \mu = 0, 1, 2, 3$$

the 2+2 decomposition results in

$$(r^2 G_A^B)_{1B} - r r_{;A} G_a^a = 0 \quad A = 0, 1,$$

and

$$(r^2 t_A^B)_{1B} - r r_{;A} t_a^a = 0 \quad A = 0, 1 \quad \left. \begin{matrix} \text{on} \\ M^2 \end{matrix} \right\}$$

which is the conservation of momentum,

and hence Euler equations for spherically

symmetric hydrodynamics or hydrostatics

for any spherically symmetric systems.

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c) The 2+2 Decomposition process

is based on

(i) the solutions to Problem 1 and

Problem 2 of last semester's

(Math 5756) final exam.

(The solution to Problem 1 is given in

Exercise 14.16 in MTW),

(ii) the solution to the addition

problems 3, 4, and 5 found

in the Appendix to Lecture 3/

APPENDIX A TO LECTURE #31

Computation of Einstein tensor

Field equations for spherical systems

- 1 -

**APPENDIX A: COMPUTATION OF
EINSTEIN TENSOR FIELD
EQUATIONS FOR SPHERICAL
SYSTEMS**

Relative to appropriately chosen coordinates the spacetime geometry of any spherical configuration is expressed by the metric

$$(ds)^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Here one of the geometrical degrees of freedom, the metric coefficient g_{rr} , governing the spatial geometry is determined by the proper mass-energy density:

$$g_{rr} = \frac{1}{1 - \frac{2m}{r}} \quad \text{with } m(t, r) = \int^r 4\pi r^2 \omega t_0^0(t, r) dr + \text{const}$$

To determine the other gravitational degree of freedom, $g_{tt}(t, r)$, we again must use the Einstein field equations G0 to P2

$$-2r r_{AB}^{(0)} + g_{AB} (2r r_c^{(2)} + r_c r_c^{(3)}) \equiv r^2 G_{AB} = 8\pi r^2 t_{AB}. \quad (1)$$

$$\frac{r_c}{r} - R = \frac{1}{2} G_a^a = 4\pi t_a^a \quad (2)$$

and the implied Euler equations for the matter degrees of freedom,

$$\square -2\tau \overset{(1)}{\gamma_{AB}} + g_{AB}(2\tau \overset{(2)}{\gamma_c}{}^C + \overset{(3)}{\gamma_c}{}^C - 1) = r^2 G_{AB} = \frac{8\pi G}{c^2} r^2 t_{AB} \quad (1)$$

$$\frac{\overset{(1)}{\gamma_c}{}^C}{\tau} - R \equiv \frac{1}{2} G_a{}^a = \frac{4\pi G}{c^2} t_a{}^a \quad (2)$$

These field equations imply the Euler equations for the matter degrees of freedom, namely

$$(r^2 t_A{}^B)_{IB} - r \overset{(1)}{\gamma_A} t_a{}^a = 0 \quad (3)$$

In other words, the divergence of (1) + Eq. (2) imply Eq. (3).

Problem 1:
Show that $w_{AIBIC} - w_{AICIB} = w_p R^D{}_{ABC}$ (4)

Problem 2:
With the help of Eq. (4) show that

$$(1) + (3) \Rightarrow 2$$

$$(1) + (2) \Rightarrow (3)$$

wish to

We exhibit explicitly the Einstein field equations relative to the coordinates ($x^0 = t$, $x^1 = r$) relative to which the metric has the form

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with

$$g_{rr} = \frac{1}{1 - \frac{2m}{r}} ; m(t, r) = \int_0^r 4\pi r'^2 (-) t_0^0(t, r') dr' + \text{const}$$

For computational ease we let

$$g_{tt} = -e^{2\Phi(r, t)} = \frac{1}{g^{tt}}$$

$$g_{rr} = e^{2\Lambda(r, t)} = \frac{1}{g^{rr}}$$

and use the solution to Problem 2 of the final exam for Math 5756

Σ_{AIB}

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

$$\begin{aligned}\gamma_{010} &= \gamma_{0,0} - \gamma_c \Gamma_{00}^c \\ &= 0 - 1 \Gamma_{00}^1 = -e^{2\phi-2\Lambda} \phi'\end{aligned}$$

$$\begin{aligned}\gamma_{011} &= \gamma_{0,1} - \gamma_c \Gamma_{01}^c \\ &= 0 - 1 \Gamma_{01}^1 = -1\end{aligned}$$

$$\gamma_{110} = \text{same} = -1$$

$$\begin{aligned}\gamma_{112} &= \gamma_{2,1} - \gamma_c \Gamma_{11}^c \\ &= 0 - 1 \Gamma_{11}^1 \\ &= -1\end{aligned}$$

$$\begin{aligned}\gamma_c^{1c} &= \gamma_{010} g^{00} + \gamma_{112} g^{rr} \\ &= (\phi' + \Lambda') e^{-2\Lambda}\end{aligned}$$

With these expressions the tensor field equations on M^2 become simply

$$\boxed{r^2 G_0^0 :}$$

$$\begin{aligned} \textcircled{1} + \textcircled{2} + \textcircled{3} &= -2r r_{j0}^{10} + (2r r_{j0}^{10} + 2r r_{j1}^{11} + r_{j0} r_{j1}^0 + r_{j1} r_{j0}^1 - 1) \\ &\quad \cancel{\textcircled{1}} \quad \cancel{\textcircled{2}} \quad \cancel{\textcircled{3}} \\ &\quad \text{cancel} \\ &= \frac{8\pi G}{c^2} r^2 t_0^0 \end{aligned}$$

$$= 2r(-\lambda' e^{-2\lambda} + 0 + e^{-2\lambda} - 1) = \frac{8\pi G}{c^2} r^2 t_0^0$$

$$= (r e^{-2\lambda})' - 1 = 11$$

$$= \boxed{(r - 2m)' - 1} = \frac{8\pi G}{c^2} r^2 t_0^0$$

$$\boxed{r^2 G_{01}^0 :}$$

$$\textcircled{1} + \textcircled{2} + \textcircled{3} = -2r r_{j012}^0 + 0 + 0 = \frac{8\pi G}{c^2} r^2 t_{01}^0$$

$$= \boxed{2r i^1 = \frac{8\pi G}{c^2} r^2 t_{01}^0}$$

$$\boxed{r^2 G_1^1 :}$$

$$\begin{aligned} \textcircled{1} + \textcircled{2} + \textcircled{3} &= -2r r_{j1}^{12} + (2r r_{j0}^{10} + 2r r_{j1}^{12} + r_{j0} r_{j1}^0 - 1) \\ &\quad \cancel{\textcircled{1}} \quad \cancel{\textcircled{2}} \quad \cancel{\textcircled{3}} \\ &= \frac{8\pi G}{c^2} r^2 t_1^1 \end{aligned}$$

$$+ 2r e^{-2\lambda} \phi^1 + e^{-2\lambda} = \frac{8\pi G}{c^2} r^2 t_1^1$$

$$(2r - 4m)\phi^1 - \frac{2m}{r} = \frac{8\pi G}{c^2} r^2 t_1^1$$

$$\boxed{(r - 2m)\phi^1 = \frac{m}{r} + \frac{(4\pi G/c^2)r^2 p}{r}}$$

LECTURE 31 (APPENDIX A cont'd)

The $2+2$ Decomposition Process!

Five Problems

Comment! The ensuing problems 3, 4 and 5 were not part of the final in Math 5756, but their solutions are part of a lecture in MATH 5757.

MATH 5756: FINAL EXAM

Pt. I

Problem 1

Consider the metric for an arbitrary spherically symmetric space-time (of, say, a collapsing or exploding star, a black hole):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where

$$\Phi = \Phi(T, R)$$

$$\Lambda = \Lambda(T, R)$$

$$r = r(T, R)$$

are as-yet unspecified functions of T and R only.

a) COMPUTE (using the Cartan-Misner approach) the non-zero orthonormal basis components of the Riemann curvature tensor.

Helpful suggestion: denote partial derivatives by dots and primes

$$\text{e.g. } \frac{\partial \Phi}{\partial T} = \dot{\Phi}; \quad \frac{\partial \Phi}{\partial R} = \Phi'; \text{ etc.}$$

b) FIND the coordinate basis components of Riemann in terms of the o.n. basis components found in a)

A.2

Problem 2, "Covariant derivative"

Consider the metric for the space-time manifold orthogonal to any given two sphere of the previous problem:

$$ds^2 = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2 \equiv g_{AB}(x^G) dx^A dx^B$$

$\left. \begin{matrix} A \\ B \\ C \end{matrix} \right\} = 0, 1 \text{ only}$

a) EXHIBIT all non-zero Christoffel symbols

$$\Gamma_{BC}^A = \frac{1}{2} g^{AD} (g_{DB,C} + g_{DC,B} - g_{BC,D})$$

for this metric

b) Consider the covector $\underline{v} = v_A(T, R) \underline{dx}^A$. COMPUTE explicitly all four components of the covariant derivative of \underline{v} , i.e. COMPUTE v_{AIB}^J ($A = 0, 1; B = 0, 1$), where " J " denotes covariant derivative based on the Γ_{BC}^A of part a).

c) Let $r(T, R)$ be a function defined on the space-time plane defined above.

Let $v_p = \left(\frac{x}{r}, \frac{r'}{r} \right)$ and then SHOW.

that $v_{AIB}^J = v_{BIA}^J$

d) STATE a condition on a covector, say $W_\alpha dx^\alpha$, which guarantees that $W_{AIB}^J = W_{BIA}^J$!

The following problems illustrate the enormous simplification that results when there are two directions (e.g. angular) of symmetry in the (linear and / or non-linear) system.

A.5

Problem 3 Geometry of spherically symmetric A.4
Space-time.

Consider $ds^2 = g_{AB}(x^a) dx^A dx^B + r^2(x^c)(d\theta^2 + \sin^2\theta d\phi^2)$
where $\{^A_B\} = \{0, 1\}$ so that $x^A; (x^0, x^1)$ are the
time and radial coordinate

Let $g_{AB} dx^A dx^B$ and r^2 be independent of the
angular coordinates. Thus $g_{AB} dx^A dx^B$ is a
tensor on the 2-dim. manifold, spanned by
 $x^A; (x^0, x^1)$.
(call it M^2)

i) Show that ${}^{(4)}\Gamma_{AB}^C = {}^{(2)}\Gamma_{AB}^C$

where ${}^{(4)}\Gamma_{AB}^C$ are obtained from $ds^2 = g_{AB} dx^A dx^B$

$${}^{(2)}\Gamma_{AB}^C = \frac{1}{2} g^{CD} (\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB})$$

i.e. ${}^{(2)}\Gamma_{AB}^C$ are the Christoffel symbols for M^2 .

ii) Show that ${}^{(4)}\Gamma_{ab}^c = -v_A^a g_{ab}$

$$\text{and } {}^{(4)}\Gamma_{cb}^a = v_c^a \delta_b^a$$

where $v_A^a = \frac{x_A}{r}$; $g_{ab} = r^2 \delta_{ab}$; $\delta_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$

i.e. $\delta_{ab} dx^a dx^b = d\theta^2 + \sin^2\theta d\phi^2$, the metric of
the unit sphere

iii) For $G_\mu^\nu = \begin{bmatrix} G_A^B & 0 \\ -\frac{1}{r} & -\frac{1}{r} \\ 0 & G_\phi^{\phi} \end{bmatrix}$ show that $G_\mu^\nu ;_r = 0 \Rightarrow$

$$r^{-2} (r^2 G_A^B)_{IB} - v_A^c G_c^B = 0$$

Problem 4 : star exploding or collapsing into a
black hole! Curvature: 2+2 split into
"transverse" (angular) and longitudinal
parts.

Consider the metric on a sphere of surface area $4\pi r^2$,

$$ds^2 = r^2 (d\theta^2 + \sin^2\theta d\phi^2) \equiv {}^{(2)}g_{ab} dx^a dx^b; \{^a_b\} = \{\theta, \phi\}$$

Let $\{^a_b\}$ be the unit tensor ("Kronecker delta") on this sphere

a) Examine the results of problem 1b and 2b

Consolidate the results of 1b by expressing them in
terms of v_A^a , v_{AB}^a , ${}^{(2)}g_{ab}$, $\{^a_b\}$, and r^2 as suggested

below; in other words, show that the coordinate basis
components of Riemann have the form

$$R_{A0Bb} = P_{AB}^a {}^{(2)}g_{ab}$$

$$R^a_{bcd} = P \left[\delta_c^{(2)} g_{bd} - \delta_d^{(2)} g_{bc} \right] \text{ where } \{^a_b\} = \{T, R\} \text{ only}
so that \{x^a\} = \{T, R\},$$

$P_{AB} = -v_A^a v_B^b + ???$ is a rank $(2, 0)$ tensor field on
the manifold spanned by $x^0 = T$

and $P = ??$ is a scalar "connection in the longitudinal
manifold" on the "Longitudinal manifold".

FIND: P_{AB} and P !

Problem 4 (cont'd)

b) Continuing the examination of (b) results, show that

$$R^A_{\ BCD} = R \left[\delta^A_C g_{BD} - \delta^A_D g_{BC} \right]$$

where (i) g_{AB} are the components of the metric on the "Longitudinal manifold" in problem 2b
(ii) R is a certain expression in Φ and λ .

WHAT is R ?

Note: $R, [\delta^A_C g_{BD} - \delta^A_D g_{BC}]$, and $R[\delta^A_C g_{BD} - \delta^A_D g_{BC}]$ may be viewed as a scalar field, (the coordinate components of) a tensor of rank(1), and again (the coordinate components of a tensor of rank(1)) on the "Longitudinal manifold spanned by $x^A, A=0, 1$.

(c) COMPUTE by the method of your choice the curvature invariant associated with the $g_{AB} dx^A dx^B = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2$, the longitudinal part of the metric on the "longitudinal 2-D space-time manifold spanned by $x^A; x^0 = t, x^1 = R$.

VERIFY that

$$(\text{curvature invariant}) = c R$$

WHAT is c ?

A.6

1

A.7

Problem 5. Einstein tensor for an exploding star or for a star collapsing into a black hole.

The metric of 4-dimensional space-time with spherical symmetry has the form

$$g_{\mu\nu} dx^\mu dx^\nu = \underbrace{g_{AB}(x^C) dx^A dx^B}_{\text{"LONGITUDINAL PART"}} + \underbrace{r^2(d\theta^2 + \sin^2\theta d\phi^2)}_{\text{"TRANSVERSE PART"} \rightarrow \underbrace{g_{ab} dx^a dx^b}_{\text{(a)}}$$

$$\text{i.e. } [g_{\mu\nu}] = \begin{bmatrix} g_{AB} & | & 0 \\ - & - & - \\ 0 & | & g_{ab} \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}^{\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix}}$$

NOTE:

Thus $g_{AB}(x^C) dx^A dx^B$ may be viewed as the metric on the 2-dimensional ^{longitudinal} space-time manifold spanned by x^0 and x^1 .

ALSO NOTE:

$$(2) g_{ab} g^{bc} = \delta_a^c, \quad \{b, c\} = 0, \varphi.$$

Using the formulas ; ∵ derived in 4a and 4b

FIND a) all non-zero components of Ricci for the 4-dimensional space-time

$$R_{\mu\nu} = \begin{cases} R_{AB} & \text{with } \{A, B\} = 0, 1 \\ R_{AA} \\ R_{ab} \end{cases}$$

Problem 5 (cont'd)

in terms of g_{AB} , $v_A v_B$, v_{AIB} , ${}^{(2)}g_{ab}$

Note:

$$[R_{\mu\nu}] = \begin{bmatrix} R_{AB} & R_{Aa} \\ R_{aA} & R_{ab} \end{bmatrix}^o_1$$

$\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix}$

so that $R_{AB} dx^P dx^B$ may be viewed as
a rank $\binom{0}{2}$ tensor on the 2-dimensional space-
time spanned by x^A .

FIND b) the curvature invariant

$$R = R^\mu_\mu = R^A{}_A + R^a{}_a = ???$$

by expressing it in terms of r^2 , $v_A v_B$, v_{AIB} &

FIND c) The Components of the Einstein
tensor

$$G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R$$

$$G_{Aa} = R_{Aa} - \frac{1}{2} g_{Aa} R = R_{Aa} \quad (g_{Aa} = 0 !)$$

$$G_{ab} = R_{ab} - \frac{1}{2} {}^{(2)}g_{ab} R$$

in terms of r^2 , $v_A v_B$, v_{AIB} , ${}^{(2)}g_{AB}$, v_{AIB}

A, 8

A, 9

Comments: (1) Problem 5 is easy because only algebraic manipulations (contraction, addition) on tensors in the T-R (longitudinal) plane and the θ - ϕ plane are required.

(2) Problem 4 is also easy; no computation is required for 4a and 4b. Only 4c requires computation.

SOME ANSWERS TO PROBLEMS 4&5

A.10

$$R_{AB} = R^M_{AB} = R^C_{ACB} + R^a_{AaB}$$

$$= R \left(\delta^C_C g_{AB} - \delta^C_B g_{AC} \right) + (-) \left[v_{AIB} + v_{AB}^r \right] \delta^a_a$$

$$R_{AB} = R g_{AB} - 2(v_{AIB} + v_A v_B)$$

$$R_{Aa} = R^M_{Aka} = R^C_{ACA} + R^b_{Aba} = 0$$

$$R_{ab} = R^M_{ab\mu b} = R^A_{\mu A b} + R^d_{\mu d b}$$

$$\stackrel{P^A}{=} \left(- (v_A^{1A} + v_A v^A) \right) g_{ab} + \underbrace{\left(\frac{1}{r^2} - v_A v^A \right)}_P (2g_{ab} - g_{ab})$$

$$R_{ab} = - (v_A^{1A} + 2v_A v^A - \frac{1}{r^2}) g_{ab}$$

$$R_{ab} = \left(\frac{1}{r^2} - v_A^{1A} - 2v_A v^A \right)^{(a)} g_{ab}$$

$$R = R^a_{\mu} = R^A_A + R^a_a = 2(R + \frac{1}{r^2} - 2v_A^{1A} - 3v_A v^A)$$

$$G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R = -2(v_{AIB} + v_A v_B) + (2v_c^{1C} + 3v_c v^C - \frac{1}{r^2})$$

$$G_{Aa} = 0$$

$$G_{ab} = v_c^{1C} + v_c v^C - R$$

g_{AB}