

LECTURE 33

Non Euclidean geometry in the equatorial
plane of a spherical star

Read, MTW Section 238

Consider a star in equilibrium having

surface area $4\pi R^2$.



The mass function

$$m(r, t) = - \int_0^r \frac{4\pi G}{c^4} r^2 t_0^0(r, t) dr; \quad t_0^0 = (\text{mass density}) c^2 = \text{energy density}$$

is independent of time, and so is t_0^0 !

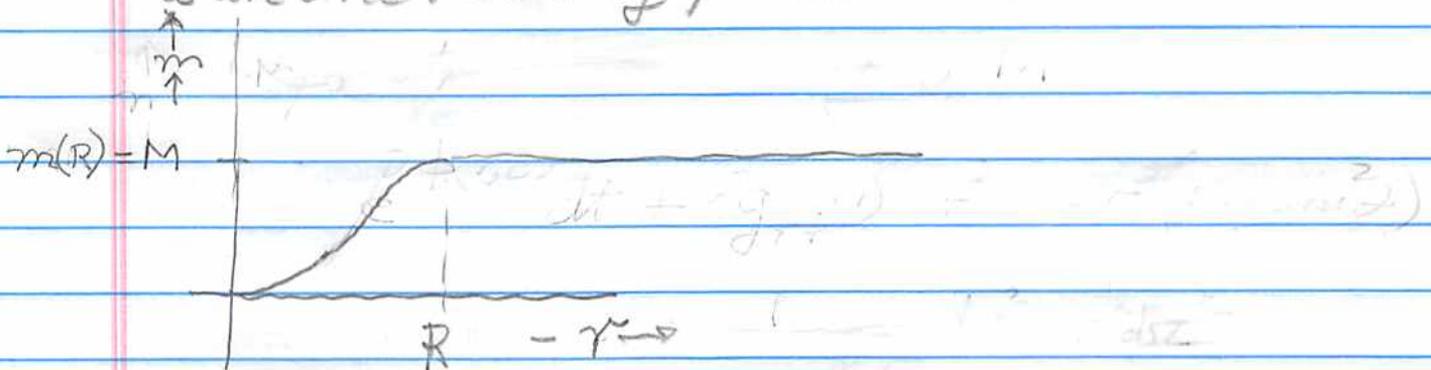
$$\frac{1}{c^2} t_0^0(r, t) = \begin{cases} \rho(r) > 0, & r \leq R & \text{INSIDE} \\ 0, & R < r & \text{OUTSIDE} \end{cases}$$

Thus

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr \quad \text{INSIDE } r \leq R$$

$$= \int_0^R 4\pi r^2 \rho(r) dr \equiv M \quad \text{OUTSIDE } R < r$$

is an increasing function of r .



The spacetime geometry is

$$ds^2 = -e^{2\phi(r,t)} dt^2 + g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

$$T = dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 ds^2$$

$$1 - \frac{2m(r)}{r}$$

The spatial geometry is

$$ds^2 \Big|_{t=\text{fixed}} = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

In the equatorial plane $\theta = \frac{\pi}{2}$ it is

$$ds^2 \Big|_{\substack{t=\text{fixed} \\ \theta = \frac{\pi}{2}}} = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\varphi^2 \equiv d\sigma^2 \text{ (Non-Euclidean)}$$

to be compared with

$$ds^2 = dr^2 + r^2 d\varphi^2 \quad \text{(Euclidean)}$$

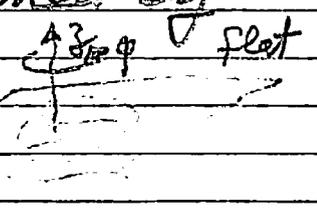
Imbedding space.

To obtain a geometrical picture of this non-Euclidean geometry, we use the technique of the imbedding diagram.

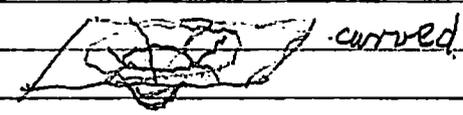
It consists of viewing the plane as a surface of revolution in a 3-D

fictitious imbedding space with a Euclidean geometry and spanned by

its three coordinates z, r, ϕ :
(Metric for imbedding space)



$$dl^2 = dz^2 + dr^2 + r^2 d\phi^2$$



On the surface of revolution

$$z = f(r)$$

this Euclidean geometry induces the metric

$$(dl)_{z=f(r)}^2 = \left[\left(\frac{dz}{dr} \right)^2 + 1 \right] dr^2 + r^2 d\phi^2$$

Identify the metric on this surface with the metric on the equatorial plane of the spherically symmetric

The Imbedding function $z(r)$ -7-
33.4

spacetime. This identification gives rise to the differential equation

$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1 - \frac{2m}{r}}$$

The solution to this differential equation

$$z(r) = \int_0^r \left[\frac{2m}{r-2m} \right]^{1/2} dr$$

PICTURE from P33.6
HERE

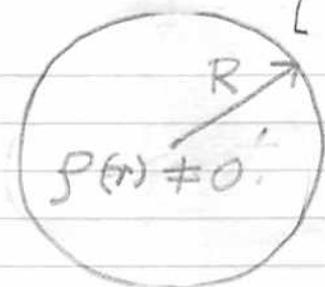
yields a 2-D surface of revolution from a fictitious 3-D perspective. This perspective is very useful. It allows us to visualize the inner 2-D spatial geometry on the equatorial plane or - because of spherical symmetry - any other rotated plane of the spherically symmetric space.

Example:

As an example, consider at some fixed time ($t = \text{const}$) a star with the mass density

$\rho(r)$ in its interior, and vacuum on the outside

$$m = \begin{cases} \int_0^r 4\pi r'^2 \rho dr' & \text{inside } r < R \\ M & \text{outside } R < r \end{cases} \quad \begin{matrix} -10- \\ 33,5 \end{matrix}$$



$$\rho = 0$$

$t = \text{fixed}$

For such a spherical configuration

$$z(r) = \int_0^r \left[\frac{2m}{r'-2m} \right]^{\frac{1}{2}} dr' \quad \text{where } m = \begin{cases} \int_0^r 4\pi r'^2 \rho dr' & \text{INSIDE} \\ M & \text{OUTSIDE} \end{cases}$$

For $m \neq M$ we have

$$z(r) = \left[8M(r-2M) \right]^{\frac{1}{2}} + \text{const} \quad \text{outside } (m=M)$$

a) Thus OUTSIDE the star we have

$$\boxed{(z-c)^2 = 8M(r-2M)}$$

which is a parabola of revolution.

$$\frac{\frac{1}{c^2} \text{energy}}{(\text{length})^3}$$

b) INSIDE the star, near the center

$$m(r) = \frac{4\pi \rho_c}{3} r^3$$

$$\rho_c = \frac{G}{c^2} (\rho_c) \quad \left[\frac{\text{mass}}{(\text{length})^3} \right] = \frac{1}{(\text{length})^2}$$

Letting

so that $\frac{2m}{r} = \frac{r^2}{a^2}$

$$z = \int_0^r \sqrt{\frac{\frac{2m}{r'}}{1-\frac{2m}{r'}}} dr'$$

$$z = \int_0^r \sqrt{\frac{\left(\frac{r'}{a}\right)^2}{1-\left(\frac{r'}{a}\right)^2}} dr' = \int_0^r \frac{\frac{r'}{a}}{\sqrt{1-\left(\frac{r'}{a}\right)^2}} dr'$$

$$= (-) a \sqrt{1-\left(\frac{r}{a}\right)^2} \Big|_0^r = a - \sqrt{a^2 - r^2} \quad \text{for } r \ll a \text{ (near the center)}$$

Indeed

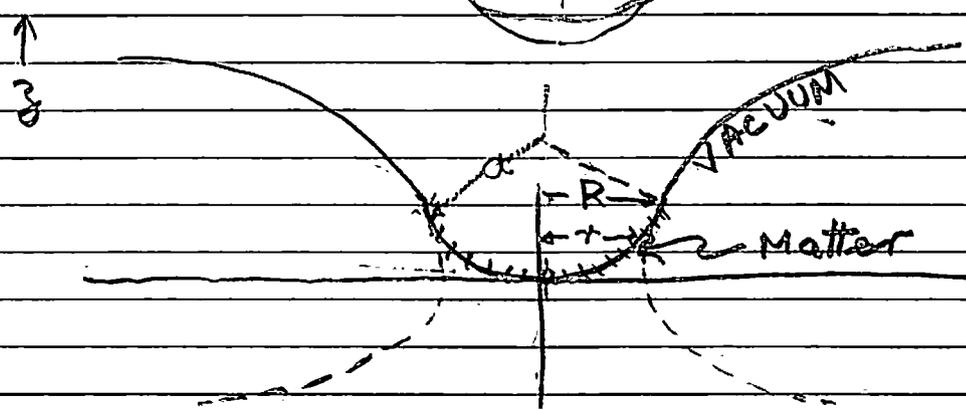
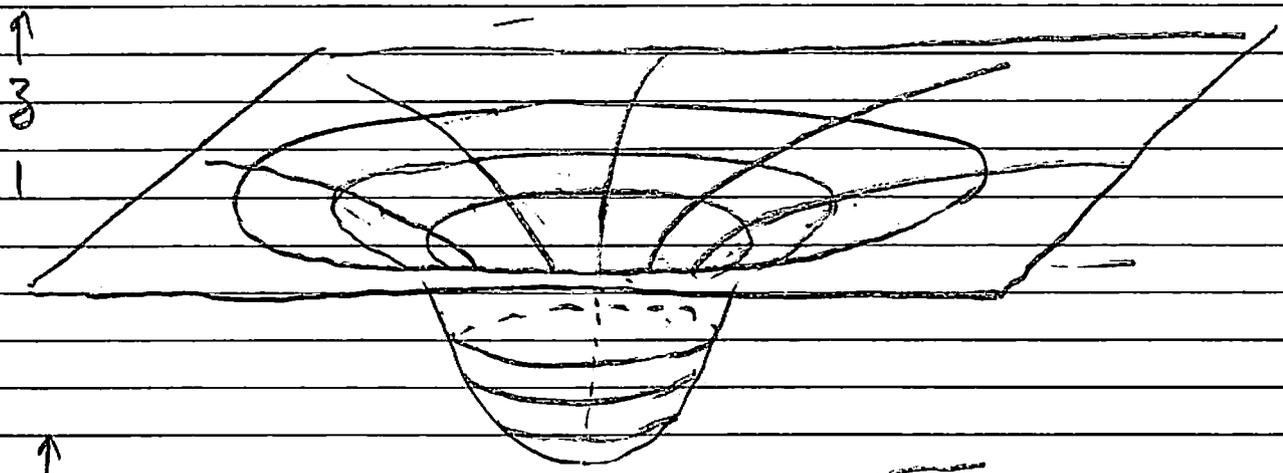
$$\frac{8\pi \rho_c}{3} = \frac{1}{a^2} \quad \left(\frac{8\pi G}{c^4} \rho_c \right) = \frac{8\pi G}{c^4} \frac{\text{energy}}{\text{density}} = \frac{8\pi G}{c^4} \frac{\text{mass}}{c^2 (\text{length})^3} = \frac{\text{length}}{(\text{length})^3} = \frac{1}{(\text{length})^2}$$

This is a circle of revolution,
 $(z-a)^2 + r^2 = a^2$.

c) At the boundary, $r = R$, $\frac{dz}{dr} = \sqrt{\frac{m}{r-2m}}$ is continuous
 because m is continuous.

The geometry of a star is therefore characterized by a circle of revolution near its center, and a parabola of revolution outside its interior joined to its surface $r = R$ without any kink because $m(r)$ is continuous there.

Embedding diagram for a ^{homogeneous} star at $t = \text{fixed}$.



This diagram high lights the curved nature of the equatorial plane of the star, consider a plumb line of length l from the surface of the star to its center

$$\int_0^l ds = \int_0^R \sqrt{g_{rr}} dr = \int_0^R \frac{dr}{\sqrt{1 - \frac{2m(r)}{r}}} \equiv l$$

The equatorial circumference is $C = 2\pi R$

$$\text{But } \frac{2\pi l}{2\pi R} = \frac{1}{\sqrt{1 - \frac{2m(r)}{r}}} \times 2\pi \int_0^R \frac{dr}{\sqrt{1 - \frac{2m(r)}{r}}} > 1$$

(area)

The fact that this ratio exceeds unity indicates the non-Euclidean nature of space,