

LECTURE 34

Relativistic stars:

1. Their equations for hydrostatic equilibrium
2. Their geometrical environment.
3. Equilibrium: Stable or Unstable?
4. Superdense stars

I SPHERICALLY SYMMETRIC SYSTEMS 34.1

It is a fact that there exist a multitude of gravitating systems which are spherically symmetric.

How does one distinguish between them?
How does one classify them?

The Einstein field equations adapted to spherical symmetry.

$$(34.1) \quad -2r \gamma_{,A}^B + \delta_A^B (2r \gamma_{,c}^c + \gamma_{,c} \gamma^{c-}) \equiv r^2 G_A^B = 8\pi r^2 t_A^B \quad (1)$$

$$(34.2) \quad \frac{\gamma_{,c}^c}{r} - R \equiv \frac{1}{2} G_a^a = 4\pi t_a^a \quad (2)$$

$$\left[\frac{1}{2} G_a^a \equiv \frac{1}{2} (G_\theta^\theta + G_\phi^\phi) = G_\theta^\theta = G_\phi^\phi \right],$$

together with their implied hydrodynamical

Euler equations of motion

$$(34.3) \quad (r^2 t_A^B)_{,B} - r \gamma_{,A} t_a^a = 0, \quad (3)$$

answers these questions mathematically.

Let us apply them to a relativistic star.

II RELATIVISTIC STAR

Consider a spherical self-gravitating configuration consisting of a perfect fluid. The components of its momentum-energy

are

$$t_{\mu}^{\nu} = (\rho + p) u_{\mu} u^{\nu} + p \delta_{\mu}^{\nu}$$

(see LECTURE 16)

Here ρ , p , and u^{μ} are the pressure, energy density, and 4-velocity components of the fluid. Their distribution in the star are governed by the law of momentum-energy conservation

$$t_{\mu}^{\nu}{}_{;\nu} = 0 \quad ; \quad u^{\mu} t_{\mu}^{\nu}{}_{;\nu} = 0.$$

These are the equations that govern the dynamics of a relativistic fluid.

For a spherically symmetric configuration there is a single vectorial equation on $M^2 = M^4/S^2$

$$(34.4) \quad u_{c|B} u^B (p + \rho) - (\delta_c^A + u_c u^A) \frac{\partial p}{\partial x^A} = 0 \quad c = 0, 1$$

The metric for any spherically symmetric configuration (by an appropriate choice of coordinates) has the diagonal form

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{2\Phi(t,r)} dt^2 + e^{2\Lambda(t,r)} (dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2))$$

$g_{AB} dx^A dx^B$ on $M^2 = M^4/S^2$

Our focus is on a star in equilibrium. Consequently, there is no explicit time dependence in all matter and geometrical variables. In particular

$$p = p(r); \rho = \rho(r); \{u^A\} = \{u^0, u^1 = 0\}$$

34.4

Accordingly, the two component vector equation (34.4) yields

$$(34.5) \quad \epsilon = \begin{cases} 0 & \Rightarrow \quad \theta = 0 \\ 1 & \Rightarrow \quad \frac{dp}{dr} = -\frac{\partial \phi}{\partial r} (p + \rho) \end{cases}$$

Furthermore, the 3-component Einstein's field Eqs. (34.1) are

$$r^2 G_0^0 \equiv -2 \frac{\partial m}{\partial r} = 8\pi r^2 t_0^0 \quad (= -8\pi r^2 \rho) \quad (34.6a)$$

$$r^2 G_0^1 \equiv 2r \frac{\partial \lambda}{\partial t} = 8\pi r^2 t_{01} \quad (= 0) \quad (34.6b)$$

$$\frac{1}{2} r^2 G_1^1 \equiv (r-2m) \frac{\partial \phi}{\partial r} - \frac{m}{r} = 4\pi r^2 t_1^1 \quad (= 4\pi r^2 p) \quad (34.6c)$$

These field equations imply

$$r^2 G_0^0: \quad \frac{dm}{dr} = 4\pi r^2 \rho \quad (34.7a)$$

$$r^2 G_0^1: \quad \dot{\lambda} = 0 \quad (34.7b)$$

$$\frac{1}{2} r^2 G_1^1: \quad \frac{d\phi}{dr} = \frac{m + 4\pi r^3 p}{r(r-2m)} \quad (34.7c)$$

COMMENT:

Furthermore, the application of these equations to the identity

$$(r^2 G_A^B)_{;B} - 2r r_{;A} G_a^a = 0$$

yields the non-trivial Euler Eq. (34.5)

on page 34.5 for the pressure gradient, namely

$$(34.8) \quad \frac{dp}{dr} = -(p+\rho) \frac{d\phi}{dr}.$$

which is, of course, Eq. (34.5) on p (34.4)

With the help of the $G_{,1}' = 8\pi t_{,1}'$ equation one obtains

$$(34.9) \quad \boxed{\frac{dp}{dr} = - \frac{m + 4\pi r^3 p}{r(r-2m)} (p + \rho)} \quad (15)$$

Thus one has three coupled non-linear ordinary d. equations (for m , ϕ , p)

- a) two for the gravitational degrees of freedom, $m(r)$ and $\phi(r)$, and
- b) one for the matter degree of freedom, $p(r)$

These equations govern any static spherically symmetric perfect fluid configuration.

III, HOW TO SOLVE THE EQUATIONS OF HYDROSTATIC EQUILIBRIUM. 34/7

The structure of a star in equilibrium is determined by

$$\frac{dp}{dr} = - \frac{(m + 4\pi r^3 \rho)}{r(r-2m)} (p + \rho) \quad \text{with } p(r=0) = p_c$$

Within a Newtonian framework this equation expresses a balance between a force due to a pressure gradient and the gravitational force acting on a small volume of fluid in the star, namely $\frac{dp}{dr} = - \frac{m}{r^2} \rho$

The mass enclosed in a sphere of surface area $4\pi r^2$ is

$$m(r) = \int_0^r 4\pi \rho r^2 dr \quad \text{with } m(0) = 0$$

These two equations together with an equation of state

$$\rho = \rho(p)$$

determine the equilibrium structure of the star

To find the structure of a star integrate from the center $r=0$ (where we must have $m(0)=0$ so that the pressure gradient $\frac{dp}{dr}$ and hence p stays finite at $r=0$)

outward until we get to that radius, call it $r=R$, where the pressure vanished:

b.c. for R is $p(R)=0$
At $r=R$

surface of the star

$m(R)=M$

"Total mass" of the star.

IV. EXTERNAL GRAVITATIONAL FIELD OF A STAR.

A. Outside the star, where $p=0, \rho=0$ we have

1. $m(r) = m(R) = M$ constant outside

thus

$g_{rr} = \frac{1}{1 - \frac{2M}{r}}$

$r > M$ outside

2. $p = 0$ outside (USE Eq. (34.7c) on p 34.5.

$\therefore \frac{d\Phi}{dr} = \frac{M}{r^2(1 - \frac{2M}{r})} = \frac{1}{2} \frac{d}{dr} \ln(1 - \frac{2M}{r})$

$2\Phi = \text{Erc}\left(1 - \frac{2M}{r}\right) \cdot C$

subject to $\Phi(r \rightarrow \infty) = 0$

(i) $-g_{tt} = e^{2\Phi} = C \left(1 - \frac{2M}{r}\right)$

where we have to choose that integration

constant $C=1$, which assumes us that the boundary condition

$$\boxed{\Phi(r=\infty) = 0}$$

is satisfied.

22. Newtonian correspondence limit compels us to call $M (= M_{\text{conv}} \frac{G}{c^2})$ the mass of the star - the mass which determines the planetary orbits.

B. Metric outside any spherical star.

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

to kinetic energy):

$$\delta(\text{total energy}) = \delta(G.E.) + \delta(C.E.) = \left(\frac{k}{3}V^{-4/3} - P\right)\delta V$$

4. Equilibrium $\Rightarrow \delta(\text{Tot. E.}) = 0 \Rightarrow P = \frac{k}{3}V^{-4/3}$

B. Stability of these configurations.

$$\delta^2(\text{total energy}) = \left(\frac{k}{3}V^{-4/3}(-) \frac{4}{3}V^{-1} - \frac{dP}{dV}\right)(\delta V)^2$$

$$-\frac{dP}{dV} \approx \frac{P}{V} \quad \text{where } P = \frac{\text{const.}}{V}$$

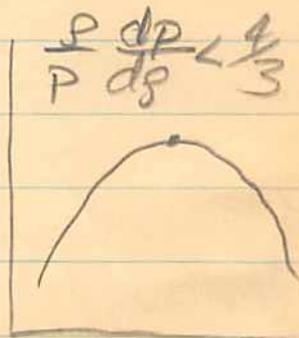
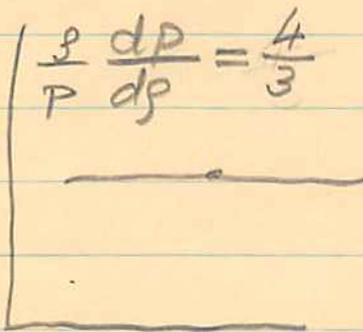
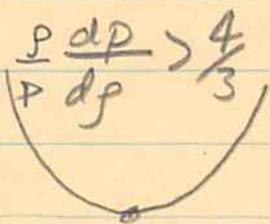
$$= \left(-\frac{4}{3} \frac{P}{V} + \frac{dP}{dV} \frac{P}{V}\right)(\delta V)^2$$

$$= \frac{P}{V} \left(-\frac{4}{3} + \frac{dP}{dV} \frac{P}{P}\right) \delta V^2$$

> 0 stable

< 0 unstable

↑
Total
energy
at a moment
of time
symmetry



← Volume →

stable

marginally
stable

unstable

V. EQUILIBRIUM; STABLE OR UNSTABLE - 1 -

34.10

T Stability of Newtonian Configuration of a Fluid ball, see also Ch. 24 in MTW

A. Equilibrium Configurations,

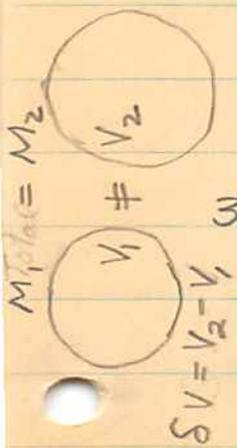
1. Consider now ^{how} the gravitational energy changes as we focus on ^{two} configurations with different total volumes but with the same total particle number (in Newtonian gravity this implies that the total mass of each configuration is constant):

$$2. \quad \delta(G, E) = \frac{1}{3} k V^{-4/3} \delta V$$

The corresponding change in the compressional energy

$$\text{comp. energy} = \int \left(\frac{\text{compress. energy}}{\text{mass}} \right) \left(\frac{\text{mass}}{\text{volume}} \right) d(\text{volume}) = \int \epsilon \rho dV$$

is
$$\delta(\text{total compress energy}) = -P_{\text{mean}} \delta V$$



3. The change in the total energy (we focus attention only on those configurations that are at their moment of time symmetry, e.g. maximum expansion or compression; hence there are no contribution due

VI Equilibrium & Stability of Superdense Stars,

A. Eq'n of structure

$$\frac{dp}{dr} = - \frac{(p+\rho)(m+4\pi r^3 p)}{r(r-2m)}$$

$$p(r=R) = 0$$

$$m = \int_0^r 4\pi \rho r^2 dr$$

$$m(r=0) = 0$$

$$p(n, z_1, z_2, \dots, z_n)$$

$$\rho(n, z_1, z_2, \dots, z_n)$$

$$\frac{d\Phi}{dr} = \frac{m+4\pi r^3 p}{r(r-2m)}$$

$$2\Phi(r=R) = \ln\left(1 - \frac{2M}{R}\right)$$

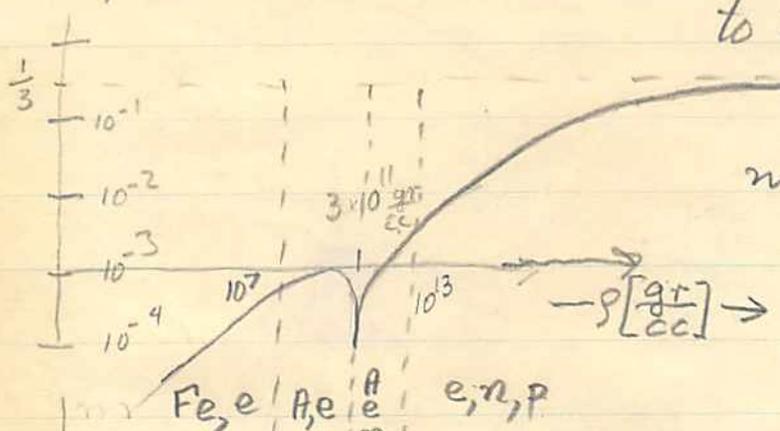
B. Eq'n of state

$$p = p(\rho)$$

cold matter catalyzed to the end point of all thermonuclear reactions.

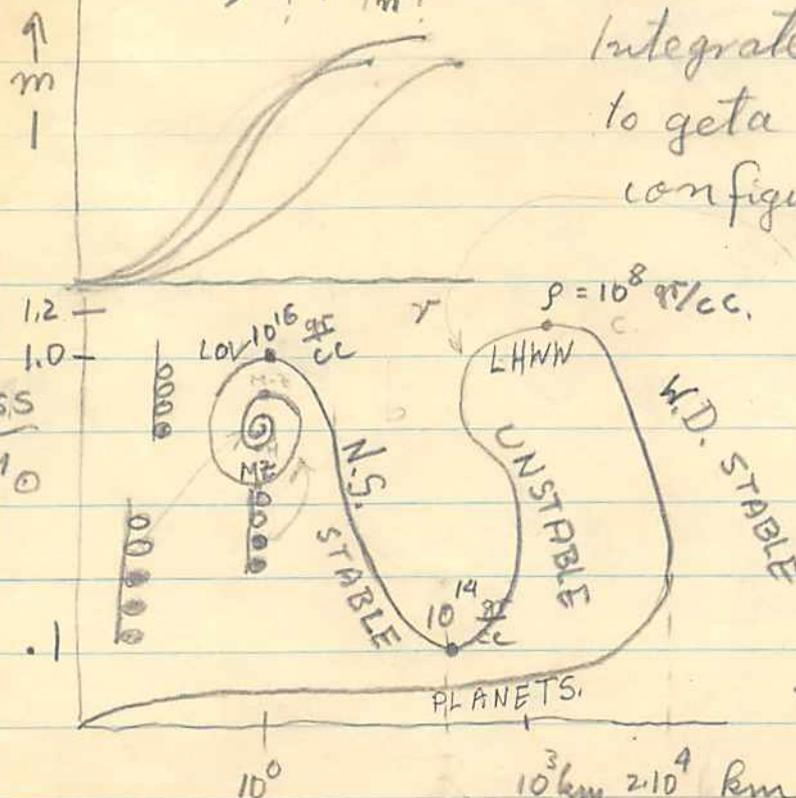
$$\mu = \frac{E}{n} + \frac{P}{n} = \frac{dp}{dn}$$

$\mu = \frac{dE}{dn} = \frac{\text{energy}}{\text{particle volume}} \frac{dp}{d\rho}$



A = nucleus having A baryons.

C.



Integrate the eq'n.s. to get a catalogue of configuration.

rest mass + internal energy + gravitational energy

Mass M_0

$$SM = M_0 \left(1 - \frac{2M}{R}\right)^{1/2} SA$$

Radius

10^0 10^3 km $2 \cdot 10^4$ km