

LECTURE 35

Ultimate purpose:

Integrate wave dynamics with particle dynamics.

(K-G, Schrödinger)

1. Wave equation and its solutions in the shortwave-high frequency asymptotic approximation

Read MTW §25.5

MTW Exercise 35.15

MTW Box 25.3

A side comment about + and - signs

a) With the metric having signature $-+++$,

the Klein-Gordon wave operator is

$$\left(\square^2 - \frac{m^2}{\hbar^2} \right) \psi = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{m^2}{\hbar^2} \psi = 0$$

This is the sign convention used by MTW,

and for good reasons I think it is a good

one because $\square^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2$ is closest to

reality in the hierarchy of concept.

b) By contrast, a $+---$ would be associated

with $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m^2}{\hbar^2} \right) \psi = 0$

GO TO P35, 2

I, Wave Equation in the Geometrical Optics Approximation

In describing the manner in which geometry affects the motion of particles, our starting point is the quantum principle which demands that their state of motion be described by a wave-function ψ which for a single particle is governed by the K-G equation

$$(1) \quad \partial_t^2 \psi - \frac{\partial^2 \psi}{c^2} + V^2 \psi - \frac{m^2}{c^2} \psi = q^2 \psi_{exp} - \frac{m^2}{c^2} \psi$$

for mechanics, and by letting

$$\psi = e^{i\phi} \quad \text{and focusing on } \frac{m}{c} |\phi| \gg \left| \frac{\partial \phi}{\partial t} \right|$$

$$(1*) \quad -\frac{\partial^2}{c^2} r^2 \phi = i \hbar \frac{\partial \phi}{\partial t};$$

for non-relativistic mechanics. These equations govern the manner in which the wave function evolves in spacetime, which in general is curved. This wave function can be reexpressed in terms of the phase function S ("eikonal," "Schrödinger phase," "dynamical phase")

and amplitude \mathcal{A} :

$$\psi(x^\alpha) = \mathcal{A}(x^\alpha) e^{iS(x^\alpha)/\hbar} \quad (35.1)$$

"slowly varying" "rapidly varying"

Introducing such a function into the wave equation

$$g^{\alpha\beta} \psi_{,\alpha;\beta} - \frac{m^2}{\hbar^2} \psi = 0, \quad (35.2)$$

one finds

$$0 \underbrace{\{ g^{\alpha\beta} \left[f_{,\alpha;\beta} + \frac{2i}{\hbar} \frac{\partial S}{\partial x^\alpha} \frac{\partial \mathcal{A}}{\partial x^\beta} + \frac{i}{\hbar} S_{,\alpha;\beta} \mathcal{A} - \frac{1}{\hbar^2} \left[\frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} \mathcal{A} \right] - \frac{m^2}{\hbar^2} \mathcal{A} \} e^{iS/\hbar}}_{(1)} = 0 \quad (35.3)$$

The wave equation (35.2) governs the
(relativistic and wave-mechanical)
dynamics of a particle of mass m
in the same way that Newton's

$$m \frac{d^2 \vec{x}}{dt^2} = \vec{F} \quad (35.4)$$

governs the dynamics of a non-relativistic

classical particle of mass m :

Even though the mathematizations

the dynamics of a particle, Eqs(35.2) & (35.4),

are in terms a ^(single) p.d.e. and a system

of o.d.e's, both laws are parametrized
by the mass m of a particular
particle.

In fact, we shall show mathematically that (35.4) is the asymptotic
shortwave-highfrequency limit.

To this end one observes that the
wave Eq.(35.2) applies to all masses m .

We mathematize this observation by

introducing the dimension less variable ϵ into Eq.(2) and any of its solutions by letting $m = \frac{m_0}{\epsilon}$:

$$(35.5) \quad \boxed{g^{\alpha\beta} \psi_{,\alpha;\beta} - \frac{1}{\epsilon^2} \frac{m_0^2}{\hbar^2} \psi = 0} \quad (5)$$

so that

$$\epsilon^2 g^{\alpha\beta} \psi_{,\alpha;\beta} - \frac{m_0^2}{\hbar^2} \psi = 0 \quad \text{and hence}$$

$$(35.6) \quad \psi = (f_0 + f_1 \epsilon + f_2 \epsilon^2 + \dots + f_N \epsilon^N + \dots) e^{i \frac{LS}{\epsilon \hbar}}$$

We shall show that in the asymptotic limit, $\epsilon \rightarrow 0$, which corresponds to the limit of very heavy particles

$$\frac{m_0}{\epsilon} \rightarrow \infty, \quad (n)$$

(with $0 < m_0$)

the Eq.(35.2) solutions ψ approach

the Eq.(35.4) solutions \vec{x} :

$$\psi(x^\alpha) \rightarrow \left\{ x^\alpha(\tau) \right\}_{\alpha=0}^3$$

35.5.

In other words, the asymptotic limit of wave dynamics consists of the mechanics of particles executing their space-time trajectories.

The Hamilton-Jacobi theory is the bridge which connects wave mechanics with classical particle mechanics. This theory is based on applying Eq.(35.6) to Eq.(35.5), collecting equal powers of ϵ , and setting their coefficient to zero:

$$\frac{1}{\epsilon^2} : \frac{\partial \phi_0}{\hbar^2} \left\{ g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} + m_0^2 \right\} = 0$$

$$\frac{1}{\epsilon} : \frac{\partial \phi_1}{\hbar^2} \left\{ g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} + m_0^2 \right\} + \frac{i}{\hbar} \cdot$$
$$+ \frac{i}{\hbar} \left\{ \left(g^{\alpha\beta} \frac{\partial S}{\partial x^\beta} \right)_{;\alpha} \phi_0 + 2g^{\alpha\beta} \frac{\partial S}{\partial x^\beta} \phi_{0;\alpha} \right\} = 0$$

(Terms of higher order yield "post geometrical optics" corrections)

One therefore obtains (after dropping the subscript "zero" from m_0):

$$(35.7a) \quad g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} + m^2 = 0 \quad (\text{H-J eq'n})$$

$$(35.7b) \quad \left(\frac{\partial^2}{\partial t^2} \rho \frac{\partial S}{\partial x^\beta} \right)_\alpha = 0 \quad \begin{array}{l} \text{(Particle} \\ \text{conservation)} \end{array}$$

"density" "4-velocity"

The application of this asymptotic expansion method to the (non-relativistic) Schrödinger equation

yields

$$(35.8a) \quad \frac{i}{2m} \vec{\nabla} S \cdot \vec{\nabla} S + U(x) + \frac{\partial S}{\partial t} = 0$$

$$(35.8b) \quad \frac{\partial(\rho^2)}{\partial t} + \frac{i}{m} \vec{\nabla} \cdot (\rho^2 \vec{\nabla} S) = 0$$

Thus, in the short wavelength/high frequency/
1 geometrical optics approximation
the solution to the wave eq'n,

Eq. (*) or (**) on page 35, 1 is

$$\psi = f_{L_0} e^{i S/\hbar}$$

This is the bridge between wave
mechanics and classical particle
mechanics.

Eqs.(35.7a) and (35.8a) are the H-J equations for a relativistic and non-relativistic system respectively

They are first order partial differential equations whose solutions are the dynamical phase S of the given system.

Once it is known for a given system, the task of exhibiting its global spacetime particle trajectories in mathematical form is complete.

This is because ^{of} the application

of the principle of constructive interference to the phase function is mathematically trivial (although physically fundamental).

Eqs (35.7b) & (35.8b), both of which are 4-dimensional divergence conditions mathematize the law of conservation of particles. Since each particle carries a certain amount of monenergy, one finds that this law plays a key role in mathematizing the law of monenergy conservation in terms of the energy-momentum tensor of an aggregate of particles.

Summary:

In the "geometrical optics" limit of wave mechanics the wave length of a wave is so short that locally the phase fronts have the straight and parallel properties of a plane wave.

This is so despite the fact that these phase fronts exhibit curvature

outside any local neighborhood.

The shape and spacing of the phase fronts is expressed by the isograms of phase function $\delta(x^\alpha)$.

It satisfies the H-J equation

$$(35.9) \quad \frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x}, x, t\right) = 0 \quad (\text{non-relativistic}) \quad (9)$$

or

$$(35.10) \quad \mathcal{H}\left(\frac{\partial S}{\partial x^\alpha}, x^\alpha\right) \equiv g^{\mu\nu}(x^\alpha) \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0 \quad (\text{relativistic}) \quad (10)$$

If the particle has charge q and is moving in an electro-magnetic field has the electromagnetic vector potential $A_\mu(x^\alpha)$, then its H-J equation is

$$(35.11) \quad \boxed{\mathcal{H}\left(\frac{\partial S}{\partial x^\alpha}, x^\alpha\right) \equiv g^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} - qA_\mu \right) \left(\frac{\partial S}{\partial x^\nu} - qA_\nu \right) + m^2 = 0}$$

(relativistic in an e.m. field) (11)

Equation (35.9) is the low velocity (non-relativistic) approximation to the more general H-J Eqs. (35.10)