

LECTURE 38

H-J analysis of particle orbits
in a spherically symmetric static
spacetime configuration

MTW Box 25.4, § 25.5

H-J theory =

(particle mechanics) \cap (wave mechanics) \cap (Geometrical optics) \cap (wave optics) \cap

(classical relativistic mechanics) \cap (relativistic quantum mechanics)

I. THE SEPARATION OF VARIABLES METHOD 38.1

The most important way of solving the Hamilton-Jacobi equation is by the method of "separation of variables".

If the equation has the form such that one of the coordinate dependencies can be isolated, i.e. \mathcal{H} has the form

$$\mathcal{H}\left(x^i, \frac{\partial S}{\partial x^i}; \phi^0(x^0, \frac{\partial S}{\partial x^0})\right) = 0,$$

where ϕ^0 depends only on x^0 , and x^i refer to all the other coordinates, say x^1, x^2, x^3 , then we may assume that

$$S = S'(x^1, x^2, x^3) + X^0(x^0).$$

This is because upon substituting this assumed form into the H-J equation and solving for ϕ^0 we find that

$$\phi^0\left(x^0, \frac{dX^0}{dx^0}\right) = \text{expression in } x^1, x^2, x^3, \frac{\partial S'}{\partial x^1}, \frac{\partial S'}{\partial x^2}, \frac{\partial S'}{\partial x^3}$$

We see that the l.h.s is independent of x^1, x^2, x^3 , while the r.h.s. is independent of x^0 . Consequently their common value is a constant, say α_1 ;

$$\phi^0(x^0; \frac{dx^0}{dx^0}) = \alpha_1 = \text{expression in } x^1, x^2, x^3, \frac{\partial S^1}{\partial x^1}, \frac{\partial S^1}{\partial x^2}, \frac{\partial S^1}{\partial x^3}$$

This constant serves as a parameter in the sol'n S of the H- \mathcal{J} equation. The successful separation of the independent variable x^0 from all the other leads to two conclusions:

(i) The solution S by necessity depends on the parameter α_1 ,

$$S = X^0(x^0; \alpha_1) + S^1(x^1, x^2, x^3; \alpha_1)$$

Schrodinger equation is analogous.

and

(i') The reduced solution $S^1(x^i; \alpha_1)$,
satisfies the reduced H-J eq'n

$$\mathcal{H}\left(x^i, \frac{\partial S^1}{\partial x^i}; \alpha_1\right) = 0$$

This is a p.d.e.

whose number of independent
variables is one less in number.

The implementation of
successfully separating one
independent variable after another leads to

(38.1) (i') the multiparametrized sol'n

$$\begin{aligned} S(x^0, x^1, x^2, x^3; \alpha_1, \alpha_2, \alpha_3) &= X^0(x^0; \alpha_1) + X^1(x^1; \alpha_1, \alpha_2) + X^2(x^2; \alpha_1, \alpha_2, \alpha_3) + X^3(x^3; \alpha_1, \alpha_2, \alpha_3) \\ &= \int_{x_0}^{x^0} p_0 dx^0 + \int_{x_1}^{x^1} p_1 dx^1 + \int_{x_2}^{x^2} p_2 dx^2 + \int_{x_3}^{x^3} p_3 dx^3 + \text{const.} \\ &\equiv \sum_{\mu=0}^3 \int p_{\mu} dx^{\mu} + \beta(\alpha_1, \alpha_2, \alpha_3) \end{aligned}$$

where the integrand of each term

$$p_\mu = p_\mu(x^\mu) = \frac{\partial S}{\partial x^\mu} = \frac{dX^\mu}{dx^\mu} \quad \mu = 0, 1, 2, 3$$

depends on
on each respective coordinate only.

(ii) the conclusion that the "super

Hamiltonian $\mathcal{H}(x^\mu, \frac{\partial S}{\partial x^\nu})$ has the

functional form

$$\mathcal{H} = \mathcal{H} \left(\underbrace{\phi^3 \left(\underbrace{\phi^2 \left\{ \underbrace{\phi^1 \left[\phi^0 \left(x^0, \frac{\partial S}{\partial x^0} \right), x^1, \frac{\partial S}{\partial x^1} \right] x^2, \frac{\partial S}{\partial x^2} \right\}}_{\alpha_2}, x^3, \frac{\partial S}{\partial x^3} \right)}_{\alpha_3} \right)}_{\alpha_1}$$

If \mathcal{H} has this form then the H-J

$$\mathcal{H} \equiv g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0$$

can always be solved with the result
that its solution is given by Eq.(38.1)
on page 38.3

II. H-J EQ'N AND ITS SOL'N FOR THE SCH SCH.

GEOMETRY

38.5

Let us apply this separation procedure

to solving the H-J equation and finding the motion of a particle in the spacetime of a black hole.

The metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The corresponding H-J equation $\mathcal{H}\left(\frac{\partial S}{\partial x^\mu}, x^\nu\right) = g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0$

is

$$\frac{1}{1 - \frac{2M}{r}} \left(\frac{\partial S}{\partial t}\right)^2 + \left(1 - \frac{2M}{r}\right) \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2} \frac{1}{\sin^2\theta} \left(\frac{\partial S}{\partial \phi}\right)^2 + m^2 = 0$$

Note that

The coordinates t and ϕ are "cyclic", i.e. the metric does not depend on these coordinates. Consequently, the separation of variables method can be applied to both of them. We obtain

$$\frac{\partial S}{\partial t} = \text{const} = -E$$

$$\frac{\partial S}{\partial \phi} = \text{const} = P_\phi$$

The H-J equation reduces therefore to a p.d.e. of two variables only

$$\phi^\theta(\theta, \frac{\partial S}{\partial \theta}) = \alpha_\theta = l^2 \quad 39,6$$

$$\underbrace{-\frac{1}{1-\frac{2M}{r}} E^2 + \left(1 - \frac{2M}{r}\right) \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2} \left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{P_\phi^2}{\sin^2 \theta} \right] + m^2}_{\mathcal{H} = \phi^r(r, \frac{\partial S}{\partial r})} = 0$$

The separation process can now be applied to the θ -coordinate. It is therefore evident that the solution to the H-J equation separates into

$$S = T(t) + R(r) + \Theta(\theta) + \Phi(\phi)$$

where

$$\frac{dT}{dt} = -E$$

$$\frac{d\Phi}{d\phi} = P_\phi$$

$$\left(\frac{d\Theta}{d\theta}\right)^2 + \frac{P_\phi^2}{\sin^2 \theta} = l^2$$

$$-\frac{1}{1-\frac{2M}{r}} E^2 + \left(1 - \frac{2M}{r}\right) \left(\frac{dR}{dr}\right)^2 + \frac{l^2}{r^2} + m^2 = 0$$

The general separated solution is therefore

$$S = -Et + \int \frac{1}{1-\frac{2M}{r}} \sqrt{E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{l^2}{r^2} + m^2\right)} dr + \int \sqrt{l^2 - \frac{P_\phi^2}{\sin^2 \theta}} d\theta + P_\phi \phi + \text{const.}$$

$$= \int p_t dt + \int p_r dr + \int p_\theta d\theta + \int p_\phi d\phi + \beta(E, l^2, P_\phi)$$

$$= \int p_\mu dx^\mu + \beta(E, l^2, P_\phi)$$

III Constructive interference yields the worldlines

Constructive interference applied to the Hamilton-Jacobi ("dynamical") phase

$$S = \int p_\mu dx^\mu = \int p_t dt + \int p_r dr + \int p_\theta d\theta + \int p_\phi d\phi + \beta(E, \ell^2, p_\phi)$$

where

$$p_t = -E$$

$$p_r = \frac{1}{1 - \frac{2M}{r}} \left[E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{\ell^2}{r^2} + m^2 \right) \right]^{1/2}$$

$$p_\theta = \left[\ell^2 - \frac{p_\phi^2}{\sin^2 \theta} \right]^{1/2}$$

$$p_\phi = p_\phi$$

yields

$$(38.2) \quad 0 = \frac{\partial S}{\partial E} = -t + \int \frac{1}{1 - \frac{2M}{r}} \frac{E}{\left[E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{\ell^2}{r^2} + m^2 \right) \right]^{1/2}} dr + \frac{\partial \beta}{\partial E}$$

$$(38.3) \quad 0 = \frac{\partial S}{\partial \ell^2} = \int \frac{\frac{1}{2} d\theta}{\left[\ell^2 - \frac{p_\phi^2}{\sin^2 \theta} \right]^{1/2}} - \int \frac{\frac{1}{2} \frac{1}{r^2} dr}{\left[E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{\ell^2}{r^2} + m^2 \right) \right]^{1/2}} + \frac{\partial \beta}{\partial \ell^2}$$

$$(38.4) \quad 0 = \frac{\partial S}{\partial p_\phi} = - \int \frac{p_\phi d\theta}{\left[\ell^2 - \frac{p_\phi^2}{\sin^2 \theta} \right]^{1/2}} + \phi + \frac{\partial \beta}{\partial p_\phi}$$

For a given set of integration (= separation)

constants

$$(38.5) \quad E, L^2, P_\varphi; \frac{\partial \mathcal{H}}{\partial E}, \frac{\partial \mathcal{H}}{\partial L^2}, \frac{\partial \mathcal{H}}{\partial P_\varphi}$$

each of these three interference conditions defines a 3-dimensional submanifold in the ambient 4-d spacetime spanned by its (t, r, θ, φ) coordinate system.

The intersection of these submanifolds is a specific 1-d submanifold, the globally defined particle worldline. Each one is uniquely identified by the six parameters, Eq. (38.5)

The tangents to these worldlines are determined by

$$\frac{dx^\mu}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\mu}, \quad \mu = 0, 1, 2, 3$$

where $\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$

$$(38.6) \quad \frac{dt}{d\tau} = \frac{E}{1 - \frac{2M}{r}}$$

$$(38.7) \quad \frac{dr}{d\tau} = \left[E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{L^2}{r^2} + m^2 \right) \right]^{1/2}$$

$$(38.8) \quad \frac{d\theta}{d\tau} = \frac{1}{r^2} \left[L^2 - \frac{P_\phi^2}{\sin^2 \theta} \right]^{1/2}$$

$$(38.9) \quad \frac{d\phi}{d\tau} = \frac{1}{r^2 \sin^2 \theta} P_\phi$$

The constructive interference conditions

Eqs. (38.2) - (38.4) on page 38.7 do not

lack any geometrical and physical information about the dynamics

of free particle in the Schwarzschild

geometry represented relative

to the metric as represented by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 [d\theta^2 + \sin^2 \theta d\phi^2]$$

However, instead of giving a mathematically completed analysis based on Eqs (38.2) - (38.4), one can already draw important conclusions based on the requirement that classically (i.e., not wave mechanically) the particle satisfy

$$(38.10) \quad \left(\frac{\partial S}{\partial r}\right)^2 \geq 0, \quad \left(\frac{\partial S}{\partial \theta}\right)^2 \geq 0.$$

IV. Classically allowed vs classically forbidden regions.

(next page)

Because of inequalities

(38/10), space is divided into regions which are classically allowed and those which are classically forbidden.

There

$$\left(\frac{\partial S}{\partial r}\right)^2 < 0 \quad \left(\frac{\partial S}{\partial \theta}\right)^2 < 0.$$

The boundary between these regions is located where

$$\boxed{\left(\frac{\partial S}{\partial r}\right)^2 = 0} \quad \boxed{\left(\frac{\partial S}{\partial \theta}\right)^2 = 0}$$

The significance of this boundary we infer from the Hamilton's equations of motion. They imply that

$$\frac{\partial T}{\partial p_r} \frac{dr}{dt} = N \frac{\partial \mathcal{H}}{\partial p_r} = 2N(r) g^{rr} \frac{\partial S}{\partial r} = 2N(r) \sqrt{E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{L^2}{r^2} + m^2\right)}$$

$$\frac{d\theta}{dt} = N \frac{\partial \mathcal{H}}{\partial p_\theta} = 2N(r) g^{\theta\theta} \frac{\partial S}{\partial \theta} = 2N(r) \frac{1}{r^2} \sqrt{E^2 - \frac{p_\theta^2}{\sin^2 \theta}}$$

Thus the boundary between what classically is allowed and what is forbidden, is the locus of turning points of the radial or polar angle motion of the particle.

From this locus of turning points one can infer major qualitative aspects such as bounded vs unbounded motion, stable vs unstable motion. As an example, consider the radial motion as determined by its locus of turning points:

$$\frac{dr}{d\tau} = 0 \Rightarrow E^2 - V_{\text{eff}}^2(r) = 0$$

Upon considering equatorial motion

$\theta = \frac{\pi}{2}$ one has $L^2 = p_\phi^2$ so that

$$V_{\text{eff}}^2 = m^2 - \frac{2M}{r} m^2 + \frac{p_\phi^2}{r^2} - \frac{2M}{r} \frac{p_\phi^2}{r^2}$$

$$= m^2 \left(1 - \frac{2M}{r}\right) \left(1 + \frac{p_\phi^2}{r^2}\right)$$

Upon introducing dimensionless quantities

$$\frac{2M}{r} = \frac{1}{r} \quad , \quad \frac{p_\phi^2}{2Mm} = a$$

we obtain the following contributions to the radial potential:

$$\frac{E}{m^2} = 1 - \frac{1}{r} + \frac{a^2}{r^2} - \frac{a^2}{r^3} = \left(1 - \frac{1}{r}\right) \left(1 + \frac{a^2}{r^2}\right)$$

rest
mass

Newtonian
attraction

centrifugal
repulsion

Angular
kinetic
energy has
weight.

which expresses the locus of turning points that separates a classically allowed from a classically forbidden region.

$$\bar{r} = \frac{r}{2M} \quad M = M_{conv} \cdot \frac{G}{c^2} \quad E = \frac{E_{conv}}{c}$$

$$a = \frac{P_{\phi}}{2Mm} \quad m = m_{conv} c$$

In the equatorial plane
 the locus of radial turning points is
 determined by $\frac{dx^{\mu}}{d\tau} = N \frac{\partial \mathcal{L}}{\partial p_{\mu}}$, $N = \frac{m}{2}$, $\mu = r$;

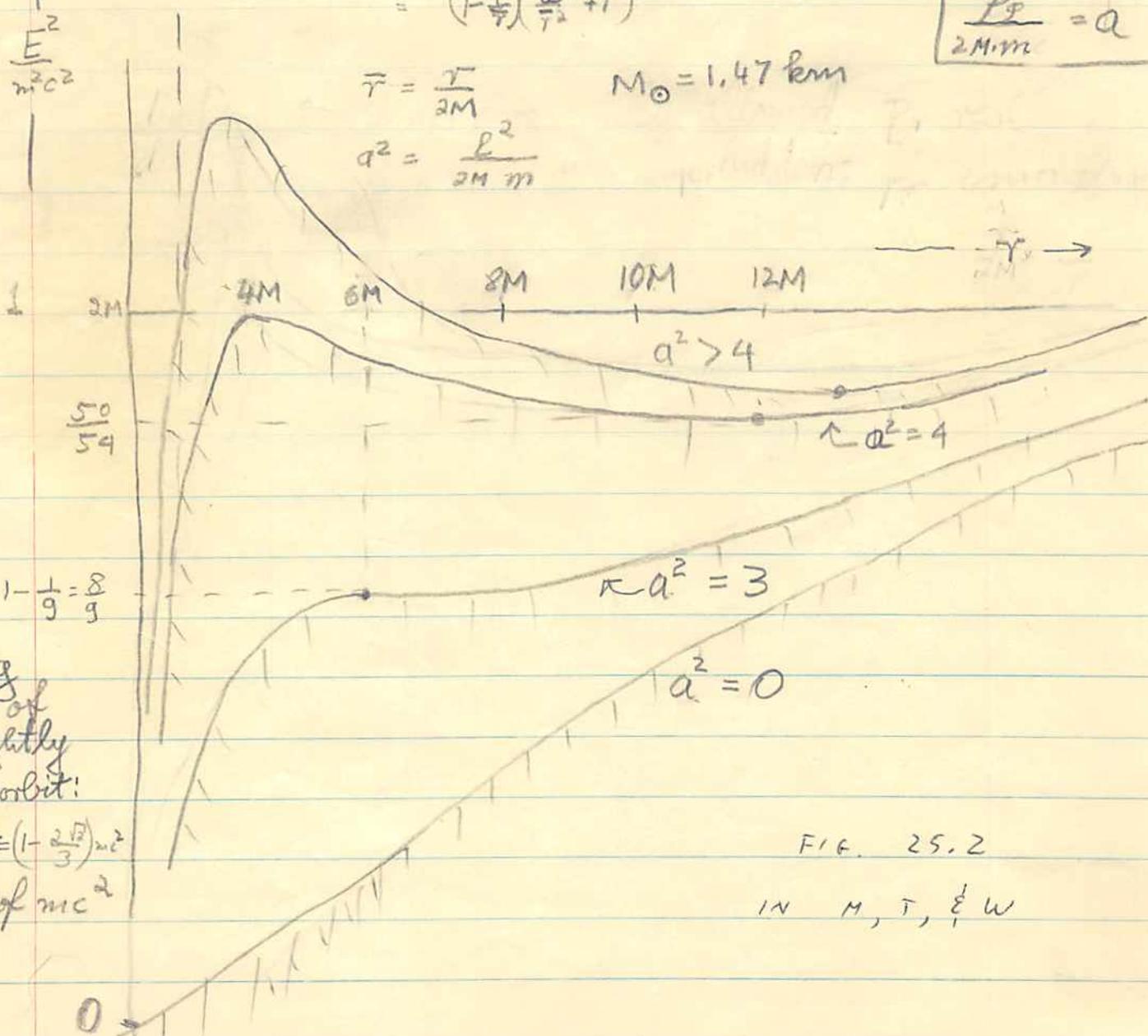
Let $\frac{2M^*}{r} = \frac{1}{r}$

$\frac{P_{\phi}}{2Mm} = a$

$$\frac{E^2}{m^2 a^2} = \left(\frac{dr}{d\tau} \frac{M}{2M} \right)^2 + \left| -\frac{1}{r} + \frac{a^2}{r^2} - \frac{a^2}{r^3} \right| \quad \text{where } \frac{2M^*}{r} = \frac{1}{r}$$

$$= \left(1 - \frac{1}{r} \right) \left(\frac{a^2}{r^2} + 1 \right)$$

$\frac{P_{\phi}}{2Mm} = a$



$\bar{r} = \frac{r}{2M}$

$a^2 = \frac{L^2}{2Mm}$

$M_{\odot} = 1.47 \text{ km}$

$1 - \frac{1}{9} = \frac{8}{9}$

binding energy of most tightly bound orbit:

$$\left(1 - \frac{1}{9} \right) mc^2 = \left(1 - \frac{2}{3} \right) mc^2$$

= 5.72% of mc^2

FIG. 25.2
 IN M, T, & W

We note that for large enough angular momentum ($p_\phi > \sqrt{3} 2M \cdot m$)

(i) there is bounded motion $E < m$

unbounded motion $E > m$

as well as motion in which the particle disappears into the black hole ($r < 2M$)

(ii) there exist stable ("Newtonian") as well as unstable ("relativistic")

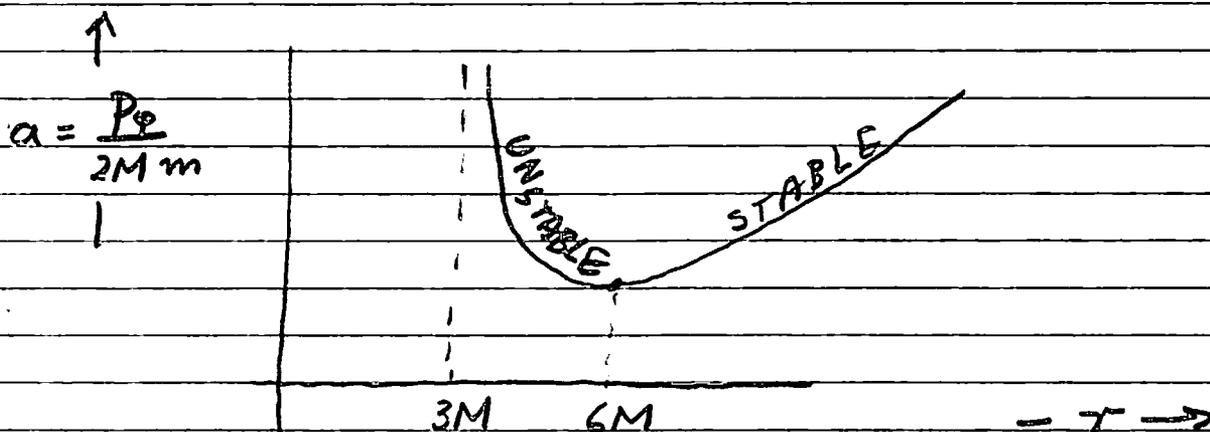
circular orbits.

They are determined by

$\frac{dE_{(r)}}{dr} = 0$, which implies

$$\frac{r}{2M} = a^2 \left(1 \pm \sqrt{1 - \frac{3}{a^2}} \right) \quad a = \frac{p_\phi}{2Mm}$$

From the catalogue of circular orbits



one can see that there exist no circular orbits, stable or unstable, for $r < 3M$.

and that the most tightly bound stable circular orbit has radius

$$r = 6M,$$