

LECTURE 39

I. Orbital motion and light deflection
in a static spherical geometry

II. The static Schwarzschild solution
to the E.F.E.

III. Black hole dynamics

For I in MTW read pages 668-669, $\left\{ \begin{array}{l} \S 31.2 (!) \\ \S 31.3 \end{array} \right.$
" do exercise 25, 16 (orbital motion)
" do exercise 25, 24 (light deflection)

For II in MTW read $\S 31.6$, Figure 31.5
" do Exercise 31.7

For III in MW read Box 31.2(!), $\S 31.5$

I)

The motion of bodies and the propagation of light in the geometrical environment of a spherical gravitating body is mathematized most efficiently within the H-J framework. It is a four-step process.

1. Solve the H-J equation

$$\mathcal{H} \equiv \frac{1}{2} \left[g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 \right] = 0$$

2. Apply the principle of constructive interference,

$$\frac{\partial S}{\partial \alpha_1} = 0, \quad \frac{\partial S}{\partial \alpha_2} = 0, \quad \frac{\partial S}{\partial \alpha_3} = 0,$$

It also implies

Hamilton's equations of motion

$$\frac{dx^\mu}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\mu}, \quad p_\mu = \frac{\partial S}{\partial x^\mu}, \quad \mu = t, r, \theta, \varphi,$$

namely

$$\frac{dr}{d\tau} = \left[E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{L^2}{r^2} + m^2 \right) \right]^{1/2} \quad (\mu = r)$$

$$\frac{d\varphi}{d\tau} = \frac{p_\varphi}{r^2 \sin^2 \theta} \quad (\mu = \varphi)$$

3. For motion confined to the equatorial

plane $\theta = \frac{\pi}{2}$ one has $p_\theta^2 = p^2 = L^2$

$$\left(\left[\text{z-component of the angular momentum} \right]^2 \right) \\ = \left[\text{total angular momentum} \right]^2$$

$$E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{p^2}{r^2} + m^2 \right) = \left(\frac{dr}{d\tau} \right)^2 = \left(\frac{dr}{d\phi} \frac{d\phi}{d\tau} \right)^2 = \left(\frac{dr}{d\phi} \right)^2 \frac{p^2}{r^4}$$

4. Upon setting $\frac{1}{r} = u$, one obtains the

differential equation for the orbit

$$\left(\frac{du}{d\phi} \right)^2 = \frac{E^2 - m^2}{p^2} + \frac{2Mm^2}{p^2} u - u^2 + 2Mu^3$$

This equation can be solved exactly in terms of elliptic integrals.

However, the properties of its solution are more easily found from the corresponding 2nd order equation

$$\frac{d^2 u}{d\phi^2} + u = \begin{cases} \frac{Mm^2}{p} + 3Mu^2 & \text{governs planetary motion} \\ 0 + 3Mu^2 & \text{governs the angular deflection of light} \end{cases}$$

a) The Newtonian planetary motion applies whenever

$$3Mu^2 \ll u$$

i.e. $\frac{3M}{r} \ll 1 \quad \forall \phi$

Under these conditions $3Mu^2$ is negligible and one has the Newtonian orbit equation

$$(39.1) \quad \boxed{\frac{d^2 u}{d\phi^2} + u = \frac{Mm^2}{P}} \quad \left(\begin{array}{l} \text{Newtonian} \\ \text{orbital motion} \end{array} \right)$$

Its solutions are the elliptical orbits.

b) The precession of the perihelion and the angular deflection of light solutions are obtained as perturbative corrections to the sol[']ns of Eq. (39.1) and of

$$\boxed{\frac{d^2 u}{d\phi^2} + u = 0} \quad \left(\begin{array}{l} \text{no angular} \\ \text{deflections} \end{array} \right)$$

II.) In 1923 Birkhoff showed that

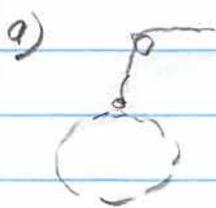
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 [d\theta^2 + \sin^2\theta d\phi^2],$$

which was first exhibited by Schwarzschild, is the only solution to the E. F. E. in vacuum which is spherically symmetric.

The singularity at $r = 2M$ is not physical.

It is a mathematical artifact due to the choice of what now is known as the Schwarzschild coordinate system.

Indeed one has the fact that



that the proper distance from r to $2M$ is

$$\int_{r'=r}^{2M} \frac{dr'}{\sqrt{1 - \frac{2M}{r'}}} = \text{finite}$$

b) In light of Eq. (38.7) a radially falling particle (or observer) which starts at $r > 2M$ reaches $r = 2M$ in a finite amount of proper ("wristwatch") time

$$\tau = \int_r^{2M} \frac{dr}{\left[E^2 - \left(1 - \frac{2M}{r}\right) m^2 \right]^{1/2}} = \text{finite},$$

even though the elapsed time as measured by the clock of distant observer is

$$t = \int_r^{2M} \frac{E dr}{1 - \frac{2M}{r}} = \infty.$$

e) The physical (o. n.) curvature (=tidal force) components

$$\hat{R}_{\alpha\beta\gamma\delta} = \left\{ \begin{array}{l} \pm \frac{2M}{r^3} \\ \pm \frac{M}{r^3} \end{array} \right\} = \text{finite (relative to Schsch. frame)}$$

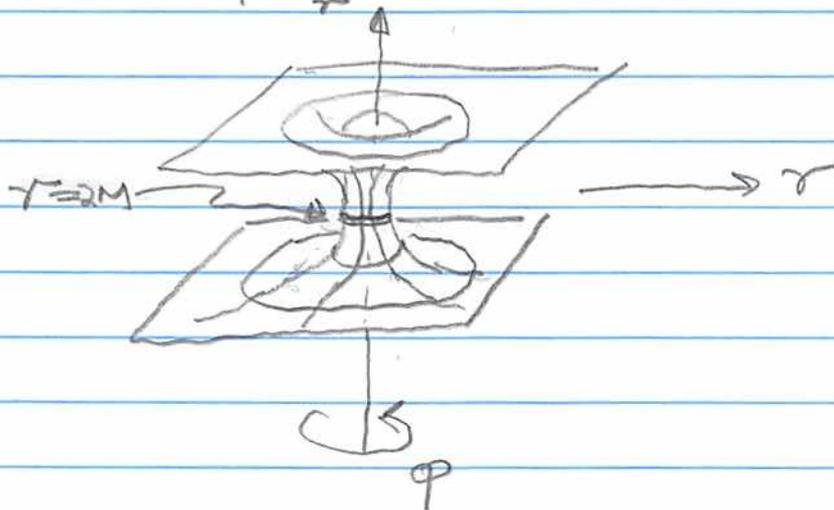
$$\hat{R}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = \left\{ \begin{array}{l} \pm \frac{2M}{r^3} \\ \pm \frac{M}{r^3} \end{array} \right\} = \text{finite (relative to any frame with arbitrary radial velocity relative to the Schsch. frame)}$$

d) the imbedding diagram of the equatorial plane $\theta = \frac{\pi}{2}$ at $t = \text{const}$ in a fictitious imbedding space

$$\begin{aligned} d\bar{s}^2 &= \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\varphi^2 \\ &= dz^2 + dr^2 + r^2 d\varphi^2 \end{aligned}$$

yields

$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1 - \frac{2M}{r}} \rightarrow z = \pm \sqrt{2M(r - 2M)}$$



e) changing to a "conformal radial coordinate"

$$r \rightarrow \rho$$

as determined by the condition that

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\phi^2 = f(\rho)^2 [d\rho^2 + \rho^2 d\phi^2] \quad \forall \phi$$

so that by equating coefficients one obtains

$$r = \rho f(\rho)$$

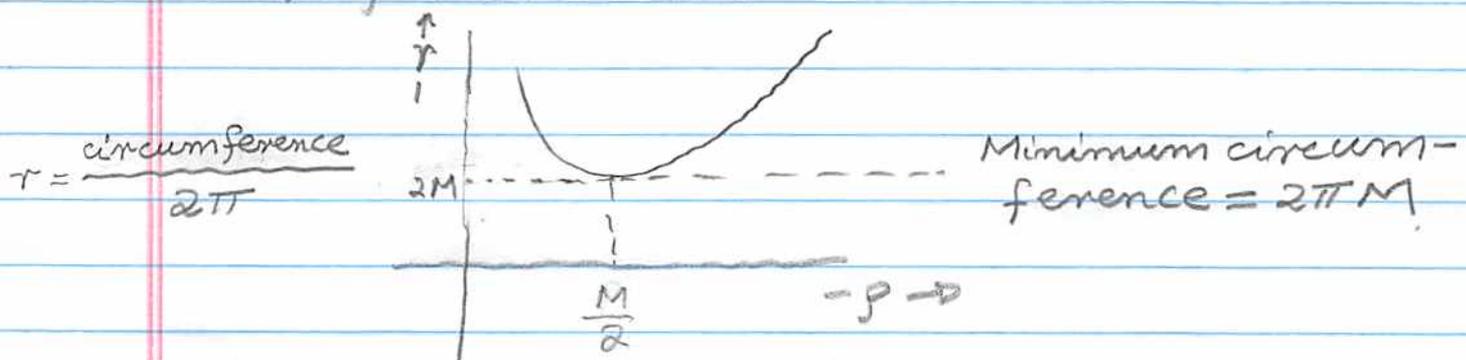
$$\sqrt{\frac{1}{1 - \frac{2M}{r}}} = f(\rho) \frac{d\rho}{dr}$$

and

$$\frac{dr}{d\rho} = \sqrt{1 - \frac{2M}{r}} \frac{r}{\rho}$$

$$r = \rho \left(1 + \frac{M}{2\rho}\right)^2; \quad f(\rho) = \left(1 + \frac{M}{2\rho}\right)^2$$

(i) The graph of the conformal radial coordinate function is



(ii) Relative to this conformal radial coordinate function the Schwarzschild metric is

$$ds^2 = \frac{\left(1 - \frac{M}{2\rho}\right)^2}{\left(1 + \frac{M}{2\rho}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 \left[d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

(iii) This geometry is symmetric around

$\rho = \frac{M}{2}$ under the isometric transformation

$$\frac{2\rho}{M} \rightarrow \frac{M}{2\bar{\rho}} \quad \left(\rho = \left(\frac{M}{2}\right)^2 \frac{1}{\bar{\rho}} \right)$$

This transformation maps the entire manifold $\rho > 0$ onto itself with the same metric

$$ds^2 = \frac{\left(1 - \frac{M}{2\bar{\rho}}\right)^2}{\left(1 + \frac{M}{2\bar{\rho}}\right)^2} dt^2 + \left(1 + \frac{M}{2\bar{\rho}}\right)^4 \left[d\bar{\rho}^2 + \bar{\rho}^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Conclusion:

The global static Schwarzschild solution to the EFE consists of two asymptotically flat isometric 3-d spaces, depicted on page 39.5.