

## LECTURE 40

The ingoing and outgoing  
Eddington-Finkelstein coordinate charts  
for the Schwarzschild geometry.

Read in MTW Box 31, 2

# I. Dynamics of the Schwarzschild geometry.

Birkoff's coordinate representation of Schwarzschild's solution to the E, F, E', s,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

is a metric which is explicitly time-invariant. This is one of its primary virtues because it leads one to the corresponding globally defined spatial geometry and topology depicted on page 39.6.

However, this time invariant representation is deficient in that it hides the causal structure as mathematized.

by the Schwarzschild coordinates.

40.2

This is because, relative to these coordinates, the time component of the metric is singular at  $r=2M$ .

Indeed, consider the causal structure of radial light cones spanned by  $(t, r)$  at  $\theta = \text{const}, \varphi = \text{const}$  on  $M^2 = M^4/S^2$ :

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}}.$$

The light cones are generated by tangents to the photon world lines (= "null rays")

$$(40.1) \quad (\Delta s)^2 = 0 : \quad \frac{dr}{dt} = \left(1 - \frac{2M}{r}\right) \text{ outgoing}$$

$$(40.2) \quad \frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right) \text{ ingoing}$$

## A) INGOING EDDINGTON-FINKELSTEIN 40.3 COORDINATES

The problem with Schesch coordinates is that the time coordinate  $t$  is a bad coordinate. This is because there are many ingoing geodesics, all of them governed by

$$(40.3) \quad \frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right).$$

They all converge to  $t = \infty, r = 2M$ ,

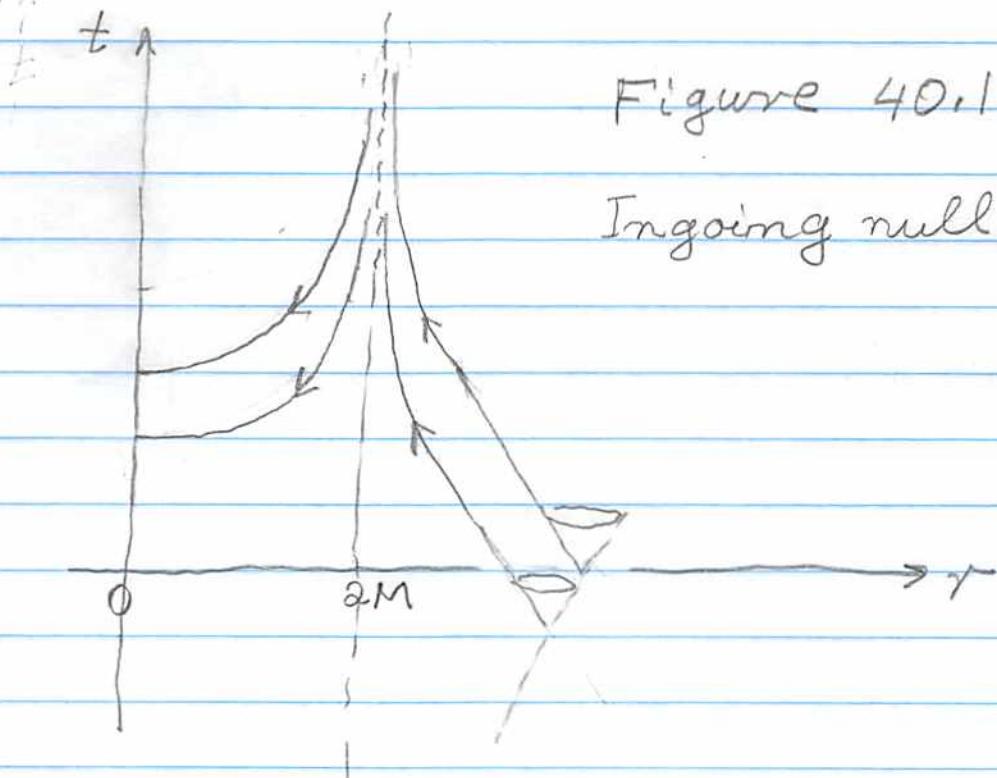


Figure 40.1

Ingoing null geodesics

and the diff'l Eq. (40.3) does not determine which ingoing null geodesic outside, i.e.

40, 4

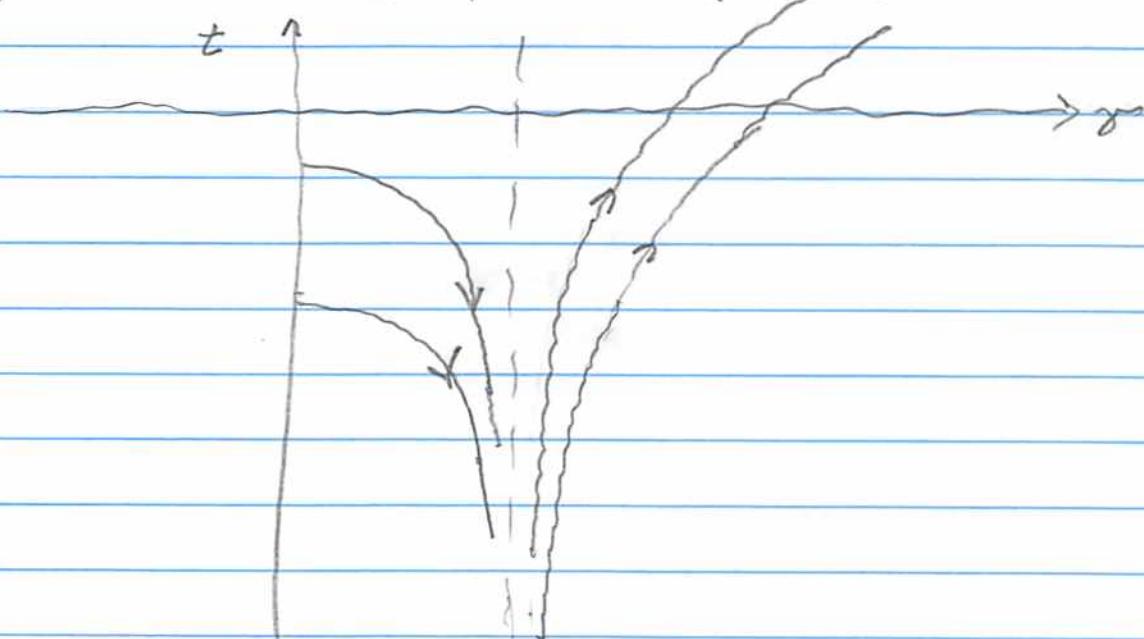
$2M < r$ , goes with which ingoing null geodesic inside, i.e.  $r < 2M$ .

Furthermore, is  $(t = \infty, r = 2M)$  a single event or is a set of distinct events?

The same ambiguity holds for outgoing radial null geodesics which are mathematized by the d.e.

$$\frac{dr}{dt} = \left(1 - \frac{2M}{r}\right).$$

They all diverge from  $(t = -\infty, r = 2M)$



These ambiguities have been resolved by Eddington (1924) and Finkelstein (1958)

They integrated Eq. 40.1) and introduced the integration constant as a new coordinate function that replaces the "bad" Schwarzschild time coordinate  $t$ . The integrals of

$$\theta = dt + \frac{dr}{1 - \frac{2M}{r}} \equiv dt + dr^*$$

are

$$\tilde{v} = t + r + 2M \ln\left(\frac{r}{2M} - 1\right) = \tilde{v}$$

The isograms of

$$\tilde{V}(t, r) = t + r + 2M \ln\left(\frac{r}{2M} - 1\right) = \text{const.}$$

$$= t + r^* = \text{const}$$

are the ingoing null geodesics:

$\tilde{V}$  is their new coordinate function. It called the advanced.

Sch sch time coordinate,

Similarly, the isograms of

$$\tilde{U}(t, r) = t - r^* = \text{const}$$

are the outgoing null geodesics,

and  $\tilde{U}$  is called the retarded

Sch sch time coordinate.

The radial variable

$$r^* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

is called the "tortoise coordinate".

Introduce  $(\tilde{V}, r)$  as the new ingoing Eddington-Finkelstein coordinates for the Schwarzschild geometry: 40.7

$$(40.4) \quad \left. \begin{aligned} d\tilde{V} &= dt + \frac{dr}{1-\frac{2M}{r}} \\ dr &= dr \end{aligned} \right\} \quad dt = d\tilde{V} - \frac{dr}{1-\frac{2M}{r}}$$

Relative to these new coordinates

the Schwarzschild metric becomes non-diagonal,

$$\begin{aligned} ds^2 &= -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2 \\ &= -(1-\frac{2M}{r}) \left( dt + \frac{dr}{1-\frac{2M}{r}} \right) \left( dt - \frac{dr}{1-\frac{2M}{r}} \right) + " \\ &= -(1-\frac{2M}{r}) d\tilde{V} \left( d\tilde{V} - \frac{2dr}{1-\frac{2M}{r}} \right) + " \end{aligned}$$

$$(40.5) \boxed{ds^2 = -(\frac{2M}{r}) d\tilde{V}^2 + 2d\tilde{V}dr + r^2 d\Omega^2},$$

but it is nonsingular at  $r=2M$ .

Thus ingoing radial null geodesics are characterized by

$$(40.6) \quad \frac{dt}{dr} + \frac{1}{1-\frac{2M}{r}} = \boxed{\frac{d\tilde{V}}{dr} = 0}$$

while outgoing radial null geodesics are characterized by

$$0 = \frac{dt}{dr} - \frac{1}{1-\frac{2M}{r}} = \frac{d\tilde{V}}{dr} - \frac{2}{1-\frac{2M}{r}} -$$

(40.7) or 
$$\boxed{\frac{d\tilde{V}}{dr} = \frac{2}{1-\frac{2M}{r}}}$$

The two boxed Eqs.(40.6)-(40.7) are the tangents to the two sets of null

geodesics

↑  
time ( $\tilde{V} + r$ )

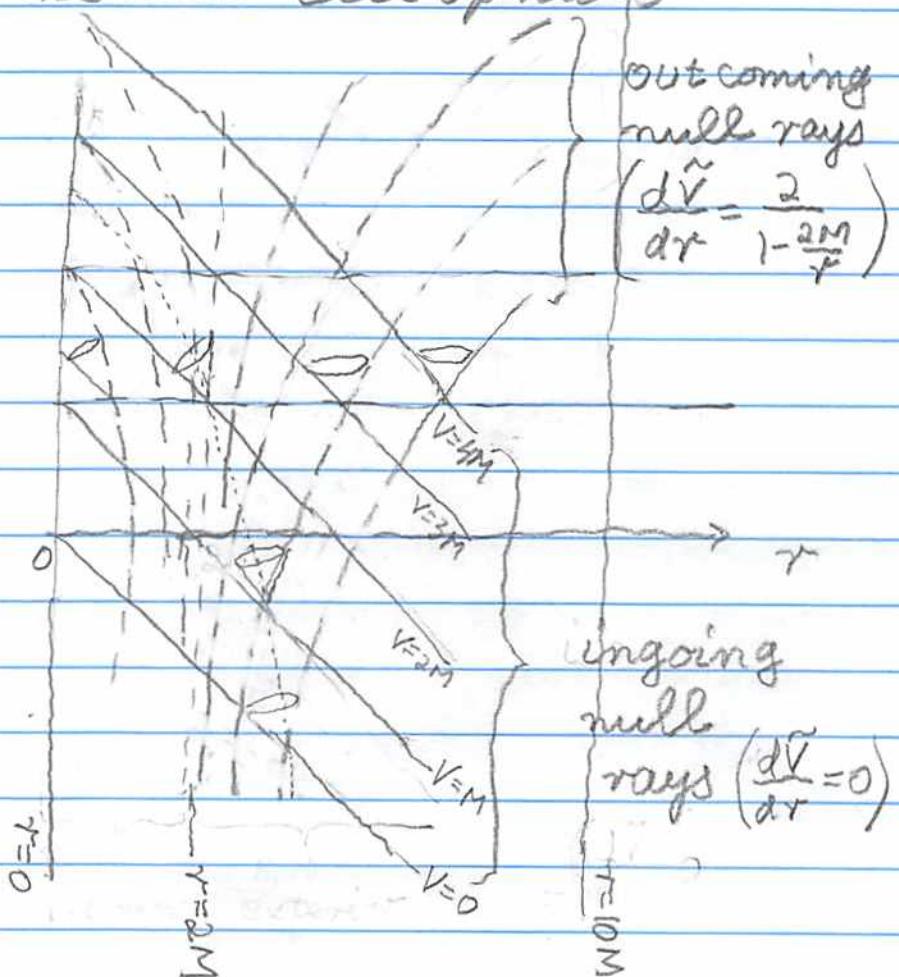


Figure 40.2

Ingoing Eddington-Finkelstein coordinate system ( $\tilde{V}, r$ ).

## B) OUT GOING EDDINGTON-FINKELESTEIN

COORDINATES

These coordinates

are based on outgoing null geodesics

$$\frac{dr}{dt} = 1 - \frac{2M}{r},$$

the integrals of

$$0 = dt - \frac{dr}{1 - \frac{2M}{r}}$$

$\underbrace{dr}_{dr^*}$

are

$$\tilde{\sigma} = t - \overbrace{(r + 2M \ln(\frac{r}{2M} - 1))}^{r^*}$$

The isograms of

$$\tilde{\sigma}(t, r) = t - (r + 2M \ln(\frac{r}{2M} - 1))$$

are the outgoing null geodesics,and  $\tilde{\sigma}$  is called the retarded Schesch,

time coordinate

Relative to the  $(\tilde{U}, \tilde{r})$  coordinate system  
the Schwarzschild metric has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{U}^2 - 2d\tilde{U}d\tilde{r} + r^2 d\tilde{s}^2$$

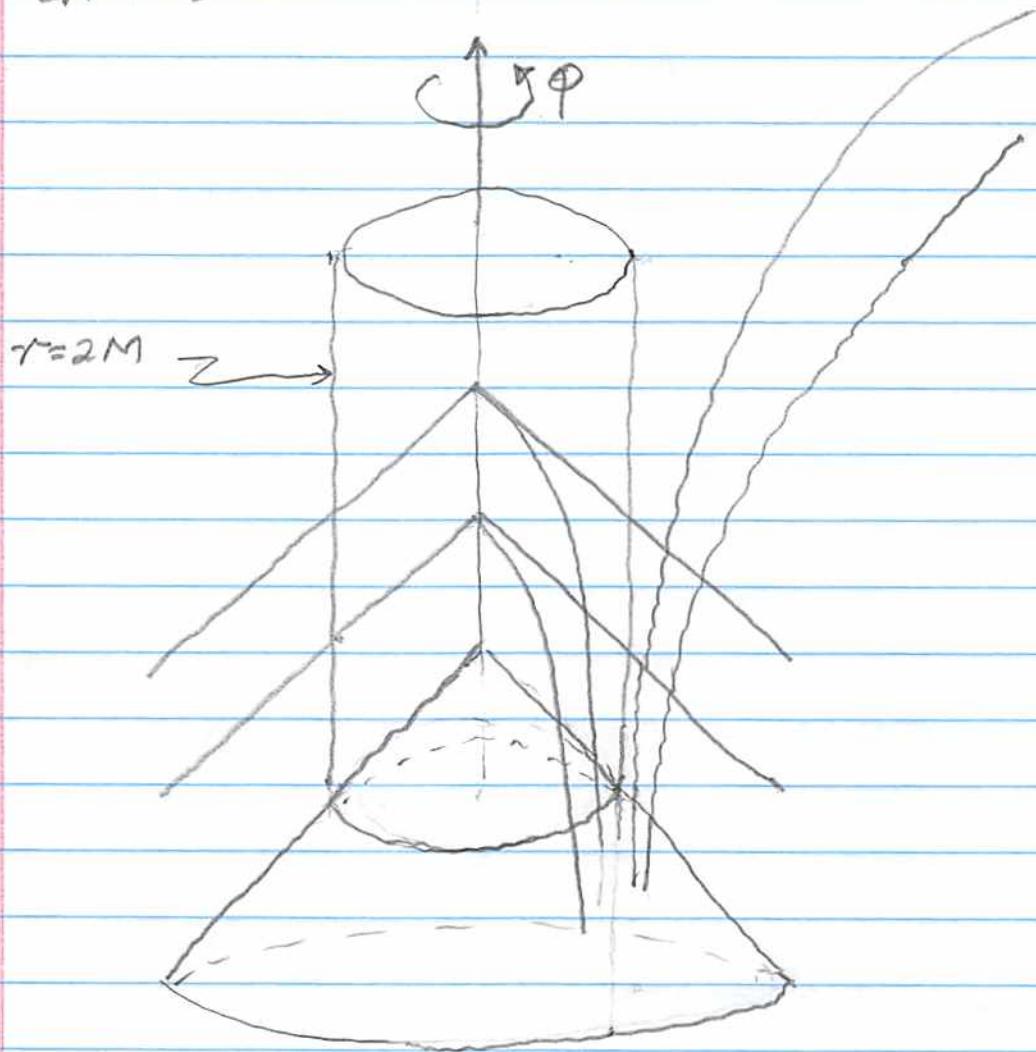
The outgoing null geodesics obey

$$\frac{d\tilde{U}}{d\tilde{r}} = 0$$

while the ingoing null geodesics obey

$$\frac{d\tilde{U}}{d\tilde{r}} = \frac{-2}{1 - \frac{2M}{r}}$$

OUTGOING Eddington-Finkelstein  
coordinates  $(U, r)$  40.11



OUTGOING Eddington-Finkelstein coordinates

$$(\tilde{V}, r)$$