

LECTURE 41: APPENDIX

Globally defined Kruskal-Szekeres

coordinates via the Rindler coordinate
atlas:

The mathematical nexus between the
Rindler spacetime of an accelerated
observer and the Kruskal spacetime
of the Schwarzschild geometry.

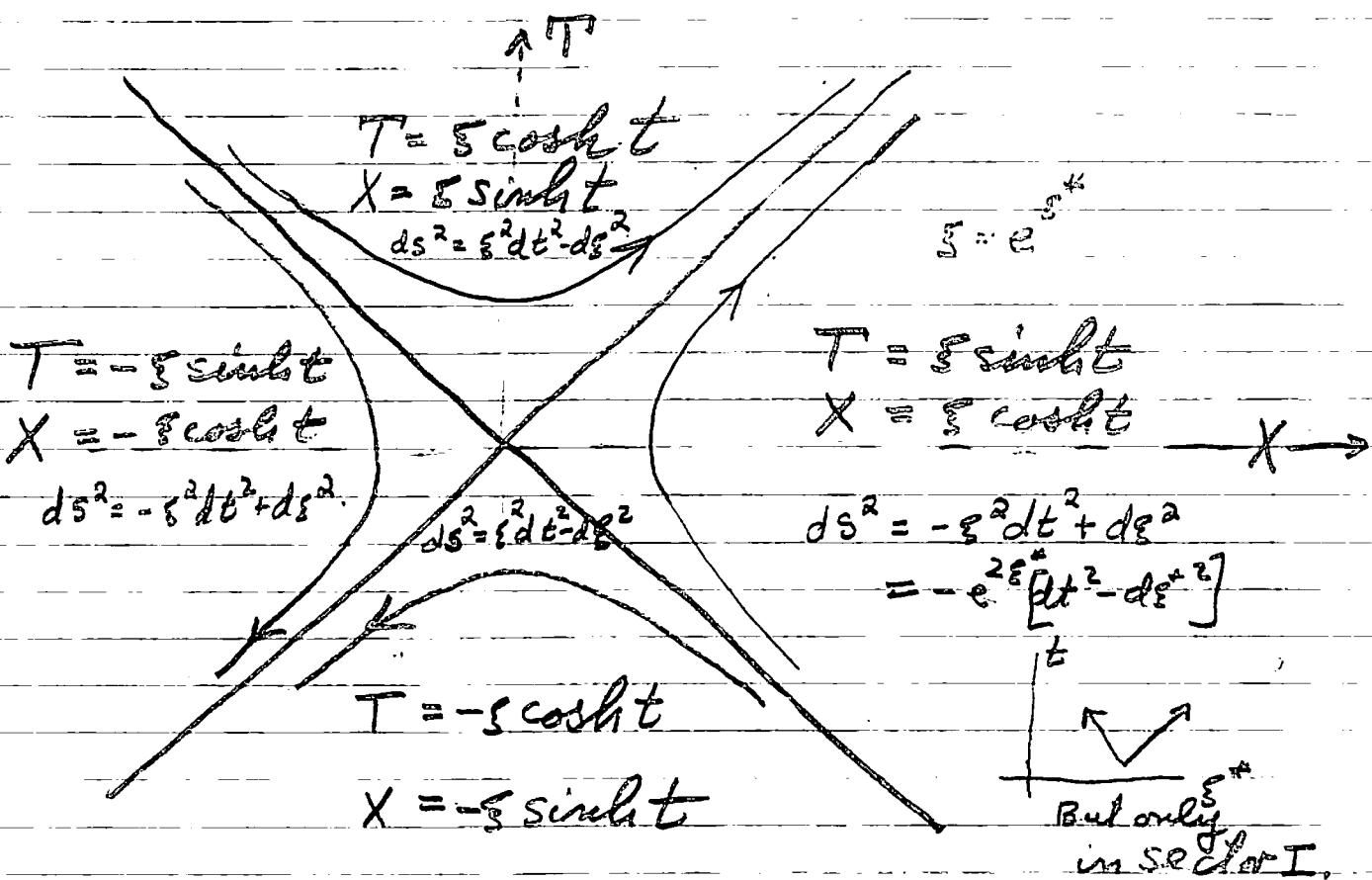
we know that a Rindler coordinate (t, ξ) frame

$$\begin{aligned} T' &= \xi \sinh t & 0 < \xi < \infty \\ X &= \xi \cosh t & -\infty < t < \infty \end{aligned}$$

covers only the limited spacetime sector $T' < X$ of Minkowski space time, whose metric is

$$\begin{aligned} ds^2 &= -dT'^2 + dX^2 + dy^2 + dz^2 \\ &= -\xi^2 dt^2 + d\xi^2 + dy^2 + dz^2 = \end{aligned}$$

The remaining three sectors are covered by related coordinate transformations



with $0 < \xi < \infty$, $-\infty < t < \infty$ in each coordinate sector.

Given that the Schwarzschild coordinates (t, r) for $r > 2M$ play the same role that the Rindler coordinates (t, s) play for Minkowski spacetime, the natural question that arises is this: Does there exist a corresponding coordinate transformation, say

$$\begin{aligned} T &= f(r) \sinh \alpha t & (\alpha \text{ is a } \left. \begin{array}{l} \text{const} \\ \text{metric} \end{array} \right\} \text{S.t.,} \\ R &= f(r) \cosh \alpha t, & \text{is nonsing'l'r} \end{aligned}$$

which can be extended to a global spacetime in the same way that the Rindler coordinates can be extended to Minkowski spacetime?

Such a coordinate system can be shown to exist, if one can exhibit (i) functions $f(r)$ and (ii) a constant α such that the metric tensor is nonsingular relative this new global (T, R) coordinate system.

The coordinate transformation from the inextendible Schwarzschild (t, r) coordinates -- inextendible because the metric is singular at $r=2M$ -- to the to-be-constructed extendible Kruskal (T, R) coordinates is determined by the differential equation for $f(r)$.

The computation is simple and is parallel to that for Minkowski spacetime:

$$dT = f(r) \sinh \alpha t \, dr + \alpha f(r) \cosh \alpha t \, dt$$

$$dR = f(r) \cosh \alpha t \, dr + \alpha f(r) \sinh \alpha t \, dt$$

$$dR^2 - dT^2 = (f')^2 dr^2 - f^2 \alpha^2 dt^2$$

$$dT^2 - dR^2 = f^2 \alpha^2 dt^2 - (f')^2 dr^2$$

Because it is not flat, the Schwarzschild metric has a form for its time and radial part which is given by

$$[-h(R, T) [dT^2 - dR^2]] = ds^2 = -(1 - \frac{2M}{r}) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}}$$

In order that the (T, R) coordinates be extendible we require that h be continuous at $r=2M$, i.e. the conformal factor h satisfy

$$\lim_{r \rightarrow 2M} h = \text{finite and nonzero}.$$

The transformation which relates (T, R) and (t, r) yields

$$-h f^2 \alpha^2 [dt^2 - (\frac{f'}{f})^2 dr^2] = -(1 - \frac{2M}{r}) [dt^2 - \frac{dr^2}{(1 - \frac{2M}{r})^2}]$$

Identifying the coefficients, one has

$$\text{Thus } dt^2 : h f^2 \alpha^2 = 1 - \frac{2M}{r}$$

$$dr^2 : h f^2 \alpha^2 = \frac{1}{1 - \frac{2M}{r}}$$

$$\left(\frac{f'}{f}\right)^2 = \alpha^2 \frac{1}{(1 - \frac{2M}{r})^2} \rightarrow \boxed{\frac{f'}{f} = \frac{\alpha}{1 - \frac{2M}{r}}}$$

The equations implied by this requirement is

$$hf^2\alpha^2 = \left(1 - \frac{2M}{r}\right) \Rightarrow h = \frac{1}{f^2\alpha^2} \left(1 - \frac{2M}{r}\right)$$

$$\boxed{\frac{f'}{f} = \frac{\alpha}{1-\frac{2M}{r}}} = \alpha \left(1 + \frac{2M}{r-2M}\right) \quad \text{for } 2M < r$$

$$\ln|f| = \alpha r + 2M\alpha \ln\left(\frac{r}{2M} - 1\right)$$

The last three boxed equations are three equations for $f(r)$, $h(R, T)$ and the constant α .

With an appropriate integration constant the solution for f is

$$f = \pm e^{\alpha r} \left(\frac{r-2M}{2M}\right)^{2M\alpha} \quad (\text{dimensionless})$$

The limit requirement or

$$h = \frac{1}{\alpha^2 f^2} \left(\frac{r-2M}{r}\right)^{-1} = \frac{(2M)^{4M\alpha}}{e^{2\alpha r} (r-2M)^{4M\alpha}} = \frac{16M^2 \cdot 2M}{r} \frac{1}{e^{r+2M}}$$

demands that the only allowed value for α is finite, to

$$4M\alpha = 1 \Rightarrow \alpha = \frac{1}{4M} \Rightarrow h = \frac{1}{\alpha^2 e^{2M}}$$

This must hold with razor sharp precision! If α deviates ever so slightly from the value $\frac{1}{4M}$, then $h(R, T)$ would

either $\rightarrow 0$ or ∞ as $r \rightarrow 2M$.

For this value of α

$$F(r) = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}}$$

$$h(R, T) = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(= \frac{1}{\alpha^2} \frac{2M^3}{e^{\frac{r}{2M}}}\right)$$

The corresponding Kruskal-Szekeres form of the Schwarzschild metric is

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} (dR^2 - dT^2) + r^2 (\theta^2 + \sin^2 \theta d\phi^2)$$

The coordinate transformations giving rise to this form are

$$T = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \sinh \frac{t}{4M} \quad 1) 2M < r$$

$$R = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \cosh \frac{t}{4M} \quad 2) \text{ upper signs go together}$$

These coordinates are dimensionless, and they imply that

$$e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) = R^2 - T^2$$

Thus we see that, inspite of its r -dependence, the metric coefficient function $h(R, T)$ is a function of R and T ; in fact, it is a function of $R^2 - T^2$.

The computed coordinate transformations for $r > 2M$ are not enough because we need another pair which applies to $r < 2M$.

To obtain it we repeat the same procedure:

Following the lead from a uniformly accelerated coordinate frame in Minkowski spacetime, we require that

$$\begin{aligned} T &= g(r) \cosh \beta t \\ R &= g(r) \sinh \beta t \end{aligned} \quad \left\{ \begin{array}{l} \text{yields the same} \\ \text{K-S metric} \end{array} \right.$$

$$h \{ dR^2 - dT^2 = (g')^2 dr^2 - g^2 \beta^2 dt^2 \} = - \frac{dr^2}{\frac{2M}{r}-1} + \left(\frac{2M}{r} - 1 \right) dt^2,$$

These are to apply in the spacetime region where $r < 2M$.

This requirement, together with

$$\lim_{r \rightarrow 2M} h = \text{finite and nonzero!}$$

$$h (dR^2 - dT^2) = h(g')^2 dr^2 - hg^2 \beta^2 dt^2 = - \frac{dr^2}{\frac{2M}{r}-1} + \left(\frac{2M}{r} - 1 \right) dt^2$$

yields $\beta = \frac{1}{4M}$

$$g(r) = \pm e^{\frac{T}{4M}} \left(1 - \frac{T}{2M}\right)^{1/2}$$

$$h = \frac{32M^3}{r} e^{-\frac{T}{2M}}$$

Thus we again obtain another pair of (dimensionless) coordinates

$T = \pm e^{\frac{T}{4M}} \left(1 - \frac{T}{2M}\right)^{1/2} \cosh \frac{t}{4M}$	$r < 2M$
$R = \pm e^{\frac{T}{4M}} \left(1 - \frac{T}{2M}\right)^{1/2} \sinh \frac{t}{4M}$	

relative to which the metric has the same desired form as before, namely

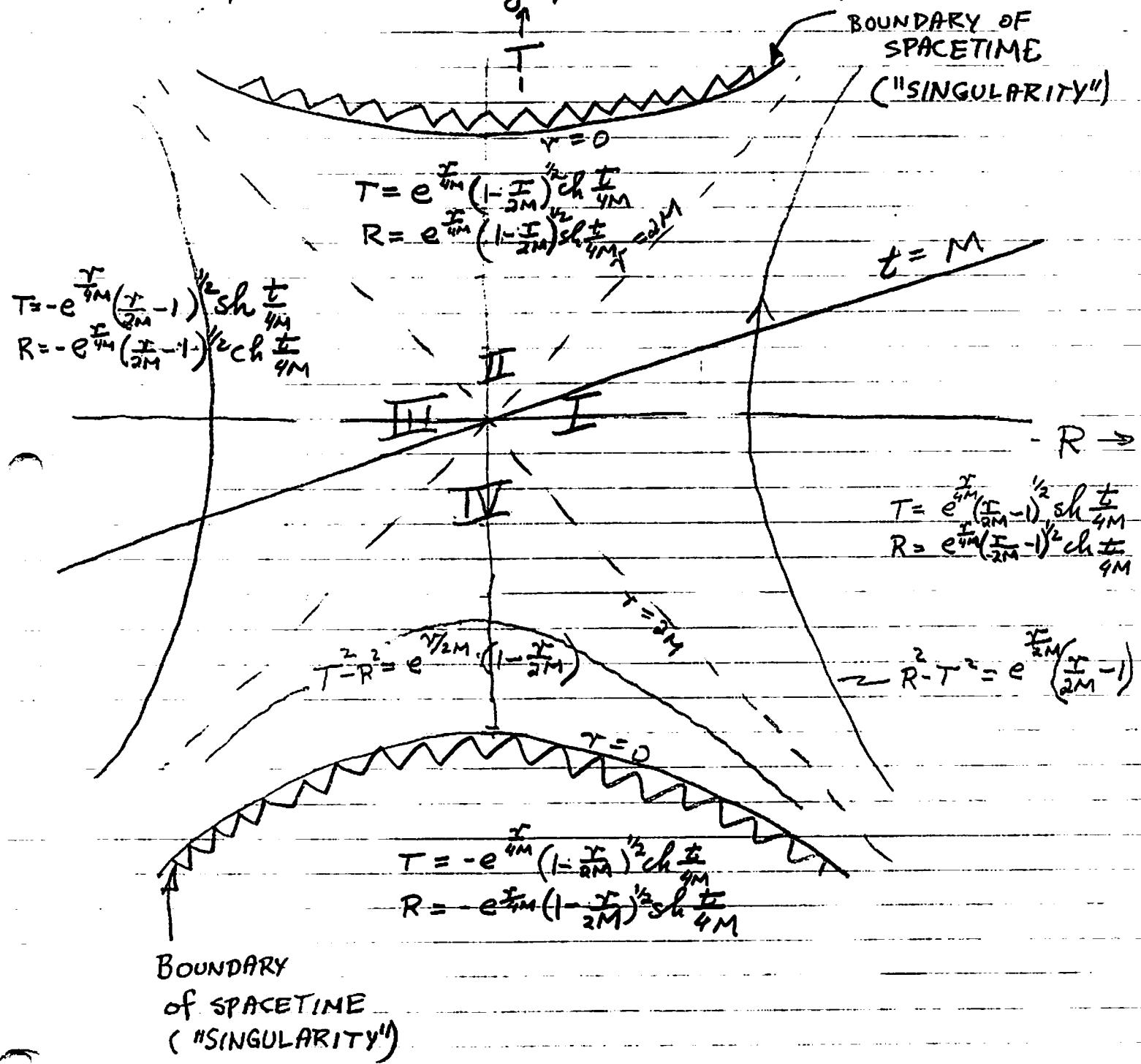
$$ds^2 = \frac{32M^3}{r} e^{-\frac{T}{2M}} (dR^2 - dT^2) + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

This is the globally defined Kruskal-Szekeres form of the metric on the Schwarzschild spacetime. This form is analogous to the Minkowski form of the metric

$$ds^2 = -dT^2 + dx^2 + dy^2 + dz^2$$

for flat spacetime.

The global structure of Schwarzschild spacetime is most transparent relative to these Kruskal-Szekeres coordinates



The Schwarzschild spacetime has an interior region, $r < 2M$, and an exterior region, $r > 2M$. These regions are different geometrically and physically:

1. The Schwarzschild coordinates (t, r) yield hyperbolae and straight lines relative to the K-S coordinates (T, R) .

Indeed in the exterior

$$R^2 - T^2 = e^{\frac{T}{2M}} \left(\frac{T}{2M} - 1 \right) \quad (\text{hyperbole})$$

$2M < T$: in I and II

$$\frac{T}{R} = \tanh \frac{t}{4M} \quad (\text{str. line})$$

while in the interior

$$T^2 - R^2 = e^{\frac{T}{2M}} \left(1 - \frac{T}{2M} \right)$$

$r < 2M$: in III and IV

$$\frac{T}{R} = \coth \frac{t}{4M}$$

2. The exterior ($2M < r$) is static, while the interior ($r < 2M$) is dynamic:

Indeed; for $r > 2M$. there exist timelike (observer) worldlines, namely

$$r = \text{const.}$$

relative to which the metric

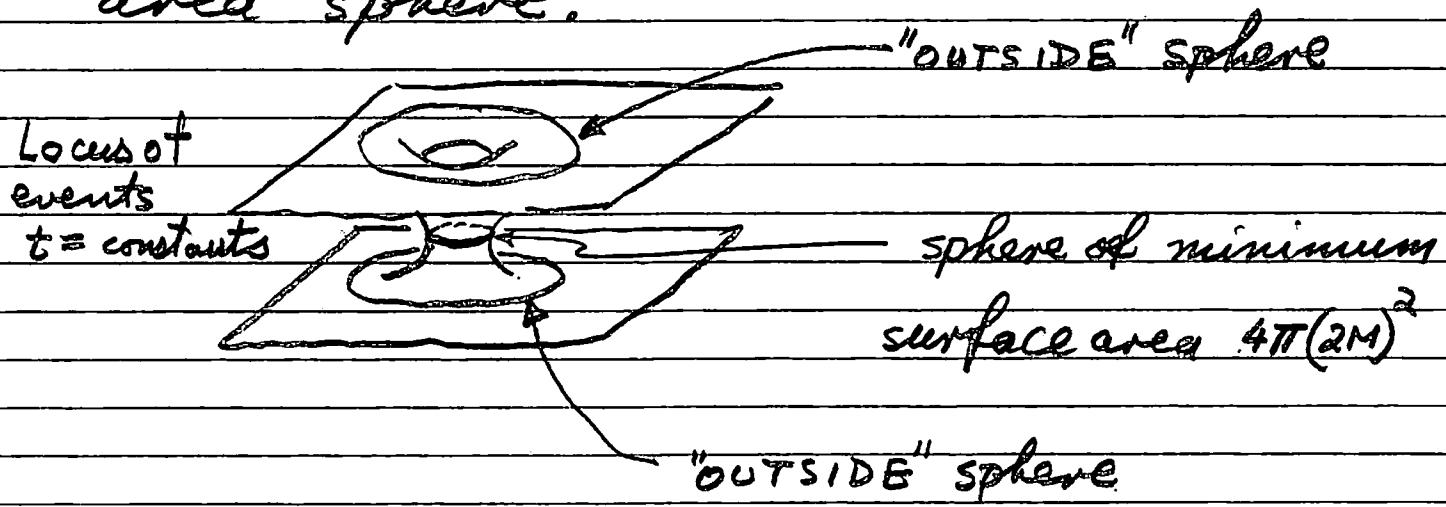
is static, i.e independent of t.

However for $r < 2M$, in the interior, there do not exist any time like worldlines relative to which the metric is static.

For example, $r = \text{const}$, are spacelike curves which do not represent the worldline of any observer.

3. The geometry on each of the spacelike hypersurfaces $t = c$, $-\infty < c < \infty$, is that of a static Schwarzschild throat. Each hypersurface $t = \text{constant}$ consists of a sequence of nested spheres. Their surface area $4\pi r^2$ is not a monotonic function of their proper separation. Instead there is a sphere of minimum area $4\pi(2M)^2 = 16\pi M^2$. This sphere, as well as all the others, have no center.

The spheres on one side of this minimum area sphere, as well as those on its other side, form a sequence with ever increasing surface area. Both sequences of spheres are on the "outside" of the minimum area sphere.



Together the two sequences of spheres form a three-dimensional passage ("Schwarzschild throat") between two asymptotically flat spatial regions.

- (i) One sees from the Kruskal diagram that these two regions are causally disjoint: an event in one region can not influence

or be influenced by an event in the other region.

(ii) one also sees that if one follows the evolution of this Schwarzschild throat on successive hypersurfaces $T = \text{const}$ over the coordinate time interval $0 \leq T \leq 1$, the throat evolves from its (maximum) area $16\pi M^2$ at $T=0$ to zero area at $T=1$; in other words the throat pinches off in a finite interval of time.

