

LECTURE 41

The global Kruskal coordinate system
for the Schwarzschild geometry

- A) The tortoise coordinate
- B) Ingoing & outgoing E-F coordinates
- C) Kruskal null coordinates via rescaling
- D) Global Kruskal atlas from
double images
- E) The Kruskal spacetime diagram

I.) THE TORTOISE COORDINATE.

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The Schwarzschild spacetime has the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) \left[dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \right] + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

whose radial null rays can be straightened

out by introducing the real-valued "tortoise"

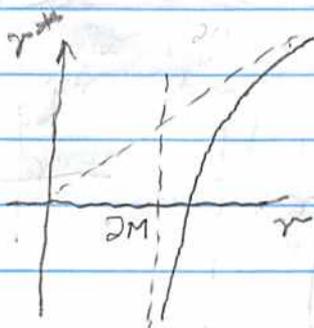
(a.k.a. Regge-Wheeler) coordinate

$$dr^* = \frac{dr}{1 - \frac{2M}{r}}$$

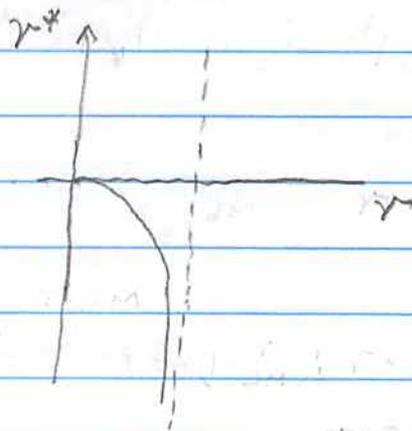
$$(41.1) \quad r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \rightarrow e^{\frac{r^*}{2M}} = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \quad \text{when } 2M < r$$

$$(41.2) \quad r^* = r + 2M \ln\left(1 - \frac{r}{2M}\right) \rightarrow e^{\frac{r^*}{2M}} = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}} \quad \text{when } r < 2M$$

The tortoise coordinate is a monotonic fn of r :



$$2M < r < \infty$$

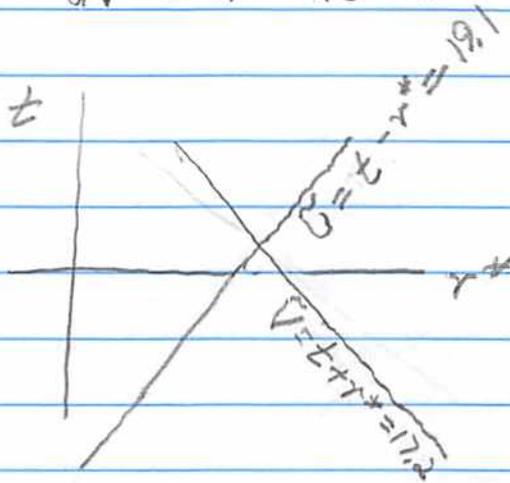


$$0 < r < 2M$$

This coordinate straightens out both the ingoing and the outgoing radial null rays because the metric assumes the form

$$(41.2) \quad ds^2 = -\left(1 - \frac{2M}{r}\right) \left[\underbrace{(dt + dr^*)}_{d\tilde{V}=0} \underbrace{(dt - dr^*)}_{d\tilde{U}=0} \right] + r^2(r^*) (d\theta^2 + \sin^2\theta d\phi^2)$$

$(ds)^2 = 0$



II.)

THE INGOING AND OUTGOING EDDINGTON-FINKELSTEIN

This suggests that one introduce

$$(41.2) \quad \tilde{V} = t + r^* \quad (\text{ingoing, "advanced" time})$$

$$(41.3) \quad \tilde{U} = t - r^* \quad (\text{outgoing, "retarded" time})$$

as the new coordinates. Introduce their differentials. The new representation of the metric, Eq. (41.2), has therefore the simplified form

(NEXT page.)

$$(41.4) \quad ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{V}d\tilde{U} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad 41.3$$

II.) MASTERING THE SINGULAR BEHAVIOUR OF ds^2

All that needs to be done is to express the coefficient $\left(1 - \frac{2M}{r}\right)$ in terms of r^* and hence in terms of $\tilde{V} - \tilde{U}$. Indeed, in light of Eqs (41.1) and (41.2)

$$e^{\frac{r^*}{2M}} = \frac{r}{2M} e^{\frac{r}{2M}} \begin{cases} \left(1 - \frac{2M}{r}\right) > 0 \text{ when } 2M < r \\ \left(-\right)\left(1 - \frac{2M}{r}\right) > 0 \text{ when } r < 2M \end{cases}$$

These are two different functions with two different domains, referred to as

the "exterior" when $2M < r$

and

the "interior" when $r < 2M$

Each domain accommodate both null coordinates \tilde{V} and \tilde{U} , by means of their Eqs. (41.2) - (41.3),

their exponentials. On the exterior one has

$$(41.5a) \quad e^{\frac{\tilde{V}}{2M}} \equiv e^{\frac{t+r^*}{2M}} = e^{\frac{t}{2M}} e^{\frac{r}{2M}} \left(1 - \frac{2M}{r}\right) > 0$$

and

$$(41.5b) \quad e^{-\frac{\tilde{U}}{2M}} \equiv e^{-\frac{t+r^*}{2M}} = e^{-\frac{t}{2M}} e^{\frac{r}{2M}} \left(1 - \frac{2M}{r}\right) > 0$$

Exterior: $2M < r$

On the interior one has

$$(41.6a) \quad e^{\frac{\tilde{V}}{2M}} \equiv e^{\frac{t+r^*}{2M}} = e^{\frac{t}{2M}} e^{\frac{r}{2M}} \left(-\left(1 - \frac{2M}{r}\right)\right) > 0$$

and

$$(41.6b) \quad e^{-\frac{\tilde{U}}{2M}} \equiv e^{-\frac{t+r^*}{2M}} = e^{-\frac{t}{2M}} e^{\frac{r}{2M}} \left(-\left(1 - \frac{2M}{r}\right)\right) > 0$$

Interior: $r < 2M$

Each of the exponential factors is positive.

Consequently,

$$e^{\frac{\tilde{V}-\tilde{U}}{4M}} \equiv e^{\frac{r^*}{2M}} = e^{\frac{r}{2M}} \begin{cases} \left(1 - \frac{2M}{r}\right) > 0 & \text{for } 2M < r \\ \left(-\left(1 - \frac{2M}{r}\right)\right) > 0 & \text{for } r < 2M \end{cases}$$

on page 14.3

so that the Schwarzschild metric, Eq. (41.4), becomes

$$(41.7) \quad ds^2 = -\frac{2M}{r} e^{-\frac{r}{2M}} e^{\frac{\tilde{V}}{4M}} d\tilde{V} e^{-\frac{\tilde{U}}{4M}} d\tilde{U} + \text{etc. for } 2M < r$$

and

$$(41.8) \quad ds^2 = -\frac{2M}{r} e^{-\frac{r}{2M}} \left(-\right) e^{\frac{\tilde{V}}{4M}} d\tilde{V} e^{-\frac{\tilde{U}}{4M}} d\tilde{U} + \text{etc. for } r < 2M$$

III INTRODUCING THE FOUR KRUSKAL COORDINATE CHARTS.

A.) Introduce the new coordinates

$$(41.9) \quad \tilde{v} = e^{\frac{\tilde{V}}{4M}} \quad \text{and} \quad \tilde{u} = -e^{-\frac{\tilde{U}}{4M}}$$

$$4M d\tilde{v} = e^{\frac{\tilde{V}}{4M}} d\tilde{V} \quad 4M d\tilde{u} = e^{-\frac{\tilde{U}}{4M}} d\tilde{U}$$

They rescale the coordinate domains of \tilde{V} and \tilde{U} so that

$$1 - \frac{2M}{r} \rightarrow 0 \quad \Rightarrow \quad \tilde{V} \rightarrow -\infty \quad \text{and} \quad \tilde{U} \rightarrow +\infty$$

Consequently the exterior and the interior metric, Eqs. (41.7) and (41.8) on pag 41.4 become

$$(41.10a) \quad ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} d\tilde{v} d\tilde{u} + \text{etc} \quad \text{for } 2M < r$$

and

$$(41.10b) \quad ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} d\tilde{v} d\tilde{u} + \text{etc} \quad \text{for } r < 2M$$

The new coordinates \tilde{v} and \tilde{u} , Eqs. (41.9), are related to the old (t, r) Schwarzschild coordinates by using the square roots of Eqs. (41.5) - (41.6).

- a)
(i) Using the positive square roots of (41.5a) and (41.5b)

$$(41.12a) \left. \begin{aligned} \tilde{v} = (+) \sqrt{e^{\frac{\tilde{v}}{2M}}} &= e^{\frac{t}{4M}} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \\ \tilde{u} = - (+) \sqrt{e^{\frac{\tilde{u}}{2M}}} &= -e^{-\frac{t}{4M}} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \end{aligned} \right\} \begin{array}{l} \text{EXTERIOR:} \\ 2M < r \end{array}$$

- (ii) Using the negative square roots of (41.6a) and (41.6b)

$$(41.12b) \left. \begin{aligned} \tilde{v} = (+) \sqrt{e^{\frac{\tilde{v}}{4M}}} &= e^{\frac{t}{4M}} e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \\ \tilde{u} = - (+) \sqrt{e^{\frac{\tilde{u}}{4M}}} &= -e^{-\frac{t}{4M}} e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \end{aligned} \right\} \begin{array}{l} \text{INTERIOR:} \\ r < 2M \\ \text{"black hole"} \end{array}$$

As already pointed out on page 41.5, the application of these two coordinate charts to the ^{two} singular representations of the

metric, Eqs. (41.7) - (41.8), results in

in the single Kruskal representation,

Eq. (41.10a) and (41.10b), which is analytic across the boundary $r=2M$ common to the exterior $r>2M$ and the interior $r<2M$.

b) (i) Using the negative square root of (41.5a) and (41.5b) on page 41,4

$$(41.12c) \left\{ \begin{array}{l} \tilde{v} = (-) \sqrt{e^{\frac{\tilde{v}}{2M}}} = -e^{\frac{t}{4M}} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \\ \tilde{u} = -(-) \sqrt{e^{-\frac{\tilde{u}}{2M}}} = e^{-\frac{t}{4M}} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \end{array} \right\} \begin{array}{l} \text{Exterior:} \\ 2M < r \end{array}$$

(ii) Using the negative square root of (41.6a) and (41.6b)

on page 41,4

$$(41.12d) \left\{ \begin{array}{l} \tilde{v} = (+) \sqrt{e^{\frac{\tilde{v}}{2M}}} = -e^{\frac{t}{4M}} e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \\ \tilde{u} = -(-) \sqrt{e^{-\frac{\tilde{u}}{2M}}} = e^{-\frac{t}{4M}} e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \end{array} \right\} \begin{array}{l} \text{Interior:} \\ r < 2M \end{array}$$

The driving force behind all four coordinate charts, Eqs. (14.12a) - (14.12d), is that the metric has the same analytic form,

$$(41.13) \quad ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} d\tilde{v} d\tilde{u} + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

relative to the new global arena

of (\tilde{v}, \tilde{u}) -values. These

$$(41.14a) \quad \tilde{v} \tilde{u} = e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1 \right)$$

and

$$(41.14b) \quad \frac{\tilde{v}}{\tilde{u}} = -e^{\frac{t}{2M}}$$

Both the Schwarzschild exterior ($2M < r$) and

its interior ($r < 2M$) have distinct double images in this arena. Indeed, introduce

the Kruskal time and space coordinates

$$T = \frac{1}{2}(\tilde{v} + \tilde{u}) \quad \tilde{v} = T + R$$

$$R = \frac{1}{2}(\tilde{v} - \tilde{u}) \quad \tilde{u} = T - R.$$

Relative to these coordinates the single globally defined metric, Eq. (4.13), is represented by

$$(41.16) \quad ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} (dT^2 - dR^2) + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

and

$$(41.17a) \quad T^2 - R^2 = e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1 \right)$$

$$(41.17b) \quad \frac{T}{R} = \tanh\left(\frac{t}{4M}\right).$$

The ingoing and the outgoing null rays,

$$dT \pm dR = 0,$$

are represented as straight 45 degree lines. The double images mathematized by Eqs (41.14a) or (41.17b) are isometric spacetime quadrants.

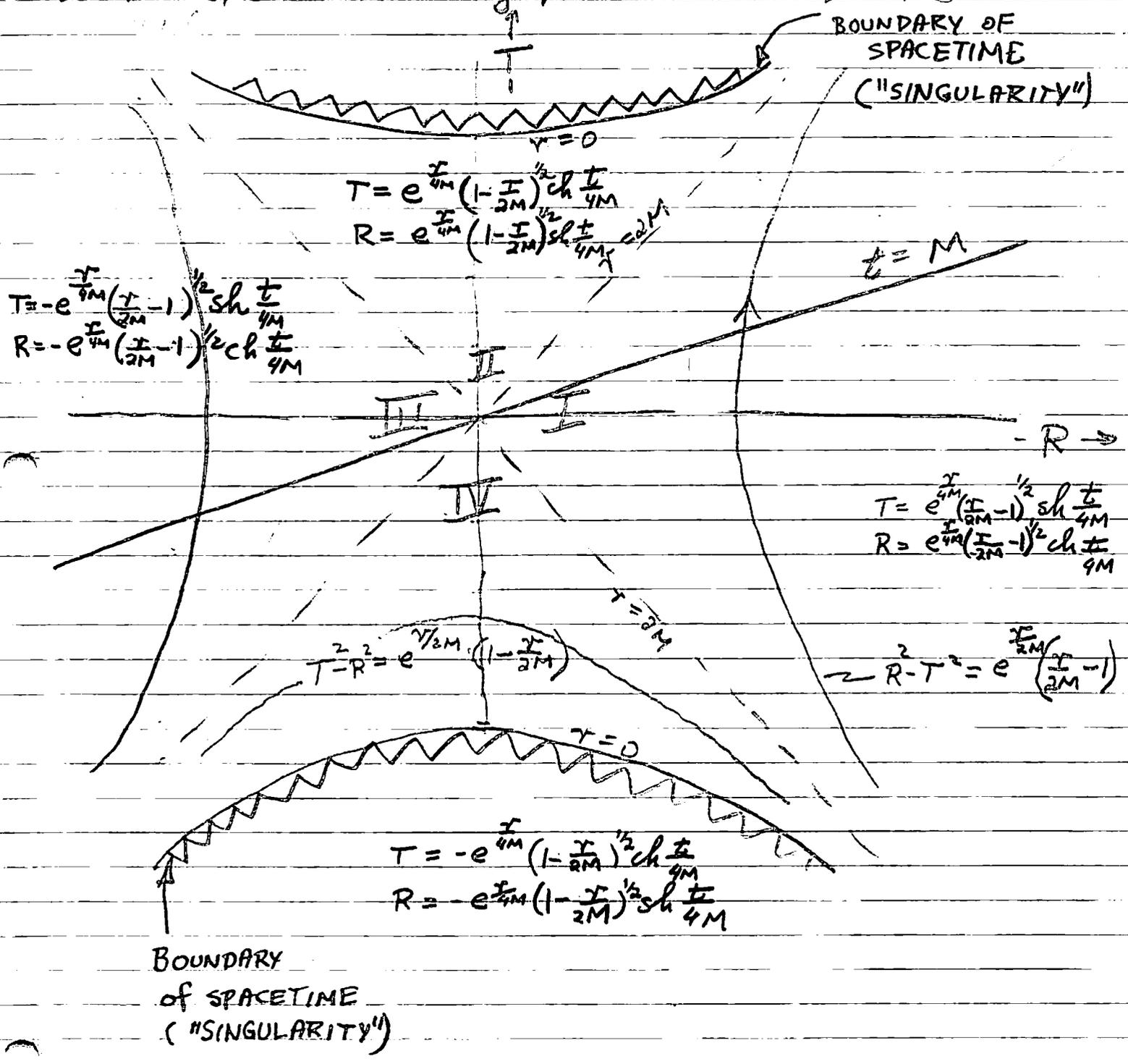
The double image of the Schwarzschild exterior
is coordinatized by

$$\left. \begin{array}{l} T \\ R \end{array} \right\} = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1 \right)^{1/2} \left\{ \begin{array}{l} \sinh\left(\frac{t}{4M}\right) \\ \cosh\left(\frac{t}{4M}\right) \end{array} \right. ,$$

while that of the interior is

$$\left. \begin{array}{l} T \\ R \end{array} \right\} = \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M} \right)^{1/2} \left\{ \begin{array}{l} \cosh\left(\frac{t}{4M}\right) \\ \sinh\left(\frac{t}{4M}\right) \end{array} \right. .$$

The global structure of Schwarzschild spacetime is most transparent relative to these Kruskal Szekeres coordinates



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