

# LECTURE 5

I. Geometrization of free body motion  
in an accelerated frame

II. Geometrization of gravity

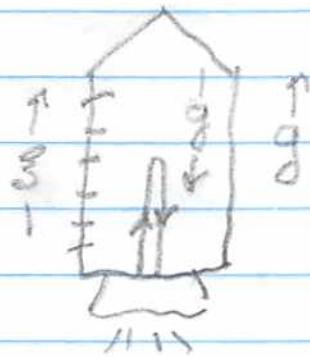
1. The Equivalence Principle.

2. Geometrized gravity

(via geometrized inertial motion)

and the Equivalence Principle.

Compare the free-body motion in  
an accelerated frame



$$\frac{d^2 \xi}{dt_{\text{conv}}^2} = -g_{\text{conv.}} \quad (5.1)$$

with the  $\mu=1$  component of the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \quad \text{by setting } x^1 = \xi. \quad \text{Low}$$

velocities imply  $\frac{dx^\alpha}{d\tau} = 1, \quad \tau = c t_{\text{conv}}$

$$\frac{d^2 \xi}{d\tau^2} = -\Gamma^1_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

$$\frac{d\xi}{(ct_{\text{conv}})^2} = -\Gamma^1_{00} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} + \text{negligible terms}$$

Comparison with Eq. (5.1) for all possible motions yields

$$\Gamma^1_{00} = \frac{g_{\text{conv}}}{c^2}$$

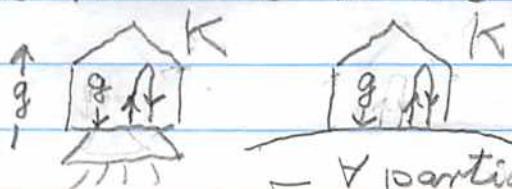
## II GEOMETRIZATION of GRAVITY

(gravity)

QUESTION: Why must one formulate gravity in geometrical terms?  
 i.e. Why would one be guilty of a self-contradiction if one denied the fact that gravity must be mathematized geometrically?

ANSWER

1. Our line of reasoning starts with the observation that the inertial mass,  $m_{\text{inert}}$ , of body equals its gravitational mass,  $m_{\text{grav}}$ , regardless of the composition of the body. This observed equality,  $m_{\text{inert}} = m_{\text{grav}}$ , is known as the equivalence principle.



"inertial force" =  $m_{\text{inert}} g$        $\Downarrow$       "gravitational force" =  $m_{\text{grav}} g$

$$K = K'$$

In general:  $\overrightarrow{\text{inertial force}} = \overrightarrow{\text{gravitational force}}$   
 $\Downarrow$   
 $m_{\text{inert}} = m_{\text{grav}}$        $\forall \text{ particles}$       (5.1)

## 2. Geometrized free-body motion +

+ Equivalence principle = geometrized gravity

a) The gravitational force field is conservative implies

$$(\text{grav'l force}) = m_{\text{grav}} \rightarrow \vec{\nabla} \phi_{\text{grav}}$$

where  $\phi_{\text{grav}}$  is the gravitational potential. (Newtonian)

Example: For a spherical body of mass  $M$  one has

$$\phi_{\text{grav}}(x, y, z) = -\frac{GM}{r} = -GM \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

b) Apply the equivalence principle to the one-dimensional geometrized inertial motion

$$\min \left( \frac{d^2 \xi}{dt^2} \text{conv} \frac{1}{C^2} = -\Gamma_{\infty}^1 = -\frac{g_{\text{conv}}}{C^2} \right)$$

$$= m_{\text{grav}} \left( -\frac{1}{C^2} g_{\text{conv}} = -\frac{1}{C^2} (\rightarrow \vec{\nabla} \phi_{\text{grav}})^2 \right) \quad (5.2)$$

c) Recall that

$$\Gamma_{00}^{ii} = \frac{1}{2} \sum_{\nu=0}^3 g^{i\nu} (g_{\nu 0,0} + g_{0,0,\nu} - g_{00,\nu})$$

(i) We are considering uniform, time-independent acceleration. Thus

$$g_{\nu 0,0} = g_{0,0,\nu} = 0$$

(ii) We are using rectilinear cartesian coordinates. Thus

$$g^{ij} = \delta^{ij} \quad i, j = 1, 2, 3$$

Consequently,

$$\boxed{\Gamma_{00}^{ii} = -\frac{1}{2} g_{00,ii} = -\frac{1}{2} (\vec{\nabla} g_{00})^i} \quad (5.3)$$

d) Insert this into Eq. (5.2) on page 5.3, and obtain

$$\frac{1}{2} (\vec{\nabla} g_{00})^i = \frac{m_{grav}}{m_{inert}} = \frac{1}{c^2} (\vec{\nabla} \phi_{grav})^i \text{ for } i=1,$$

use the experimental result that

$$\frac{m_{grav}}{m_{inert}} = 1,$$

with the result that

5.5

$$r_{00}^i = \frac{1}{c^2} (\vec{\nabla} \phi_{\text{grav}})^i$$

and

$$g_{00} = \text{const} - 2 \frac{\phi_{\text{grav}}}{c^2}$$

e) Impose the observed boundary cond'n

that, in the absence of gravitation,  
the spacetime metric is flat, i.e. is that  
of a global inertial reference frame:

$$ds^2 = g_{00} dt^2 + \sum_{i=1}^3 dx^i dx^i = -dt^2 + dx^2 + dy^2 + dz^2;$$

i.e.  $g_{00} = -1$  whenever  $\phi_{\text{grav}} = 0$

It follows that


$$g_{00} = -1 - 2 \frac{\phi_{\text{grav}}}{c^2} = -1 - \frac{2}{c^2} \begin{cases} \text{Newtonian} \\ \text{gravitational} \\ \text{potential} \end{cases}$$

### 3.) CONCLUSION

a) Gravitation is to be mathematized in terms of  
geometrical concepts. The gravitational  
potential is the Newtonian limit of

a geometrical formulation in terms of  
the metric tensor

b) The five step line of reasoning implies  
that fundamentally

$$\left( \begin{array}{l} \text{set of generalized} \\ \text{gravitational} \\ \text{potentials} \end{array} \right) = \left( \begin{array}{l} \text{set of components} \\ \text{of a metric} \\ \text{tensor field} \end{array} \right)$$

or more briefly

"gravitation = geometry" (properly understood)

c)

Newton's 1st Law  
geometrized relative  
to rotating & acc'd frames

$$\left\{ \begin{array}{l} \text{inertial} \\ \text{forces} \end{array} \right\} = \left\{ \Gamma^{\alpha}_{\alpha\beta} \right\}$$

$$\downarrow \text{Equivalence Principle} \rightarrow \Gamma^i_{00} = \frac{1}{c^2} (\nabla_\alpha \phi_{\text{grav}})^i$$

Gravity  $\neq 0 \Rightarrow$  Metric tensor  
is not flat

5.7

d) What is the geometrized generalization  
of Newton's gravitational field  
equation?

$$\nabla^2 \phi = 4\pi G \rho_g ?$$

Here  $\rho_g$  is the mass density

III) Looking ahead, we have the  
1. Mathematical Structure of Gravitation

5.8

geometrized motion  
of matter

(1) particle  
geodesics

(2) dynamics  
of matter  
in motion

Field eq'n's

geometrized  
gravitation

[metric  
parallel transport  
curvature]

2. Mathematical Structure of Electromagnetism,  
motion of  
charged matter

Lorentz  
equations  
of motion

Maxwell's  
field  
eq'n's

e.m. field