

## LECTURE 7

I Momentum conservation + P, of R, implies  
energy conservation

II The momenergy (= "energy-momentum")  
four-vector

Read Ch. 7.1-7.7 in Taylor & Wheeler's  
Spacetime Physics  
2<sup>nd</sup> Edition

I.)

## 1. Momentum in Relativity

Collisions of particles is the observational basis for momentum and its conservation in Newtonian dynamics as well as its generalization to relativistic dynamics. In that context the 3-d relativistic momentum

vector is

$$\vec{\text{momentum}} = \frac{m\vec{\beta}}{\sqrt{1-\beta^2}} \quad (7.1)$$

## 2. Energy in Relativity

In Newtonian mechanics momentum conservation and energy conservation are distinct physical principles

In relativistic mechanics, by contrast, momentum conservation implies

energy conservation,

The line of reasoning leading to this conclusion is based on the fact that 3-d relativistic momentum, Eq. (7.1), of a particle is proportional to the spatial components of the unit tangent

$$\{u^{\mu}\} = \left\{ \frac{dx^{\mu}}{d\tau} \right\} = \left\{ \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right\}$$

to the particle's world line. This tangent is a 4-vector, and its components obey the well-known transformation laws between inertial frames of reference.

(2) The components of this four-vector are not independent. This is because of the

change of the proper ("wristwatch") time along the worldline is related to the corresponding spacetime displacement by

$$-(d\tau)^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

Consequently,

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 \equiv 1 - \beta^2$$

or

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}$$

and hence

$$\frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt} \frac{dt}{d\tau} = \vec{\beta} \frac{1}{\sqrt{1-\beta^2}}$$

The components of the four-velocity

are therefore

$$\{u^\mu\} \equiv \left\{ \frac{dx^\mu}{d\tau} \right\} = \left\{ \frac{1}{\sqrt{1-\beta^2}}, \frac{\vec{\beta}}{\sqrt{1-\beta^2}} \right\} \equiv \left\{ \frac{1}{\sqrt{1-\beta^2}}, \vec{v} \right\} \quad (7.2)$$

(ii) Combine Eqs (7.1) and (7.2) and obtain

$$\vec{\text{momentum}} = m \text{ (space components of } u^\mu)$$

or

$$\boxed{\vec{p} = m\vec{u}} = m \frac{\vec{v}}{\sqrt{1-\beta^2}}$$

(iii) Apply this definition to the momentum conservation principle in two

inertial frames in relative motion.

The result of this application is the

following

Theorem (Conservation of energy)

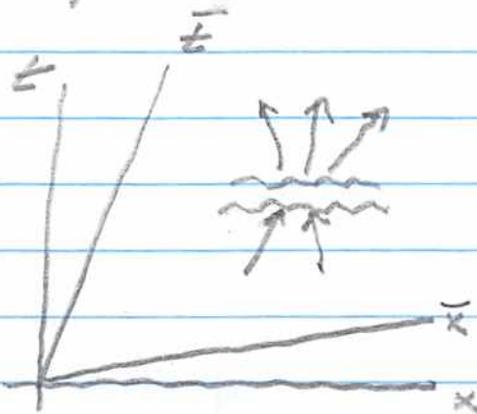
Given: (a) The Principle of Relativity

(b) Conservation of spatial momentum  
in a collision of particles

Conclusion: Total energy of particles  
is conserved.

proof:

- 1.) Consider a collision process in two inertial frames



Let

$\{u^\mu\}$  = 4-velocity of a particle

- 2.) Apply the transformation law to  $u^\mu$  to get the transformation law for the momentum of each particle

$$p_x = m u_x = m (\bar{u}_x \cosh \theta + \bar{u}_t \sinh \theta)$$

$$= \bar{p}_x \cosh \theta + m \frac{1}{\sqrt{1-\beta^2}} \sinh \theta$$

- 3.) Apply the momentum conservation principle to the particle momenta in

each inertial frame

$$\begin{aligned}
 0 &= \sum_i P_{x(i)} \Big|_{\text{after}} - \sum_j P_{x(j)} \Big|_{\text{before}} \\
 &\text{in } (t, x) \text{ frame} \\
 &= \left( \sum_i \bar{P}_{x(i)} \Big|_{\text{after}} - \sum_j \bar{P}_{x(j)} \Big|_{\text{before}} \right) \cosh \theta \\
 &\text{in } (\bar{t}, \bar{x}) \text{ frame} \xrightarrow{\quad} 0 \\
 &+ \left( \sum_i \frac{m_i}{\sqrt{1-\beta_i^2}} \Big|_{\text{after}} - \sum_j \frac{m_j}{\sqrt{1-\beta_j^2}} \Big|_{\text{before}} \right) \sinh \theta
 \end{aligned}$$

4 Conclusion:

$$\sum_i \frac{m_i}{\sqrt{1-\beta_i^2}} \text{ is conserved}$$

4.) Go to the Newtonian limit.

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2 \left( 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} \dots \right)$$

$$\underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{\frac{1}{2}mv^2 + \dots}_{\text{kinetic energy}} = \text{(mass-energy of the particle)}$$

Summary:

Momentum conservation + P, of R

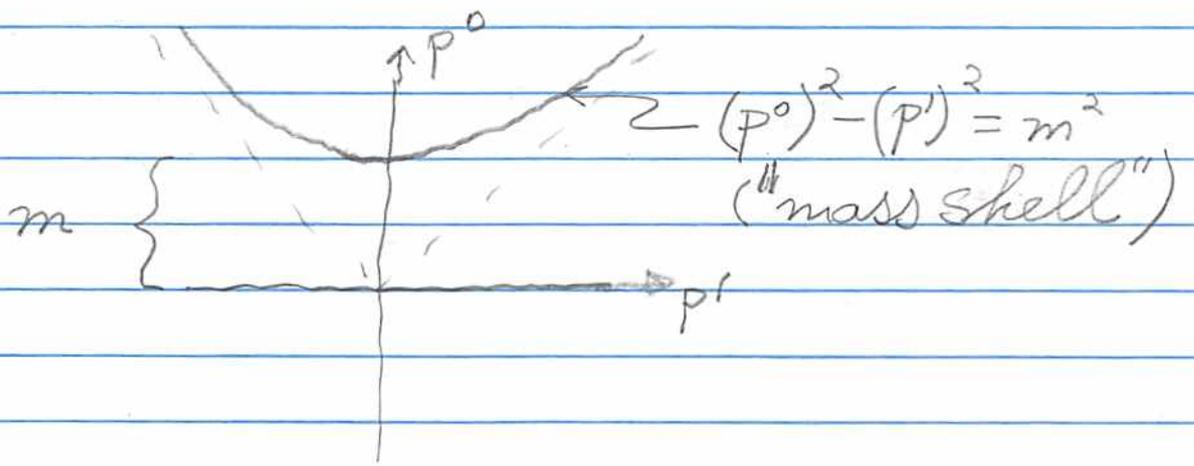
$$\Rightarrow \left[ \sum_i (\text{mass-energy})_i \equiv \text{Total mass-energy is conserved} \right]$$

## II) Proposition (Energy-momentum 4-vector)

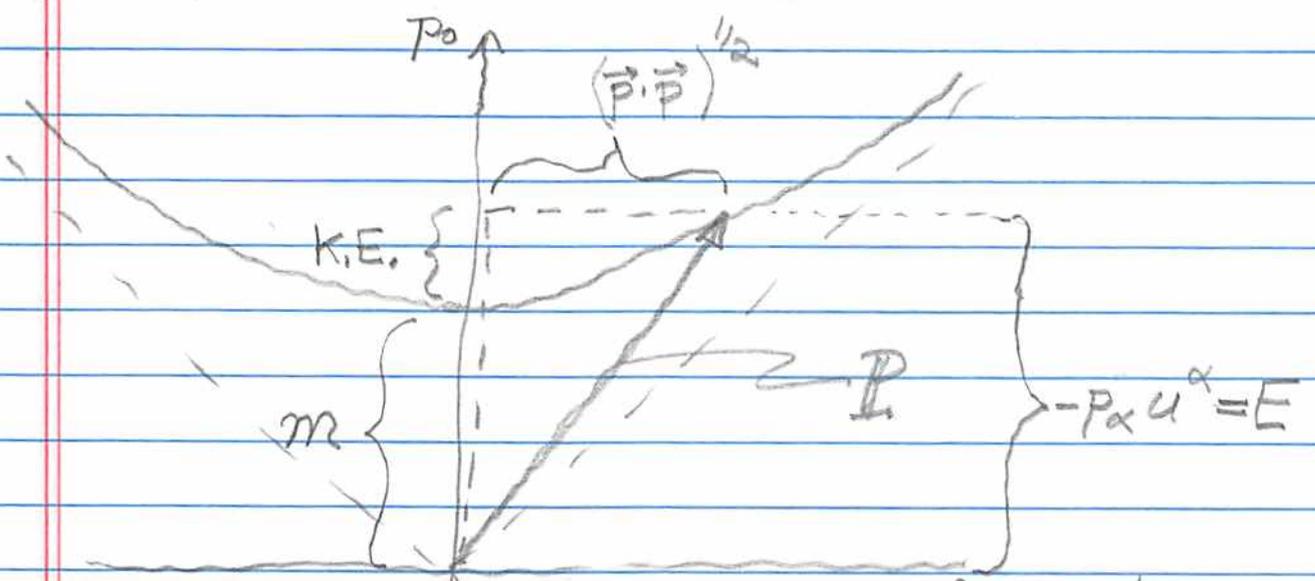
a) Definition

$$P: \{p^\alpha\} \equiv \left\{ \frac{m}{\sqrt{1-\beta^2}}, \frac{m\vec{\beta}}{\sqrt{1-\beta^2}} \right\} \begin{matrix} \text{mom energy} \\ \text{for each} \\ \text{particle} \end{matrix}$$

b) (Magnitude)<sup>2</sup> = (p<sup>0</sup>)<sup>2</sup> - (p<sup>1</sup>)<sup>2</sup> - (p<sup>2</sup>)<sup>2</sup> - (p<sup>3</sup>)<sup>2</sup>  
 = m<sup>2</sup> = (rest mass)<sup>2</sup>



c) Projection onto the zero-axis = E (= energy)



Kinetic energy = E - m =  $\frac{m}{\sqrt{1-\beta^2}} - m \approx \frac{1}{2} m \beta^2 + \frac{3}{8} m \beta^4 + \frac{5}{16} m \beta^6 + \dots$

e) Every application of the law of conservation of momentum in a collision is a statement about a polygon built of 4-vectors in spacetime

$$\sum_{i=1}^N P_i \Big|_{\text{before}} = P_{\text{tot}} \Big|_{\text{before}} = P_{\text{tot}} \Big|_{\text{after}} = \sum_{j=1}^M P'_j \Big|_{\text{after}}$$

