

LECTURE 8

Particle density-flux 3-Form

1.) Derivation of *S

2.) The particle current 4-vector,

In MTW : Box 4.4 (the math of *J)

Box 5.2 (the physics of *J)

Box 15.1 (the charge density-flux
3-form)

1. Definition of the concept "particle":
In physics a particle is a contextually
small entity of a specific nature
made of specific attributes (mass,
momentum, charge, spin, color, etc.)

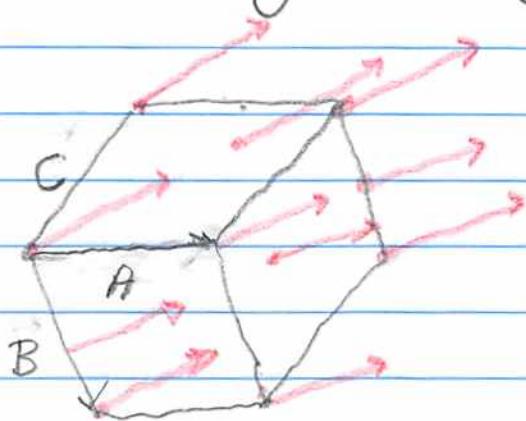
In physics the concept "particle" is
the building-block fundamental to
understanding the physical world.

I) To mathematize gravitation one must
mathematize matter in relativistic motion.

Q: What is the reasoning process to achieve this?

I) A: Consider a set of particles in a region
of space and time large enough so that
one talk about density, flux, pressure,
etc., but small enough so that

these particles can be said to have the same velocity in a given volume element



There are two frames of reference, the LAB frame and the COMOVING frame, which is determined by the particles.

The particles have zero spatial velocity in the COMOVING frame $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$. Their common 4-velocity is

$$U = U^\mu \frac{\partial}{\partial x^\mu} = 1 \frac{\partial}{\partial t} + 0 \frac{\partial}{\partial x} + 0 \cdot \frac{\partial}{\partial y} + 0 \cdot \frac{\partial}{\partial z} \equiv \frac{d}{dt}.$$

Relative to the LAB frame their 4-velocity is

$$u = \bar{u}^m \frac{\partial}{\partial \bar{x}^m} = \bar{u}^0 \frac{\partial}{\partial \bar{x}^0} + \bar{u}^1 \frac{\partial}{\partial \bar{x}^1} + \bar{u}^2 \frac{\partial}{\partial \bar{x}^2} + \bar{u}^3 \frac{\partial}{\partial \bar{x}^3}$$

The element of 3-volume is spanned by the three space-like vectors,

$$A = 0 \cdot \frac{\partial}{\partial \bar{x}} + \Delta x \frac{\partial}{\partial \bar{x}} + 0 \frac{\partial}{\partial \bar{y}} + 0 \frac{\partial}{\partial \bar{z}} = A^\alpha \frac{\partial}{\partial \bar{x}^\alpha}$$

$$B = 0 \cdot \frac{\partial}{\partial \bar{x}} + " + \Delta y \frac{\partial}{\partial \bar{y}} + " = B^\beta \frac{\partial}{\partial \bar{x}^\beta}$$

$$C = " + " + " + \Delta z \frac{\partial}{\partial \bar{z}} = C^\gamma \frac{\partial}{\partial \bar{x}^\gamma}$$

Their components are attached to respective pairs of particles in the comoving frame, with its orthonormal comoving basis

$\{e_\mu = \frac{\partial}{\partial x^\mu}\}$. Consequently, the element of proper (= comoving) volume is

$$\Delta x \Delta y \Delta z = \det \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \Delta x & 0 & 0 \\ 0 & 0 & \Delta y & 0 \\ 0 & 0 & 0 & \Delta z \end{vmatrix}$$

$$\begin{aligned}
 \Delta x \Delta y \Delta z &= \det \begin{vmatrix} u^0 u^1 u^2 u^3 \\ A^0 A^1 A^2 A^3 \\ B^0 B^1 B^2 B^3 \\ C^0 C^1 C^2 C^3 \end{vmatrix} \quad , \text{ p. 8-5, 8-8, 8-9} \\
 &= u^m \epsilon_{\alpha\beta\gamma} \langle dx^\alpha A \rangle \langle dx^\beta B \rangle \langle dx^\gamma C \rangle \\
 &= u^m \epsilon_{\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!} (A, B, C) \\
 &\Rightarrow \text{comoving volume spanned by } (A, B, C)
 \end{aligned}$$

II Consider the comoving density of particles in $\Delta x \Delta y \Delta z$,

$$N = \frac{\# \text{ particles}}{\Delta x \Delta y \Delta z} = \frac{\text{(number of)} \text{ particles}}{\text{(comoving)} \text{ volume}}$$

The number of particles in the comoving volume element is a frame invariant, e.g. independent of the observer's reference frame:

d

Thus

8.5

$$\# = N \Delta x \Delta y \Delta z.$$

$$\# = N U^M \epsilon_{\mu\nu\rho\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\delta (A, B, C) =$$

$$= N \bar{U}^{\bar{M}} \bar{\epsilon}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\gamma}} d\bar{x}^\alpha \wedge d\bar{x}^\beta \wedge d\bar{x}^\delta (A, B, C)$$

Relative to any basis $\{e_\sigma\}$ and its dual

$$\{\omega^\sigma\}$$

$$\omega^\sigma(e_\sigma) = \delta_\sigma^\sigma,$$

be it coordinate-induced, orthonormal,

oblique, the frame invariance of

the particle count is mathematized by

the statement

$$(8.1) \quad \# = N U^M \epsilon_{\mu\nu\rho\gamma} \omega^\alpha \wedge \omega^\beta \wedge \omega^\delta (A, B, C),$$

This statement says that $\#$ depends
on the spanning vectors in the manner
of a trilinear function;

Given N , $\epsilon_1 = u^{\alpha} \epsilon_{\alpha \beta \gamma}$, and $\epsilon_{\mu \nu \alpha \beta \gamma \delta} w^{\mu} w^{\nu} w^{\alpha} w^{\beta} w^{\gamma} w^{\delta}$ in Eq(8.1) on page 8.5, there must be some chosen triad of vectors (A, B, C) (such as the one on page 8.3), but there may be any such chosen triad.

Using this "some but any" principle one arrives at the concept

$${}^*S = N u^{\alpha} \epsilon_{\alpha \beta \gamma} w^{\mu} w^{\nu} w^{\delta},$$

the density-flux 3-form.

a) *S is a trilinear map

$$V \times V \times V \rightarrow \mathbb{R}^+ \quad (\# \text{ of particles})$$

$$(A, B, C) \mapsto {}^*S(A, B, C) = N u^{\alpha} \epsilon_{\alpha \beta \gamma} w^{\mu} w^{\nu} w^{\delta} (A, B, C);$$

b) *S is a tensor of rank(3).

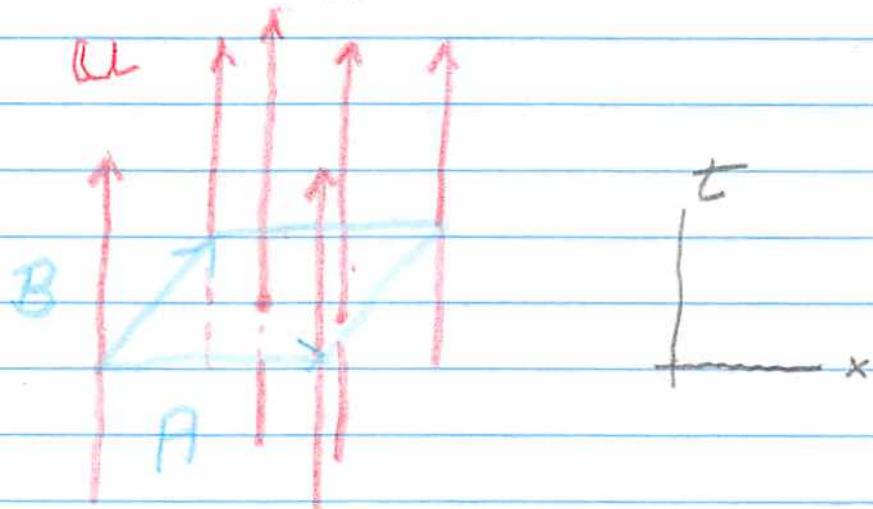
$$\textcircled{c}) {}^*S = \frac{(\# \text{of particles})}{(\text{as-yet-unspecified spacetime 3-volume})}$$

CASE(1) The spacetime 3-volume is spanned

by three space-like vectors,

In the COMOVING frame one has

$$\# = {}^*S(A, B, C) = N u^\mu \epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma$$



$$\# = N u^\mu \epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma = \text{comoving volume}$$

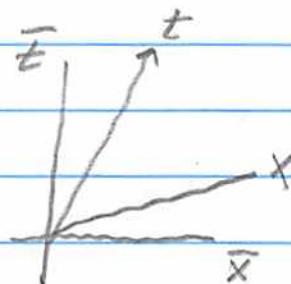
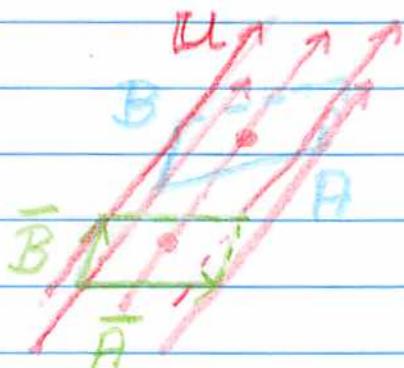
$\underbrace{\quad}_{\text{comoving volume}} \quad \underbrace{\quad}_{\text{comoving}}$

$$\tilde{S}^0 = \frac{\#}{(\text{comov. volume})} = \frac{\#}{\text{comov. volume}} = (\text{comoving particle}) \text{ density}$$

In the LAB frame

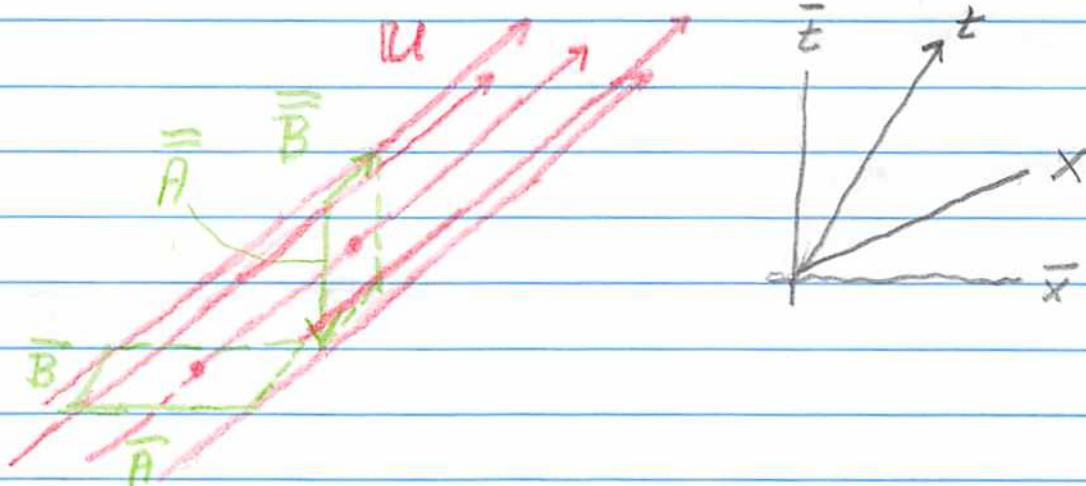
$$\# = \underbrace{N \bar{U}^0}_{\text{Lab volume}} \underbrace{\epsilon_{\bar{\alpha} \bar{\beta} \bar{\gamma} \bar{\delta}}}_{\bar{A}^{\bar{\alpha}} \bar{B}^{\bar{\beta}} \bar{C}^{\bar{\gamma}}} \bar{D}^{\bar{\delta}} + \sum_{i=1}^3 N \bar{U}^i \epsilon_{i \bar{\alpha} \bar{\beta} \bar{\gamma}} \bar{A}^{\bar{\alpha}} \bar{B}^{\bar{\beta}} \bar{C}^{\bar{\gamma}}$$

$$\equiv \bar{S}^0 = \frac{\#}{(\text{lab vol.})} = (\text{particle density in LAB frame})$$



CASE(2) The spacetime 3-volume is spanned by one timelike vector (\bar{A}) and two spacelike vectors (\bar{B}, \bar{C})

In the LAB frame



$$\# = {}^*S(\bar{A}, \bar{B}, \bar{C})$$

$$= \underbrace{N \bar{u}^{\bar{1}}}_{\#} \epsilon_{\bar{1} \bar{2} \bar{3}} \bar{\Delta t} \bar{\Delta y} \bar{\Delta z}$$

lab area spanned
by \bar{A} and \bar{B}

\bar{A} 's lab time window

$\frac{\#}{(\text{time})(\text{area})}$

d) The components of the density-flux

3-form

$${}^*S = S^0 \epsilon_{\alpha \beta \gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma + S^1 \epsilon_{1 \alpha \beta \gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} + \dots$$

consist of

$$S^0 = \frac{\text{particles}}{\text{volume}} = \text{particle density}$$

$$\left. \begin{matrix} S^1 \\ S^2 \\ S^3 \end{matrix} \right\} = S^i = \frac{\text{particles}}{(\text{time})(\text{directed area})} = \begin{pmatrix} \text{particle flux into the } i^{\text{th}} \\ \text{direction} \end{pmatrix}$$

e) The vector formed from the proper particle density N and the particle 4-velocity u ,

$$Nu = S = S^m e_m$$

is called the particle current 4-vector.