

# LECTURE 9

The particle density-flux 3-form:

Its physical, geometrical and algebraic properties.

In MTW know Box 4.4

Box 5.1

Fig. 5.1

Box 5.2

9.1

The particle density-flux 3-form

$$\begin{aligned} \mathcal{L}^* S &= N u^\mu \epsilon_{\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma \\ &= S^N \sum_\mu \end{aligned}$$

is the means of mathematizing  
the geometrical spacetime properties of

a continuous medium consisting of  
particles. This includes the types of  
media found in extreme relativistic  
astrophysical environments as well as those  
media driven by ultra-intense laser radiation.

## I. Physical Properties of the Particle Density-Flux 3-form.

The 3-form  $\mathcal{L}^* S$  is the unification of four  
physical attributes of particles in motion:

(i) Their common 4-velocity

$$(9.1) \quad \boxed{u_i = u^\mu \frac{\partial}{\partial x^\mu}}$$

(ii') Their number, which is frame invariant

$$\# = \mathcal{S}(A, B, C) = N u^{\mu} \sum_{\mu} (A, B, C)$$

$$= N u^{\mu} \epsilon_{\mu\alpha\beta\gamma} \frac{dx^{\alpha} dx^{\beta} dx^{\gamma}}{3!} (A, B, C)$$

in a spacetime 3-volume spanned by

three chosen 4-vectors A, B, C.

If all three chosen 4-vectors are spacelike

with simultaneous tip and tail events relative to  
the COMOVING frame. In that case

$$\# = N u^{\mu} \sum_{\mu} (A, B, C) = N u^{\mu} \epsilon_{\mu\alpha\beta\gamma} \frac{dx^{\alpha} dx^{\beta} dx^{\gamma}}{3!} (A, B, C)$$

$$= \det \begin{vmatrix} u^0 & 0 & 0 & 0 \\ 0 & A^1 & A^2 & A^3 \\ 0 & B^1 & B^2 & B^3 \\ 0 & C^1 & C^2 & C^3 \end{vmatrix}$$

$\times$

(9.2)  $N = \frac{\#}{A \cdot B \times C} = \frac{\#}{\text{comoving}} = \frac{\#}{\text{comoving density of particles}}$

(ii) If all three chosen 4-vectors, say  $\bar{A}, \bar{B}$ , and  $\bar{C}$  are spacelike having simultaneous tip and tail events in the LAB frame instead, then

$$\# = N u^{\bar{\mu}} \epsilon_{\bar{\mu}\bar{\alpha}\bar{\beta}\bar{\gamma}} \frac{d\bar{x}^{\bar{\alpha}}}{3!} d\bar{x}^{\bar{\beta}} d\bar{x}^{\bar{\gamma}} (\bar{A}, \bar{B}, \bar{C})$$

$$\# = N \begin{vmatrix} u^{\bar{0}} & u^{\bar{1}} & u^{\bar{2}} & u^{\bar{3}} \\ 0 & \bar{A}^{\bar{1}} & \bar{A}^{\bar{2}} & \bar{A}^{\bar{3}} \\ 0 & \bar{B}^{\bar{1}} & \bar{B}^{\bar{2}} & \bar{B}^{\bar{3}} \\ 0 & \bar{C}^{\bar{1}} & \bar{C}^{\bar{2}} & \bar{C}^{\bar{3}} \end{vmatrix}$$

$$\# = N u^{\bar{0}} (\bar{A} \cdot \bar{B} \times \bar{C}) =$$

$$\# = N u^{\bar{0}} \text{ LAB volume}$$

$$(9.3a) \boxed{N u^{\bar{0}} \equiv S^{\bar{0}}} = \frac{\#}{\text{LAB volume}} \equiv \text{particle density in LAB}$$

Comment

$u^{\bar{\mu}} = e_{\bar{\mu}} u^{\bar{\mu}} = e_{\bar{\mu}} u^{\bar{\mu}}$  is the same common 4-velocity as before. However, in the LAB frame one has  $\{u^{\bar{\mu}}\} = \left\{ \frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}} \right\}$ .

(next page)

Thus, in the LAB frame one has

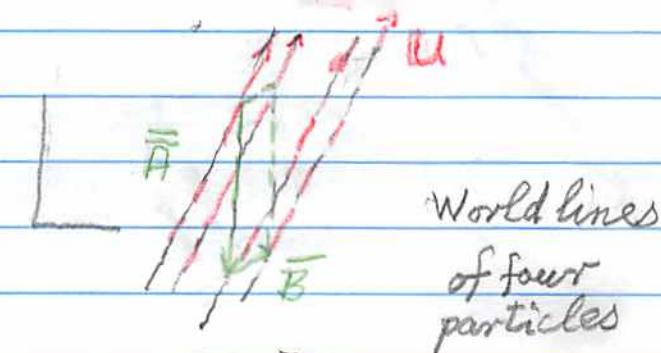
$$N\bar{u}^{\bar{0}} = \bar{s}^{\bar{0}} \geq N = Nu^0 \equiv S,$$

which is to say, the observed density of moving matter is larger than the particle comoving density.

(iv) If one of the chosen 4-vectors is purely time like, while the other two are spacelike with simultaneous tip and tail events in the LAB frame, then

$$\# = Nu^{\bar{x}} \epsilon_{\bar{A}\bar{B}\bar{C}\bar{x}} d\bar{x}^{\bar{1}} d\bar{x}^{\bar{2}} d\bar{x}^{\bar{3}} (\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}})$$

$$= N \det \begin{vmatrix} u^{\bar{0}} & u^{\bar{x}} & u^{\bar{y}} & u^{\bar{z}} \\ -\Delta \bar{t} & 0 & 0 & 0 \\ 0 & \bar{B}^{\bar{x}} & \bar{B}^{\bar{y}} & \bar{B}^{\bar{z}} \\ 0 & \bar{C}^{\bar{x}} & \bar{C}^{\bar{y}} & \bar{C}^{\bar{z}} \end{vmatrix}$$



$$= N \Delta \bar{t} [u^{\bar{x}} (\bar{B} \times \bar{C})_{\bar{x}} + u^{\bar{y}} (\bar{B} \times \bar{C})_{\bar{y}} + u^{\bar{z}} (\bar{B} \times \bar{C})_{\bar{z}}]$$

This # is the number of particles passing through the area  $B \times C$  during the LAB time  $\bar{t}$ . On a per-unit time basis one has

$$\frac{\#}{\Delta t} = N u^x (\bar{B} \times \bar{C})^x + N u^y (\bar{B} \times \bar{C})^y + N u^z (\bar{B} \times \bar{C})^z$$

$$= \frac{(\text{particle current})}{(\text{through area } (\bar{B} \times \bar{C}))}$$

The three parts contributing to <sup>points</sup>  
this current are due to the three flux components :  
(next page)

$$(9.3b) \boxed{N u^i \equiv \oint \vec{S}^i} = \frac{\#}{\Delta t (\vec{B} \times \vec{C})^i}$$

= number of particles

(LAB)  
(time) (area whose  
normal points into  
the  $i^{th}$  direction)

=  $\left( i^{th} \text{ component} \right)$        $i = x, y, z$   
of particle flux

Having identified the four physical attributes

the boxed Eqs (9.1), (9.2), (9.3a) and (9.3b) one systema-

tizes them by

(next page)

combining the LAB flux components with the LAB particle density, Eq.(9.3a), and obtain the particle current 4-vector

$$S = Nu:$$

$$Nu^0 = \frac{\text{particles}}{\text{LAB volume}}$$

$$Nu^x = \frac{\text{particles}}{\text{LAB time}(\text{area})^x}$$

$$Nu^y = \frac{\text{particles}}{\text{LAB time}(\text{area})^y}$$

$$Nu^z = \frac{\text{particles}}{\text{LAB time}(\text{area})^z}$$