MATH 5757

MODERN MATHEMATICAL METHODS IN RELATIVITY THEORY II

Applied Differential Geometry

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Volume II

PREFACE

The complexity of mathematics is a reflection of the causal relation- ships that exist in the world. This fact is predicated on the observation that the universe is ruled by natural laws and, therefore, is stable, firm and absolute – and knowable.

Knowledge is of course not automatic. Its acquisition is a matter of choice, of the right way of using one's mind, and of the perseverance to do so.

When it comes to Einstein's geometrical mathematization of gravitation, its formulation is wrought with a complexity which at first sight seems bewildering. Fortunately, however, Charles Misner, Kip Thorne, and John Wheeler have reduced that complexity into digestible form. Their book GRAVITATION, informally known as MTW, is by far the best for STEM[1] and philosophy students who wish to grasp and understand gravitation in physical and mathematical terms.

GRAVITATION is the result of an integration of physics, astro- physics, and mathematics at its best. The thoroughness of the process can be measured by the fact that the book's weight due to gravity is more than 5 pounds and it has more than 1,300 pages, which give rise to a PDF of more than 50 Megabytes. This can be an embarrassment of riches even for an advanced student.

The purpose of "A COMPANION TO THE TRACK 2 MATHEMATICS OF MTW" is to deal with this "embarrassment" by focusing primarily on the modern mathematical methods for understanding Einstein's geometrical formulation of spacetime and gravitation.

In Volume I this focus starts with the spacetime physics of Special Relativity, then integrates linear algebra with multivariable calculus on manifolds before it derives and then applies the two Cartan structure equations of modern differential geometry to a problem in astrophysics.

One of the most glaring elements missing from Einstein's original 1916 line of reasoning is the physical and geometrical meaning of the tensor on the l.h.s. of his field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu} \ .$$

That element was supplied by Cartan and Wheeler who identified the l.h.s. as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \text{“moment of rotation per spacetime 3-volume”} \ .$$

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[1] Science, Technology, Engineering, and Mathematics

The underlying concept which made this possible is the "moment" concept, which is familiar from physics in other contexts.

In order to mathematize this concept geometrically and make it generalizable to 4-d spacetime, Volume II, the ensuing lecture notes, traces the inductive path of reasoning leading to the field equations, and then chews and rechews the "moment of rotation per spacetime 3-volume" in order to integrate it with the already-familiar physics knowledge about the concept of a moment. This fact is already evident from the lecture content as spelled out in the ensuing table of contents.

CONTENTS

(As developed over 41 class periods)

28. Moment of rotation per volume= Einstein tensor [MTW 15.4]. 15.4].
29. Moment of rotation, moment of force, and the Einstein field equations.
30. Einstein's eqations ⇒ conservation of momenergy [MTW ch. 15]; integral form of the Einstein field equations; comparison with integral formulation of Coulomb's law and Ampere's law. Spherically symmetric systems.
31. Einstein's equations for spherically symmetric configurations.
32. Geometrical and matter degrees of freedom.Einstein's equations for spherically symmetric configurations.
33. Helmholtz's theorem.
34. Integration of Einstein's field equation via mass-energy conservation; Mass distribution determines spatial geometry. Inner geometry via imbedded surface; application to the space geometry of a spherical star [MTW 23.8].
35. Simplified Einstein field equations. Equations of hydrostatic equilibrium [MTW ch.23]; equilibrium configurations: stable vs. unstable [MTW ch. 24].
36. Hamilton-Jacobi theory and the principle of constructive interference [MTW Box 25.3]. Constructive interference ⇒ world lines have a finite length determined by Planck's constant. Derivation of Heisenberg's indeterminacy principle.
37. Reconstruction of classical world lines from the principle of constructive interference.
38. Hamilton- Jacobi analysis of the orbits of a particle in the spacetime of a spherically symmetric vacuum configuration [MTW 25.5, Box 25.4].
39. Precession of the perihelion and the deflection of light by the sun [MTW 25.5,25.6].
40. Schwarzschild spacetime: Regular behavior of proper time, proper distance, and curvature at the Schwarzschild radius r=2M [MTW 31.2]; geometry and topology of two assymptotically flat connected regions [MTW 31.6, 31.7].
41. Schwarzschild spacetime: dynamics, causal structure near r=2M; Eddington-Finkelstein coordinates [MTW 31.4, Box 31.2]; Kruskal-Szekeres coordinates [MTW 31.5].
42. Globally defined coordinate system for Schwarzschild spacetime.
43. Scalar, vector, and tensor harmonics, their behavior under parity transformation; geometrical objects on 2-D Lorentz spacetime.

Resource Texts (with Comments)

Physics:

1. E.F. Taylor and J.A. Wheeler, *Spacetime Physics.*, First Edition. Contains a wealth of problems, puzzles, and paradoxes for grasping the nature of the physical world.

2. E.F. Taylor and J.A. Wheeler, *Spacetime Physics.*, Second Edition. Illustrates the thinking method necessary for grasping the nature of the physical world.

3. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *GRAVITATION*, (Freeman, San Francisco, 1973). An Aristotelian (in contrast to a Platonic) approach to physics and mathematics: based on the evidence of the senses, mathematize the physical world in terms of concepts, principles, and theories so powerful that man's unaided faculty of reason can readily grasp the complexities of the universe, *if (s)he chooses to.*

4. J.A. Wheeler, *A JOURNEY INTO GRAVITY AND SPACETIME*, (W.H. Freeman and CoFreeman, New York, 1990). Formulates the Einstein field equations and solves them *without any* differential calculus.

Mathematical Physics:

1. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *GRAVITATION*, (Freeman, San Francisco, 1973). The Track-1 sections of *GRAVITATION* illustrate the fact that mathematics necessarily is the language of physics.

2. J.A. Wheeler, *A JOURNEY INTO GRAVITY AND SPACETIME*, (W.H. Freeman and CoFreeman, New York, 1990). Solves the Einstein field equations using Greek mathematics.

Physical Mathematics:

1. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *GRAVITATION*, (Freeman, San Francisco, 1973). Chapter 15 shows how the principle of energy-momentum conservation is mathematized by the topological principle $\partial\partial\mathcal{V} = 0$ ("the boundary of a boundary is zero").

2. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, (Springer, 1978).

Theoretical Mathematics:

1. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *GRAVITATION*, (Freeman, San Francisco, 1973). Chapter 9 gives an overview of Differential Topology.

2. I.M. Singer and Thorpe, *Lecture Notes on Elementary Topology and Geometry.*

3. N.J. Hicks, *Notes on Differential Geometry*, (Van Norstround, Princeton, N.J., 1964.)

4. T. Apostol, *Mathematical Analysis.*

Lecture 0

How Newton was led to his universal law of gravitation: a road map

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Abstract

We identify the roots and the fundamental premise of Newton's scientific achievements: to grasp the nature of the world, one's thinking must begin with information received from the world. Adopting it, we apply elementary calculus to three pieces of information, Kepler's three laws, to obtain Newton's universal law of gravitation.

1 Newton's fundamental premise and its origin

1.1 The fundamental premise

Q: What was the fundamental premise which paved the way towards Newton's unprecedented achievement?
Why was he successful, while others (like Descarte) were not?

A: Newton stated it thusly:

> $\cdots$ I frame no hypotheses; $\cdots$
> The word "hypothesis" is here used by me to signify only
> (i) such a proposition as is not a phenomenon
> (ii) nor deduced from any phenomena,

> but assumed or supposed – without any experimental proof [whatsoever].

To be more explicit[1], he used "hypothesis" to refer to an *arbitrary* statement, i.e. a claim unsupported by any observational *evidence*. Here are some examples:

(i) The works of Plato are being studied by a reading group of gremlins on the planet Venus (to pick an obvious example).

(ii) Colored light is produced by rotating particles and white light is less produced by nonrotating particles (Descarte).

(iii) White light is a symmetrical wave pulse (Robert Hooke).

(iv) Quarks are composed of strings in a 26-dimensional space (20th century string theorist), etc.[2]

As Wolfgang Pauli would say, none of these statements is right; they aren't even wrong.
Following Newton, what Pauli was directing attention to was that there are three types of claims:

1. Right ones, which are true because they have a positive relationship to reality,
2. false ones, which are untrue because they have a negative relation to reality, and
3. arbitrary ones, for which there is no evidence whatsoever: they are detached from reality.

and it is the arbitrary ones "which aren't even wrong".

Q: What cognitive value, if any, did Newton see in one's contemplation of arbitrary claims?

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[1] Newton did not mean to reject out of hand all hypotheses that lacked full experimental *proof.*

[2] A continuation of this list would include astrology, intelligent design, clairvoyance, ESP, God, an afterlife, reincarnation, and other misintegrations.

A: Newton must be credited with being the first one to identify what in 20th century vernacular is called *Garbage In Gargage Out (G.I.G.O.).* Writing to a friend, he said:

> "If anyone may offer conjectures about the truth of things from the mere possibility of hypotheses, I do not see by what stipulation anything certain can be determined in any science; since one or another set of hypotheses may always be devised which will appear to supply new difficulties. Hence I judged that one should *abstain from contemplating hypotheses, as [one does] from improper argumentation.*"

In other words, one's thinking ("contemplation") should not start with *Garbage,* i.e. arbitrary claims ("hypotheses") because the output, "conjectures about the truth of things," will also be *Garbage,* just as one gets "from improper argumentation" .
Furthermore, as David Harriman puts it[3] , one cannot even achieve the misguided goal of disproving an arbitrary idea. Such a claim can always be shielded by further arbitrary assertions ("one or another set of hypotheses") There is only one way out of such a proliferating web of arbitrary conjectures, and that is to dismiss them outright as uncognitive and unworthy of attention.

This is why Newton insisted that the arbitrary be rejected *without contemplation.*

With this Newton introduced a new epistemological principle into the theory of knowledge:
The outright dismissal of arbitrary claims, without contemplation!
For this principle alone Newton deserves to be regarded as the greatest epistemologist of his era.

Q: What, then, is Newton's fundamental premise stated positively?

A: To grasp *the nature of the world* one's thinking has to start with *information received from the world.*

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[3] *THE LOGICAL LEAP: Induction in Physics*, by D. Harriman, With an Introduction by L. Peikoff (New American Library, New York, N.Y., 2010), page 65.

a) *What* is *the nature of the world?* The world is a causal realm ruled by natural law. It is not a realm of inexplicable miracles ruled by a supernatural power, nor is it an unintelligible chaos ruled by chance. Instead, *the nature of the world* is expressed by the observation that

> "Things are what they are because they were what they were, and things will be what they will be because they are what they are"

This is the law of causality, Aristotle's *law of identity* (everything has a specific nature; things are what they are; A is A)
applied to actions.

b) That one's thinking "start with information received from the world" is the starting point of Aristotle's epistemology, *the evidence of the senses,*

be they aided or unaided by specialized instruments.

1.2 Its origin

Newton did not arrive at his fundamental premise in a cultural vacuum.

Q: What was the frame of reference – the context – that led to the achievements of Newton, those before him, and those after him?

A: Here it is essential to realize that Aristotle, whose works Newton had studied as a college student, may be considered as the cultural barometer of Western History.
Whenever his influence dominated the scene it paved the way towards histories most brilliant eras, whenever it fell so did mankind.

Aristotle's revival in the 13th century brought men to the Renaissance, and the Renaissance led to the Age of Reason, the Enlightenment. Indeed, Galileo was born in the year that Michelangelo died (1564), and Newton was born on the day that Galileo died (1642).

2 Newton's universal law of gravitation

The Enlightenment was ushered in by Newton's unprecedented achievements. There were three of them:

1. His *Opticks*, an inspiration and exemplar of Induction and the Experimental Method.
2. His infinitesimal calculus.
3. His universal law of gravitation.

All three of them illustrate Aristotle's dictum which Newton adopted as his basic premise:

> "To grasp the nature of the world one's thinking has to start with information received from the world".

To grasp the nature of gravitation, Newton's thinking started with information about the dynamics of moving bodies,

$$m \times \overrightarrow{acceleration} = \overrightarrow{Force},$$

applied to the motion of planets as given by Kepler's three laws.

2.1 Kepler's three laws

(1) The radius vector sun-planet sweeps out equal areas in equal times.

(2) The trajectory of each planet is an ellipse with the sun located at one focus,

$$r = \frac{p}{1 - \epsilon \cos\theta} \ .$$

(The parameter p is called the semi-latus rectum of the ellipse. It is the vertical distance from the focus to ellipse. The parameter ϵ is the eccentricity of the ellipse.)

(3) The square of the planets' orbital periods vary as the third power of the major axes of their ellipses:

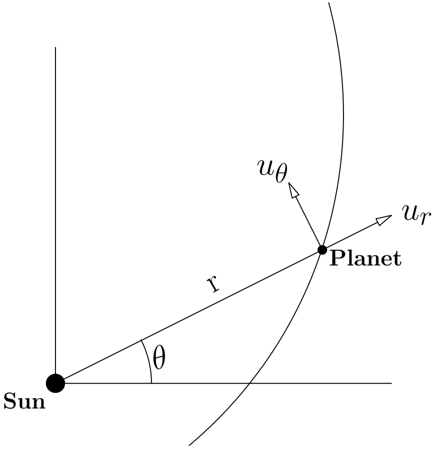
$$\frac{T^2}{a^3} = same\ const.\ for\ all\ planets\ .$$

2.2 Newton's first step: acceleration of a moving body

Using his second law of motion,

$$m\frac{d^2\overrightarrow{R}}{dt^2} = \overrightarrow{F},$$

Newton determined $\overrightarrow{F}$ by evaluating the acceleration along the trajectory of a moving body.



a) Location: $\overrightarrow{R} = r\overrightarrow{u_r}$;

$$\begin{aligned} \overrightarrow{u_r} &= \overrightarrow{i}\cos\theta &+ \overrightarrow{j}\sin\theta \\ \overrightarrow{u_\theta} &= -\overrightarrow{i}\sin\theta &+ \overrightarrow{j}\cos\theta \end{aligned}$$

b) Velocity:

$$\begin{array}{rll|rl} \frac{d\overrightarrow{R}}{dt} & = \frac{dr}{dt}\overrightarrow{u_r} & +r\frac{d\overrightarrow{u_r}}{dt} & \overrightarrow{u_\theta} & = \frac{d\overrightarrow{u_r}}{d\theta} \\ & = \frac{dr}{dt}\overrightarrow{u_r} & +r\overrightarrow{u_\theta}\frac{d\theta}{dt} & \frac{d\overrightarrow{u_\theta}}{d\theta} & = -\overrightarrow{u_r} \end{array}$$

c) Acceleration:

$$\frac{d^2\overrightarrow{R}}{dt^2} = \frac{d^2r}{dt^2}\overrightarrow{u_r} + \overbrace{2\frac{dr}{dt}\overrightarrow{u_\theta}}^{coriolis\,acc'n} \overbrace{-r\overrightarrow{u_r}\left(\frac{d\theta}{dt}\right)^2}^{centripetal\,acc'n} + r\overrightarrow{u_\theta}\frac{d^2\theta}{dt^2} \tag{1}$$

$$= \underbrace{\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]}_{a_r}\overrightarrow{u_r} + \underbrace{\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)}_{a_\theta}\overrightarrow{u_\theta} \tag{2}$$

2.3 Newton's second step: use Kepler's laws

Applying $\overrightarrow{F} = m\overrightarrow{a}$ to Kepler's three laws yields both the direction and the magnitude of the gravitational force on a planet.

2.3.1 Kepler's first law

Equal areas in equal times, $\Delta(\text{area}) \propto \Delta t$, implies $\frac{d(\text{area})}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{c}{2}$. Consequently,

$$a_\theta = 0.$$

Thus the acceleration, Eq.(2), is *purely radial* along the direction sun-planet:

$$\frac{d^2\overrightarrow{R}}{dt^2} = a_r\overrightarrow{u_r}.$$

This is the first key result.

2.3.2 Kepler's second law

Next consider one of Kepler's ellipses having semi-latus rectum p and eccentricity ϵ. Calculate the radial acceleration a_r using Kepler's first and second laws:

$$\begin{array}{llll}
r & = \frac{p}{1-\epsilon\cos\theta} & \Leftarrow \textit{Kepler } (2) & \\
\frac{dr}{dt} & = -\frac{p\epsilon\sin\theta}{(1-\epsilon\cos\theta)^2}\frac{d\theta}{dt} & & \\
 & = -\frac{\epsilon\sin\theta}{p}\underbrace{r^2\frac{d\theta}{dt}}_{c} & \Leftarrow \textit{Kepler } (1) & \\
 & = -\frac{\epsilon\sin\theta}{p}c & \Leftarrow \textit{Kepler } (1) & \\
\frac{d^2r}{dt^2} & = -\frac{c}{p}\epsilon\cos\theta\frac{d\theta}{dt} & & \\
 & = \frac{c}{p}\left(\frac{p}{r}-1\right)\frac{d\theta}{dt} & \Leftarrow -\epsilon\cos = \frac{p}{r}-1 & \Leftarrow \textit{Kepler } (2) \\
 & = \frac{c}{p}\left(\frac{p}{r}-1\right)\frac{c}{r^2} & \Leftarrow \frac{d\theta}{dt} = \frac{c}{r^2} & \Leftarrow \textit{Kepler } (1) \\
r\left(\frac{d\theta}{dt}\right)^2 & = \frac{c^2}{r^3} & \Leftarrow \textit{Kepler } (1) &
\end{array}$$

Subtracting the last two lines, one finds that the radial acceleration a_r in Eq.(2) is

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{c^2}{p}\frac{1}{r^2}\,.$$

Thus, not only is the acceleration, Eq.(2), of a moving planet *purely radial*, but its *magnitude* is inversely proportional to its squared distance, with a constant of proportionality constant (c^2/p) that depends on the square of the planet's areal velocity and the shape of the planetary ellipse.

Question: Is this acceleration the same for all planets?
The answer depends on the orbital periods of the planets.

2.3.3 Kepler's third law

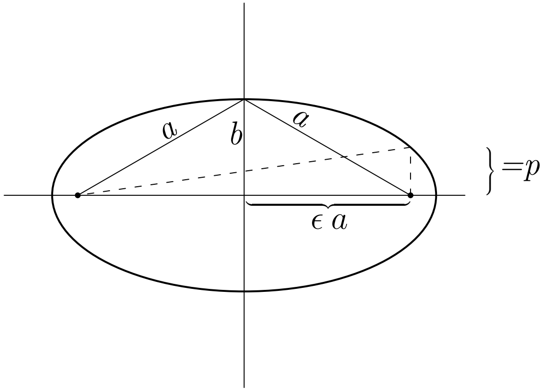
Kepler's first law implies that the orbital period is proportional to the planetary ellipse. This ellipse has major and minor axes a and b. Consequently,

$$\frac{c}{2} = \frac{d(\text{area})}{dt} \Rightarrow \frac{c}{2}T = \text{area} = \pi ab$$

Hence

$$\boxed{c = \frac{2\pi ab}{T}}$$

Q: What is the relation between the semi-latus rectum p and the two axes a and b?



A: Passing through the two foci of the ellipse are its two lati recti ("straight sides"), the vertical chords through the two focal points located at $\pm\epsilon a = \pm\sqrt{a^2 - b^2}$. The size of each latus rectum is $2p$. Thus one has a right triangle whose two sides are $2\epsilon a$ and p, and whose hypotenuse is $2a - p$. Pythagoras tells us that

$$p^2 + (2\epsilon a)^2 = (2a - p)^2.$$

Using $(\epsilon a)^2 = a^2 - b^2$one finds that the semi-latus rectum is

$$\boxed{p = \frac{b^2}{a}\ .}$$

By applying the two boxed expressions to the radial acceleration a_r

$$\begin{aligned} a_r &= -\frac{c^2}{p}\frac{1}{r^2} = -\left(\frac{2\pi ab}{T}\right)^2 \frac{a}{b^2}\frac{1}{r^2} \\ &= -4\pi^2 \frac{a^3}{T^2}\frac{1}{r^2}\ . \end{aligned}$$

Using Kepler's third law, one obtains

$$\boxed{a_r = -\frac{\gamma}{r^2}}$$

where $\gamma = \gamma(M)$ is a constant which is the same for all planets, but which depends on the mass M of of the sun in an as-yet-unspecified way.

2.4 Newton's third step: use his 2nd and 3rd law of motion.

Q: What is the value of that planet-independent constant γ?
Newton answers this question by resorting to his second and third law of motion.

(i) Applying his second law to a planet of mass m,

$$m \times \overrightarrow{(\text{acceleration})} = \vec{F}\ ,$$

one obtains the purely redial gravitational force,

$$\boxed{F_r^{SP} = -\frac{m\gamma(M)}{r^2}}$$

acting on the planet. This is the force with which the sun attracts the planet.

(ii) On the other hand, by his third law there is an equal but opposite force acting on the *sun*,

$$F_r^{PS} = -F_r^{SP} \ ,$$

which is to say that the weight of the planet towards the sun is equal to the weight of the sun towards the planet. Consequently,

$$\frac{M\,\Gamma(m)}{r^2} = \frac{m\,\gamma(M)}{r^2} \ .$$

This equality holds for all pairs of masses m and M. Consequently,

$$\boxed{F_r = -\kappa \frac{Mm}{r^2} \ .}$$

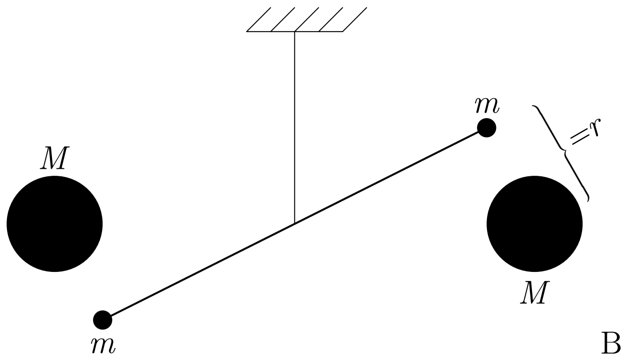
Here κ, *Newton's gravitational constant*, is a universal constant independent of M and m. The boxed equation is a mathematical statement of Newton's universal law of gravitation.

2.5 The Cavendish Experiment (1789)

The universal constant κ has the value

$$\kappa = \frac{1}{15\,000\,000} \left[\frac{cm^3}{gr\ sec^3}\right]$$

in c.g.s. units. This constant is determined by measuring the attractive force between masses separated by a known distant r. One suspends two small masses m from a torsional balance.



By bringing large masses M to each of the masses m, Cavendish measured the gravitational force F_r by measuring the angular deflection of the pendulum.

Newton's third law of motion, "For every action there is an equal and opposite reaction", applied to his law of gravitation, implies that the weight of an apple attracted by the earth's gravity equals the weight of the earth attracted by the gravity of the apples. This equality determines the mass of the earth once the weight of the apple and its distance from the (center of the) earth have been measured.

LECTURE 1

The concept "Gravitation": Where does it come from?

A. Gravitation's observational basis

B. Mathematization of gravitation according to

I. Galileo

II. Kepler

III. Newton

IV. Lagrange & Hamilton

V. Einstein

The theme of Math 5757 is to grasp the nature of the world, in particular, the existence and the nature of the concept of "gravitation", which is the subject in this theme.

A.) To develop this, our thinking must start with information received from the world.

In the case of gravitation this information is in the specific form of the laws of motion of bodies. It is precisely in terms of the observed motion of bodies that one arrives at the concept "gravitation."

More generally, the concept "gravition" is the (mental) product of the integration of a constellation of concepts each one which is formed by a process of "measurement omission". \footnote{ The process of concept formation is a process of "measurement omission". It is explained on pages 11–18 of chapter 2 ("Concept Formation") in "Introduction to Objectivist Epistemology" by Ayn Rand; also summarized in the Q&A on pages 137–139.}

The formation of the concept "gravitation" starts out by observing various kinds of motions of bodies and then singling out a particular type which is different from all the rest, e.g., the motion of the moon, or a falling apple as compared to pushing a ball makes it move. The common features which particularizes that type of motions are the characteristic imprints, the signature that gravitation imparts to the motion of bodies.

B) Historically, what nowadays is identified as the gravitation phenomenon has been the chief motivating force for the mathematization (= "mathematical formulation") of the laws of motion. This is because they are the premier tool for identifying what gravitation is. It was Galileo, Kepler, Newton, Einstein and others who; each in their own way mathematizes gravition in their own way.

I. **Galileo:** ("E pur si moeve" ⟶ "And yet it moves")

"Independence of Horizontal and Vertical Motion".

Using <u>the experimental method</u> in studying the motion of a projectile, Galileo found its horizontal and its vertical motion are independent of each other.

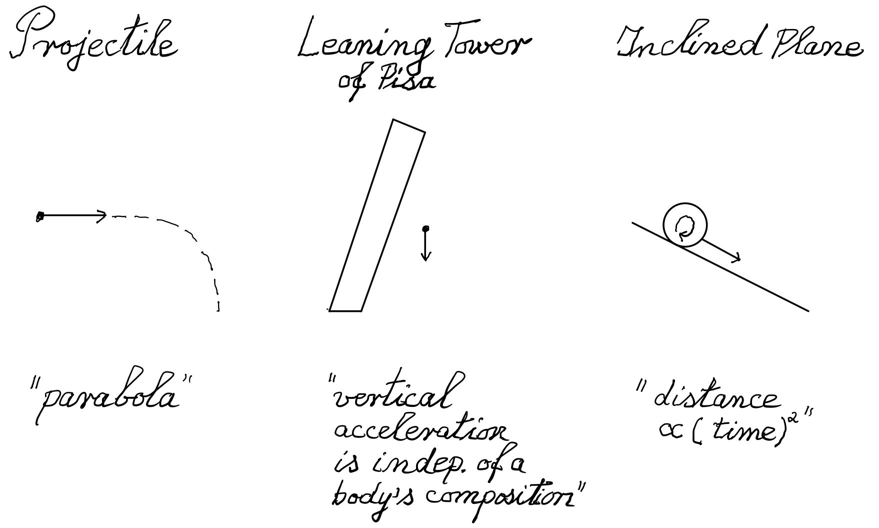


Figure 1.1: Galileo's mathematizations of the motion of bodies.

The independence of these motions are mathematized by the two statements:

1. Horizontal velocity = const.

2. Vertical acceleration = const.

   a) Vertical velocity = $g \times$ time

   b) Vertical distance = $\frac{1}{2} g \cdot (\text{time})^2$

Comment 1.1

1. The fact that the horizontal velocity is constant is a special case of the law of inertial motion of bodies ("Newton's 1st law of motion").

2. Although the concept <u>gravitation</u> was not yet known to Galileo, he identified its <u>imprint</u> on the motion of bodies by the proportionality constant g between the vertical velocity and the time of travel of the body (= point particle).

II. Kepler: "Kepler's 3 laws of planetary motion."

From Tycho Brahe's direct observations Kepler, by inductive reasoning ("from the particular to the general"), mathematized the motion of planets (and moons) into his three laws.

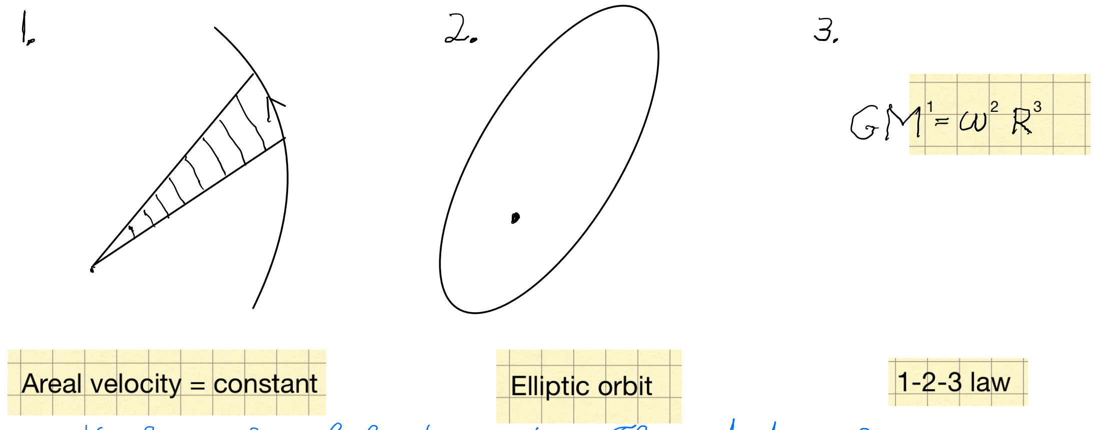


Figure 1.2: Kepler's 3 laws of planetary motion. The constant G in his 1-2-3 law is Newton's gravitational constant, $G = \frac{1}{15,000} \frac{m^3}{(kg)(sec)^2} = \frac{1}{15,000,000} \frac{cm^3}{(g)(sec)^2}$.

1. The radius vector sun-planet sweeps out equal areas in equal times

2. Planets travel in ellipses, with the sun at one of their foci.
3. $G \times (\text{Mass of the Sun}) = \left(\frac{2\pi}{\text{period}}\right)^2 \times (\text{major axis})^3$ is Kepler's "1-2-3 Law", namely

$$GM' = \omega^2 R^3 \qquad (1.1)$$

Comment: Without Kepler's Herculean work of extracting his 3 mathematical laws from Tycho Brahe's astronomical observations, it is doubtful that Newton could have formulated his law of universal gravitation in such short order. (See Lecture 0 for how he obtained his law from Kepler's 3 laws.)

III. $\boxed{\text{Newton}}$ "Universal Law of Gravitation."

(Galileo's motion of bodies) + (Local vector calculus) + $\left(\vec{F} = m\frac{d^2\vec{x}}{dt^2}\right)$ + (Kepler's 3 laws of planetary motion) + + (observational data about planets, comets, moons, cannon balls, apples, etc.)

implies

$$m_{\text{inertial}} \frac{d^2\vec{x}}{dt^2} = -m_{\text{grav'l}} \frac{GM}{r^3}\vec{x} \qquad (1.2)$$

Exercise (<u>Newton's law of gravitation</u>)

Show that

$$\{\text{Kepler's 3 laws}\} \begin{smallmatrix} a) \Longrightarrow \\ b) \Longleftarrow \end{smallmatrix} (\text{grav'l force field}) = \frac{GM}{r^3}\vec{x}$$

Comment 1.2

Showing the implication "⇒" entails only differential calculus (See P 5-10 of Lecture 0 \footnote{ Being endemic to a cultured physicist or mathematician, this is also done in OSU's Math 1181H and 4551.}.)

Comment 1.3

Showing the implication "⇐" is more challenging mathematically. This is because it entails integral calculus.

Comment 1.4

By introducing his 2nd Law,

$$\frac{d}{dt}\left(m\frac{d\vec{x}}{dt}\right) = \vec{F},$$

Newton accomplished several feats in one fell swoop:

(i) He introduced a new concept, the inertial mass of a body.

(ii) Whereas Kepler and Galileo mathematized motion in terms of global geometrical figures (ellipses, parabolas, etc.), Newton, having introduced the concept of mass and $F = ma$, did so in terms of locally defined differential equations such as the boxed Eq. (1.2) on page 1.4, or more generally

$$\boxed{m_{\text{inertial}} \frac{d^2\vec{X}}{dt^2} = -m_{\text{grav'l}}\, G \sum_i M_i \frac{\vec{X}-\vec{X}_i(t)|}{|\vec{X}-\vec{X}_i(t)|^3}} \qquad (1.3)$$

whenever there are several gravitation forces due to masses M_1 at $\vec{x}_1(t)$, M_2 at $\vec{x}_2(t)$, ...

(iii) Newton gave a local definition of acceleration by means of a <u>double limiting</u> process applied to differential equations.

Comment 1.5

Q: What is the difference between Newton's contribution to our understanding of the motion of bodies and that due to Kepler and Galileo?

A: Galileo and Kepler's geometrical figures mathematize the motion of bodies <u>kinematically</u>, i.e. <u>without</u> any references to their masses. By contrast, Newton's equations for the motion of bodies mathematize them <u>dynamically</u> <u>in terms</u> of their <u>masses</u>.

IV. Euler, Lagrange, Hamilton

The time interval between Newton and Einstein (17th, 18th, and 19th century) was marked by the development of the <u>"Hamilton's Principle"</u> of least action by Euler, Lagrange, and Hamilton. This principle uses the calculus of variations to replace the task of setting up Newton's vectorial equations of motion with the much easier task of extremizing a scalar integral, the "action" of the mechanical system,

$$\int (K.E. - P.E.)\, dt = \text{extremum!} \qquad (1.4)$$

with the implication that

$$\delta\left\{\int (K.E. - P.E.)\, dt\right\} = 0 \qquad (1.5)$$

The main virtue of this formulation of the classical laws of motion is that the action of a mechanical system (i) is a scalar and (ii) that the extremum of this scalar is <u>independent of</u>

the choice of coordinates used to describe the mechanical system. If one reexpresses the Lagrangian K.E.–P.E. in terms of different coordinates, then the resulting Euler-Lagrange equations of motion (whose solution extremizes the action) still describes the same mechanical system, but relative to the new set of coordinates.

Q1: What is the physical origin of Hamilton's Principle as formulated by Lagrange?

A1: The observation-based reasoning leading to this principle is given on P1-7 of the attached Appendix "Lagrangian Mechanics and the 3-Body Problem.

Q2: Is it possible to give a non-trivial application of this principle?

A2: Yes. Pages 9-25 in the attached Appendix develop the theory of the (i) "Restricted 3-Body Problem" and (ii) the motion a charged particle in the crossed electric-magnetic field of a magnetron, which is used in radar and microwave ovens.

V. Einstein

By the time Einstein started examining the concept "gravitation", he had at his disposal, and then made excellent use of, the highly developed art of analytical mechanics as formulated by Lagrange and Hamilton.

In 1913 he took a key step. Using Hamilton's principle of least action, he equated Hamilton's action integral to the coordinate frame independent length

$$\int d\tau = \int \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu} \qquad (1.6)$$

of a worldline between two events, and then pointed out that the metric tensor-field

$$g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu \qquad (1.7)$$

is where gravitation stamps its imprints, i.e. characterizes the gravitational field.

Thus instead of using Eq. (1.2) on page 1.4 to mathematize gravitation's imprints on the motion of bodies, Einstein applied Hamilton's variational action principle, Eq. (1.5) on page 1.5, to

$$\delta\left\{\int\sqrt{-g_{\mu\nu}(x)\,dx^\mu dx^\nu}\right\}=0\,. \tag{1.8}$$

He thus obtained

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0\,,\quad \mu=0,1,2,3\,. \tag{1.9}$$

These, we shall see,

(a) reduce to Newton's Eq. (1.2) on page 1.4,

$$m_{\text{inertial}}\frac{d^2\vec{x}}{d(ct)^2}+m_{\text{grav'l}}\vec{\nabla}\frac{\phi_{\text{grav'l}}(\vec{x},t)}{c^2}=0\,, \tag{1.10}$$

in terms of the Newtonian gravitational potential $\phi_{\text{grav'l}}(\vec{x},t)$ \footnote{The units of $\phi_{\text{grav'l}}$ are energy/mass, so that $\frac{\phi_{\text{grav'l}}}{c^2}$ is dimensionless.} and

(b) express the imprints of gravitation on the motion of bodies in the form of geodesic states of motion on a spacetime manifold with a metric tensor field, and hence

(c) mathematizes gravitation in geometrical terms.

LECTURE 2

World lines of extremal length

0. The Twin Paradox
I. Extremal length: WHY?
II. Generalization
III. The Variational Principle

LECTURE 2

IV. Parametrization Invariance
  1. Noether's Theorem Illustrated
  2. Underdetermined System
V. Torsionless metric-compatible parallel transport
VI Constant of motion

consigned to LECTURE 3

Read § 13.4 (Geodesics as world lines of extremal proper time); for a a relevant review read Section 13.3, especially the six conclusions at its end.

0. The Twin Paradox: A Reminder

2.1

The twin "paradox" is based on comparing two (biological or any other clocks in relative motion.

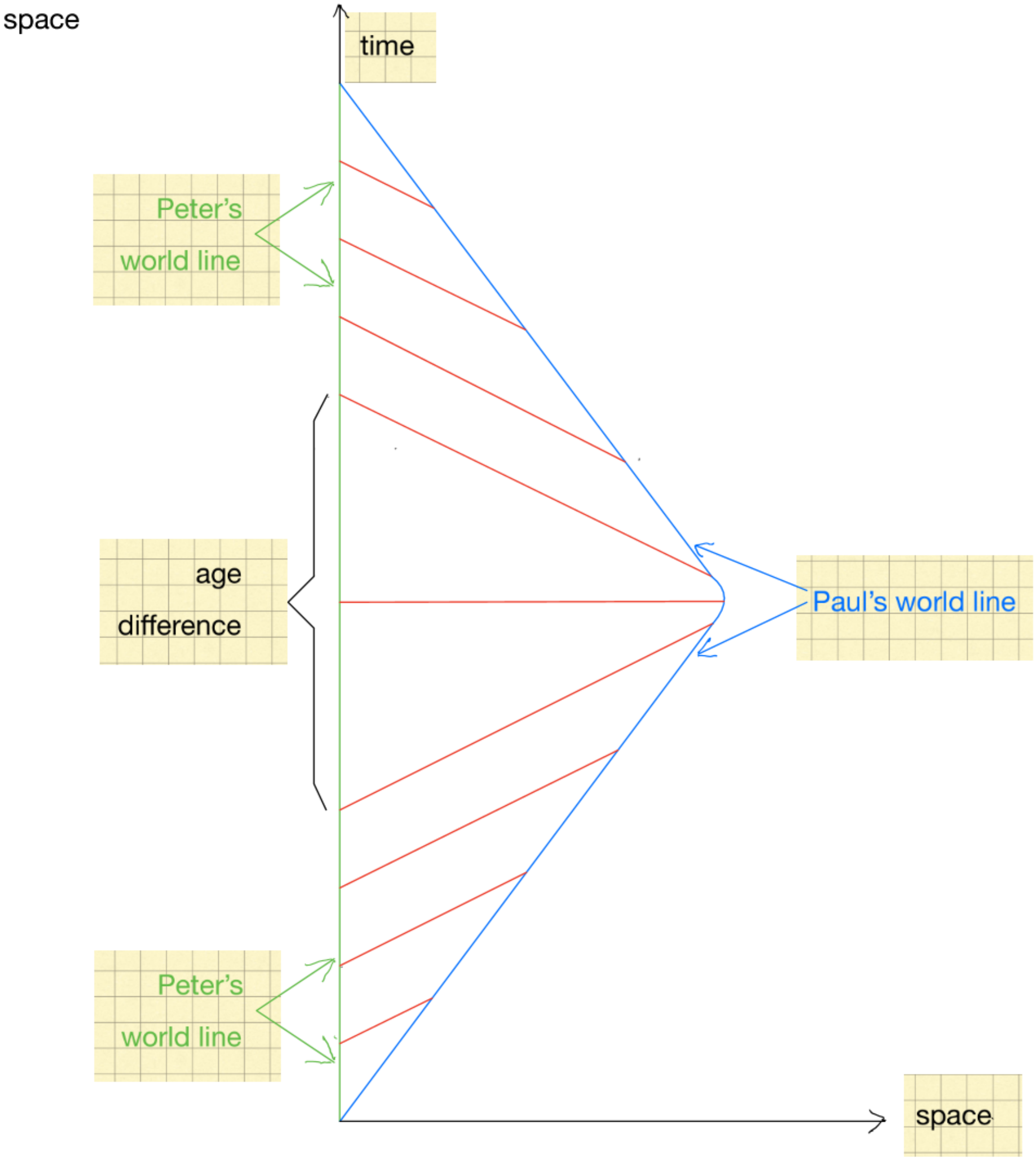


Figure 2.1: The age difference between Peter and Paul is due to the fact that Paul's lattice work of clocks records a very rapid aging in Peter during Paul's deceleration and acceleration process to be reunited with Peter. The red locus of events marks those which are simultaneous with the tickings of Paul's clock along his blue world line.

By biological standards pi-mesons, charged and neutral, have a very short life time. In their own comoving frames their life times are

$\pi^{\pm}$ life time = 2.6×10^{-8} sec

π° life time = $8 \cdot 10^{-17}$ sec

However, these particles, when created in the upper atmosphere or by an accelerator, are usually born with very high observable velocities.

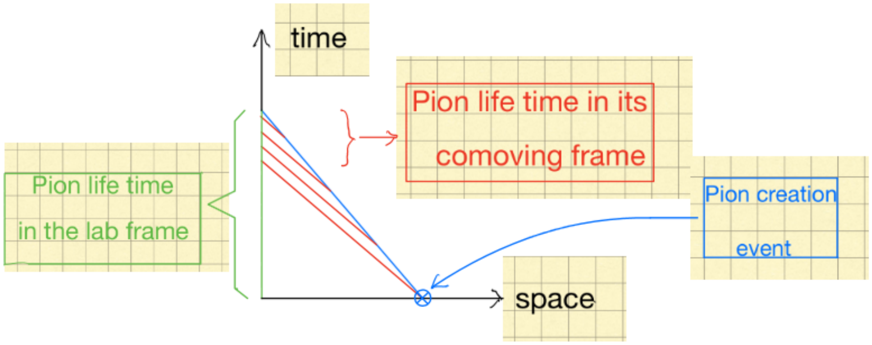


Figure 2.2 A π-meson, when created with a LAB velocity comparable to that of light, has a LAB life time considerably longer than its proper life time in its comoving frame. This time dilation relative to the LAB increases the π-meson's travel distance by a correspondingly larger amount.

Assume that the particle velocity is $v = .995c$ so that $\gamma = \frac{1}{\sqrt{1-(.995)^2}} = 10$.
The π-meson travel distance Δx predicted within the Newtonian framework is quite different from that of its relativistic extension.

$$\pi^{\pm}: \begin{cases} \Delta X_{\text{NEWTON}} = (2.6 \times 10^{-8}\,\text{sec}) \times .995 \times 3 \times 10^{10} = 7.8 \times 10^{2}\,\text{cm} = 7.8 \text{ meter} \\ \Delta X_{\text{RELAT.}} = (2.6 \times 10^{-8}\,\text{sec}) \times \gamma \times .995 \times 3 \times 10^{10} = 78 \times 10^{2}\,\text{cm} = 78 \text{ meters} \end{cases}$$

$$\pi^{0}: \begin{cases} \Delta X_{\text{NEWTON}} = (8 \times 10^{-17}\,\text{sec}) \times .995 \times 3 \times 10^{10} = 2.4 \times 10^{-6}\,\text{cm} \\ \Delta X_{\text{RELAT.}} = (8 \times 10^{-17}\,\text{sec}) \times \gamma \times .995 \times 3 \times 10^{10} = 2.4 \times 10^{-5}\,\text{cm} \end{cases}$$

Hamilton's Principle, the Twin "Paradox", and Geodesics as Word Lines of extreme Length.

I. THE WHY OF EXTREMAL LENGTH

In a Lorentz frame it is easy to distinguish a straight line from one which is not. Compare a "broken" world line with a straight one, both starting at (0,0) and finishing at (0,T).

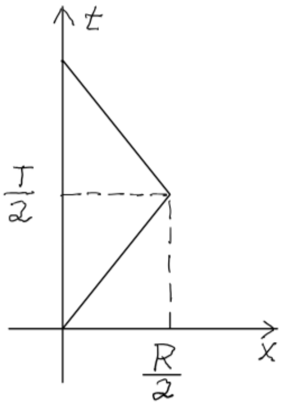


Figure 2.3 Straight vs. broken world line.

2.4

Q: What is the amount of elapsed proper time along each of these world line?

A: Along the broken world line:

$$\tau = 2\sqrt{\left(\frac{T}{2}\right)^2 - \left(\frac{R}{2}\right)^2} = \sqrt{T^2 - R^2}$$

Along the straight world line:

$$\tau = T$$

Comment 2.1: This illustrates the twin paradox: an inertial spacetime observer ages more than one whose world line is not straight.

Conclusion: The proper time along a straight world line is a maximum in relation to time along nearby broken world lines.

II. GENERALIZATION

This conclusion generalizes to the case where one compares multiply broken world lines with a straight line in a Lorentz frame.

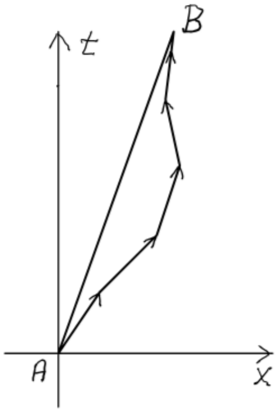


Figure 2.3 Straight vs. multiply broken world line.

In that circumstance one has

$$(a) \qquad \tau = \int_A^B d\tau = \int_A^B \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \left(\begin{array}{c} \text{maximum for a straight line} \\ \text{compared to any } \underline{\text{variant}} \\ \text{of the straight line} \end{array} \right)$$

This maximum principle holds for any Lorentz frame, even if one choses to introduce curvilinear coordinates

$$(b) \qquad \boxed{\begin{aligned} \tau_A^B &= \int_A^B d\tau = \int_A^B \sqrt{-g_{\mu\nu}\, dx^\mu dx^\nu} \\ &= \left(\begin{array}{l} \text{an extremum for a time-like} \\ \text{worldline that is straight in} \\ \text{each local Lorentz frame} \\ \text{along its path, as compared to} \\ \text{any nearby variant of this} \\ \text{world line} \end{array} \right) \end{aligned}} \qquad (2.1)$$

In a single Lorentz frame the introduction of curvilinear coordinates is optional. However, if one considers a world line passing through a sequence of distinct Lorentz frames, then the use of curvilinear coordinates is mandatory.

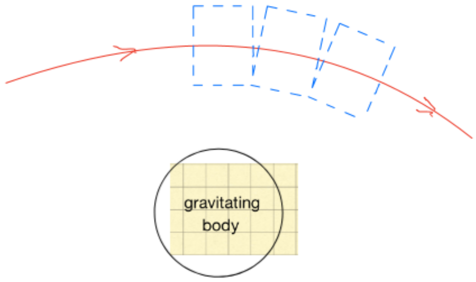


Figure 2.4 Trajectory of a particle passing through a sequence of distinct local Lorentz frames in the neighborhood of a gravitating body.

Note that in a mandatory curvilinear scenario we have replaced the "maximum" condition with an "extremum" condition. This is because there may be more than one locally straight worldline connecting the two events A and B.

III. THE VARIATIONAL PRINCIPLE

In order to determine the consequences of this variational principle, compare a locally straight world line with one of its general variants

Definition of "variant": Different in form from others of its kind.

I

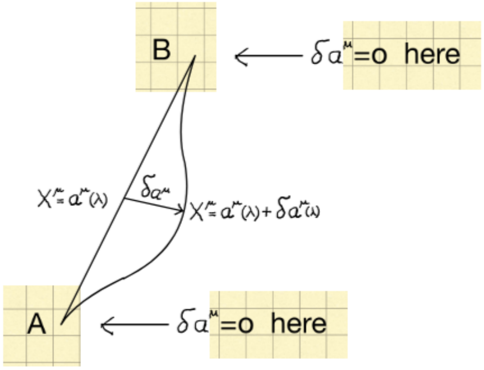


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Figure 2.5 Extremal curve $x^\mu=a^\mu(\lambda)$ and its variant $x^\mu=a^\mu(\lambda)+\delta a^\mu(\lambda)$, both passing through the same initial point event A and the final point event B.

C

Let $x^\mu = a^\mu(\lambda)$ be the world line which passes though paint events A and B and which extremizes Eq. (2.1) on page 2.5. This implies that for any variant $x^\mu = a^\mu(\lambda) + \delta a^\mu(\lambda)$ passing through the same pair of events, A and B, the integral $\tau_A{}^B$ must satisfy

$$\delta \tau_A{}^B \equiv \tau_A{}^B [a^\mu + \delta a^\mu] - \tau_A{}^B [a^\mu] = 0 \qquad (2.2)$$

to first order accuracy in δa^μ. Here

$$\tau_A{}^B [x^\alpha] = \int_0^1 \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda \qquad (2.3)$$

for both

$$x^\mu = a^\mu(\lambda)$$

and its variant

$$x^\mu = a^\mu(\lambda) + \delta a^\mu(\lambda),$$

whose variation $\delta a^\mu(\lambda)$ vanishes at the end points A and B:

$$\delta a^\mu(0) = \delta a^\mu(1) = 0 \qquad \mu = 0, 1, 2, 3, \qquad (2.4)$$

but is otherwise arbitrary.

The extremum condition Eq. (2.2) is one which is necessary for the worldline $x^\alpha = a^\alpha(\lambda)$ to be optimal. This means that it is satisfied by Eq. (2.1) on page 2.5.

In order to use Eq. (2.2), expand the integrand $I\left(a^\alpha + \delta a^\alpha; \frac{d}{d\lambda}(a^\mu + \delta a^\mu)\right) - I\left(a^\alpha; \frac{d}{d\lambda}(a^\mu)\right)$ under $\int_0^1 (\cdots)\, d\lambda$ in Eq. (2.2),

$$\Delta I \equiv \left[g_{\mu\nu}(a^\alpha(\lambda) + \delta a^\alpha(\lambda)) \frac{d(a^\mu + \delta a^\mu)}{d\lambda} \frac{d(a^\nu + \delta a^\nu)}{d\lambda} \right]^{1/2} - \left[g_{\mu\nu}(a^\alpha(\lambda)) \frac{da^\mu}{d\lambda} \frac{da^\nu}{d\lambda} \right]^{1/2},$$

in a power series in $\delta a^\alpha(\lambda)$, but retain only its Principal Linear Part.

The result is

$$\begin{aligned}\Delta I(\lambda)=\Big[-g_{\mu\nu}(a^\alpha)\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}&-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta a^\alpha\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}\\ &-g_{\mu\nu}(a^\alpha)\frac{d\,\delta a^\mu}{d\lambda}\frac{da^\nu}{d\lambda}\\ &-g_{\mu\nu}(a^\alpha)\frac{da^\mu}{d\lambda}\frac{d\,\delta a^\nu}{d\lambda}\Big]^{1/2}-\Big[-g_{\mu\nu}(a^\alpha)\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}\Big]^{1/2}\end{aligned}$$

$$=\frac{-\frac{1}{2}g_{\mu\nu}\frac{d(\delta a^\mu)}{d\lambda}\frac{da^\nu}{d\lambda}-\frac{1}{2}g_{\mu\nu}\frac{da^\mu}{d\lambda}\frac{d(\delta a^\nu)}{d\lambda}-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\delta}\delta a^\delta\frac{\partial a^\mu}{\partial\lambda}\frac{\partial a^\nu}{\partial\lambda}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \qquad (2.5)$$

Because of Eq.(2.5) the extremum condition, Eq(2.2) on page 2.7 is

$$\delta\tau_A{}^B[a^\alpha]=\int_0^1\Delta I(\lambda)\,d\lambda=\int_0^1\frac{-\frac{1}{2}g_{\mu\nu}\frac{d(\delta a^\mu)}{d\lambda}\frac{da^\nu}{d\lambda}-\frac{1}{2}g_{\mu\nu}\frac{da^\mu}{d\lambda}\frac{d(\delta a^\nu)}{d\lambda}-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\delta}\delta a^\delta\frac{\partial a^\mu}{\partial\lambda}\frac{\partial a^\nu}{\partial\lambda}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\,d\lambda=0 \qquad (2.6)$$

The change as exhibited after the second equality sign of Eq.(2.6) is only the pricipal part linear in δa^γ of the change exhibited after the first equality sign; the contributions of quadratic and higher orders in δa^γ have been suppressed.

This is because the focus of interest is only on the necessary condition for $\tau_A{}^B[a^\alpha]$ to be an extremum, and not on what kind of extremum $\tau_A{}^B[a^\gamma]$ is.

The successful completion of this partial calculation depends on isolating δa^μ from its derivative. This is achieved by partial integration of the 1st two terms and using the fact that $\delta a^\mu = 0$ at the endpoints. The result is

$$\delta \tau_A{}^B = \frac{1}{2}\int_0^1 \left\{ \frac{d}{d\lambda}\left(\frac{g_{\mu\nu}\frac{da^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \right)\delta a^\mu + \frac{d}{d\lambda}\left(\frac{g_{\mu\nu}\frac{da^\mu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \right)\delta a^\nu - \frac{\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\,\delta a^\gamma \right\} d\lambda$$

$$= \int_0^1 f_\gamma(\lambda)\, \delta a^\gamma \sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}\, d\lambda \qquad (2.7) \qquad \text{2.10 a}$$

where

$$f_\gamma(\lambda) = \frac{\frac{1}{2}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\left[\frac{d}{d\lambda}\left(\frac{g_{\gamma\nu}\frac{da^\nu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\right) + \frac{d}{d\lambda}\left(\frac{g_{\gamma\nu}\frac{da^\mu}{d\lambda}}{\sqrt{g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}}\right)\right] - \frac{1}{2}\frac{\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{da^\mu}{d\lambda}\frac{da^\nu}{d\lambda}}{\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}} \qquad (2.8)$$

An extremum is achieved when

$$\boxed{f_\gamma(\lambda) = 0} \qquad \gamma = 0, 1, 2, 3 \qquad (2.9)$$

These equations mathematize the neccessary condition for the existence of a world line of maximal proper time running through point events A and B.

COMMENT:

At first sight Eqs (2.8) and (2.9) seem to have a daunting complexity. However, a second look reveals that they have a symmetry which not only implies an important mathematical conservation law but also reveals the key geometrical attribute of the world lines. This is because of the extremum principle on which they are based.

IV. THE EULER-LAGRANGE EQUATION

The line of reasoning **threading** pages 2.7 to 2.10a, and culminating in Eq. (2.9) illustrates the logic at the base of the Euler-Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right)-\frac{\partial L}{\partial x^\gamma}=0$$

of variational calculus. The starting point for both equations is the to-be-extremized integral

$$\begin{aligned}\tau_A{}^B &= \int_{\lambda_A=0}^{\lambda_B=1}\sqrt{-g_{\mu\nu}(x^\gamma(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda\\ &=\int_0^1 L(\dot{x}^\mu(\lambda), x^\gamma(\lambda))\,d\lambda\end{aligned}$$

Its first order variation is

$$\begin{aligned}\delta\tau_A{}^B &= \int_0^1\left\{L(\dot{x}^\mu(\lambda),x^\gamma(\lambda))\Big|_{x^\gamma=a^\gamma+\delta a^\gamma}-L(\dot{x}^\mu(\lambda),x^\gamma(\lambda))\Big|_{x^\gamma=a^\gamma}\right\}d\lambda \qquad (*)\\ &=\int_0^1\left\{-\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right)+\frac{\partial L}{\partial x^\gamma}\right\}\Big|_{x^\gamma=a^\gamma}\,\delta a^\gamma(\lambda)\,d\lambda. \qquad (**)\end{aligned}$$

The expression in Eq. (**) is the Principal Linear Part (P.L.P.) of the difference as displayed by Eq. (*). It is the dominant part obtained by neglecting all non-linear terms in the series expansion of Eq. (*) in powers of δa^γ. The result is Eq. (**), namely

$$\delta\tau_A{}^B=\int_0^1 f_\gamma(\lambda)\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}\,\delta a^\gamma(\lambda)\;d\lambda,$$

which is Eq. (2.7) on page 2.10a.

The fact that $\{a^\gamma(\lambda)\}$ extremizes the variational integral implies that the PLP, Eq. (**), must vanish for arbitrary variation $\delta a^\gamma(\lambda)$. Consequently,

$$\left\{-\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right)+\frac{\partial L}{\partial x^\gamma}\right\}\Big|_{x^\gamma=a^\gamma}=0,$$

This is the Euler-Lagrange equation, i.e.

$$f_\gamma(\lambda)\sqrt{-g_{\alpha\beta}\frac{da^\alpha}{d\lambda}\frac{da^\beta}{d\lambda}}=0$$

for the variational integral $\tau_A{}^B$.

LECTURE 3

World lines of extremal length

LECTURE 2
- 0. The Twin Paradox
- I. Extremal length: WHY?
- II. Generalization
- III. The Variational Principle

LECTURE 3
- IV. Parametrization Invariance
  - 1. Noether's Theorem Illustrated
  - 2. Underdetermined System
- V. Torsionless metric-compatible parallel transport
- VI Constant of motion

V. PARAMETRIZATION INVARIANCE

1.) Noether's Theorem

Before engaging in a mathematical frontal attack on Eqs. (2.9), it is extmely rewarding to take note of the fact that the variational integral, Eq(2.3), is invariant under a transformation of its curve parameter. This invariance permits one to master the apparent complexity of Eqs. (2.8) and (2.9).

Indeed, let $\lambda \to \bar\lambda = \bar\lambda(\lambda)$, $\bar\lambda \to \lambda = \lambda(\bar\lambda)$ its inverse, with $\bar\lambda(\lambda=0)=0$ and $\bar\lambda(\lambda=1)=1$. Consider the new reparametrized trajectory $x^\alpha = \bar a^\alpha(\lambda) = a^\alpha(\bar\lambda(\lambda))$ (2.10) instead of the old one,

$$x^\alpha = a^\alpha(\lambda). \qquad (2.11)$$

The value of the variational integral Eq. (2.3) on page 2.7 is

$$\mathcal{T}_A{}^B[\bar a(\cdot)] = \int_{\lambda=0}^{\lambda=1} \sqrt{-g_{\mu\nu}\left(a^\alpha(\bar\lambda(\lambda))\right) \frac{d\,a^\mu(\bar\lambda(\lambda))}{d\lambda} \frac{d\,a^\nu(\bar\lambda(\lambda))}{d\lambda}}\, d\lambda \qquad (2.12)$$

$$= \int_0^1 \sqrt{-g_{\mu\nu}\left(a^\alpha(\bar\lambda)\right) \frac{da^\mu(\bar\lambda)}{d\bar\lambda} \frac{d\bar\lambda}{d\lambda} \frac{da^\nu(\bar\lambda)}{d\bar\lambda} \frac{d\bar\lambda}{d\lambda}}\, d\lambda$$

$$= \int_{\bar\lambda=0}^{\bar\lambda=1} \sqrt{-g_{\mu\nu}\left(a^\alpha(\bar\lambda)\right) \frac{d\,a^\mu(\bar\lambda)}{d\bar\lambda} \frac{d\,a^\nu(\bar\lambda)}{d\bar\lambda}}\, d\bar\lambda = \mathcal{T}_A{}^B(a(\cdot)) \qquad (2.13)$$

Thus the value the variational integral is independent of all worldlines connecting point events A and B in Figure 2.5, including those which are not extremal spactime trajectories.

One says that the integral is invariant under the transformation that takes a^α into $\bar{a}^\alpha$.

This transformation is one which is "global" in that it is a statement about any (extremal or non-extremal) A-to-B trajectory as a whole.

The benefit derived from this observation is that, via variational calculus (which led to Eqs. (2.9)), it implies a differential identity which holds pointwise along any curve in the domain where it is defined. This identity, which is Eq. (2.17) below on page 2.17, implies the "conservation law"

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)=0, \qquad (2.14)$$

i.e. $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ is a constant along any trajectory whenever it is parametrized by the proper time τ.

The transformation invariance of the variational integral Eqs. (2.12)-(2.13) on page 2.11 implies the corresponding "conservation law", Eq. (2.14). This conclusion is an illustration of what is known as Noether's Theorem.

2.) Underdetermined System

(a) The equations $\{f_\gamma(\lambda)=0:\gamma=0,1,2,3\}$, namely Eqs. (2.9) on page 2.10a, constitute an underdetermined system. For any solution $a^\alpha(\lambda)$ to $f_\gamma(\lambda)=0$ there are many other solutions. Indeed, for an arbitrary function $\lambda(\bar\lambda)$, in Eq.(2.7) on page 2.9 one finds that

$$f_\gamma(\lambda(\bar\lambda)) = f_\gamma(\bar\lambda).$$

Consequently, if $a^\alpha(\lambda)$ is a solution to $f_\gamma(\lambda)=0$, then $a^\alpha(\lambda(\bar\lambda)) \equiv \bar a^\alpha(\bar\lambda)$ is also a solution to $f_\gamma(\bar\lambda)=0$.

(b) The equations $f_\gamma(\lambda)=0$ on page 2.9 constitute mathematical overkill. There are more of them than necessary to express the extremal nature of the variational integral

2.15

$$\tau_A{}^B[X^\alpha] = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda . \qquad (2.15)$$

In other words, one of these equations holds for all world lines, even those that do not extremize $\tau_A{}^B$.

This fact follows from its invariance under reparametrization. The reparametrization

$$\lambda \to \bar\lambda : \ \bar\lambda(\lambda) = \lambda + h(\lambda) \qquad (2.16)$$

subject to

$$\left.\begin{aligned}\bar\lambda(0)&=0\\ \bar\lambda(1)&=1\end{aligned}\right\} \Longleftrightarrow h(0)=h(1)=0$$

does not change the value of the variational integral.

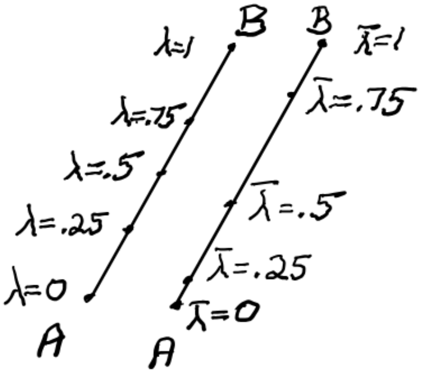


Figure 2.6 Two different parametrizations of the worldline AB.

It corresponds to a mere "repositioning of the beads along a string" (=reparametrization). This process is mathematized by the statement

$$\bar{a}^\alpha(\lambda) = a^\alpha(\bar{\lambda}(\lambda))$$

Apply this to Eq. (2.15) on page 2.15 and find that the value of the variational integral,

$$\tau_A{}^B[\bar{a}(\cdot)] = \int_{\lambda=0}^{\lambda=1} \sqrt{-g_{\mu\nu}\,(a^\alpha(\bar{\lambda}(\lambda)))\,\frac{d\,a^\mu(\bar{\lambda}(\lambda))}{d\lambda}\,\frac{d\,a^\nu(\bar{\lambda}(\lambda))}{d\lambda}}\,d\lambda = \int_{\bar{\lambda}=0}^{\bar{\lambda}=1} \sqrt{-g_{\mu\nu}(a^\alpha(\bar{\lambda})\}\,\frac{d\,a^\mu(\bar{\lambda})}{d\bar{\lambda}}\,\frac{d\,a^\nu(\bar{\lambda})}{d\bar{\lambda}}}\;d\bar{\lambda}$$

$$= \int_{\lambda=0}^{\lambda=1} \sqrt{-g_{\mu\nu}(a^\alpha(\lambda)\}\,\frac{d\,a^\mu(\lambda)}{d\lambda}\,\frac{d\,a^\nu(\lambda)}{d\lambda}}\;d\lambda = \tau_A{}^B(a(\cdot)),$$

has not changed, even though $\bar{\lambda}(\lambda)$ is not the identity function.

The first order change in $a^\gamma(\lambda)$ brought about by the reparametrization (2.16) on page 2.15 is

$$a^\gamma(\lambda) \longrightarrow a^\gamma(\bar{\lambda}) = a^\gamma(\lambda + h(\lambda)) = a^\gamma(\lambda) + \delta a^\gamma(\lambda)$$

where

$$\delta a^\gamma(\lambda) = \frac{da^\gamma}{d\lambda} h(\lambda) \qquad \left(\text{“Principal linear part”}\right)$$

and higher order terms have been neglected.

The fact that such variations can not change the variational integral for arbitrary $h(\lambda)$ implies

$$\delta \tau_A{}^B = \int_0^1 f_\gamma(\lambda) \frac{da^\gamma}{d\lambda} h(\lambda) \sqrt{\quad} \, d\lambda = 0$$

Consequently,

$$\boxed{f_\gamma(\lambda) \frac{da^\gamma}{d\lambda} = 0} \quad (\text{even if } f_\gamma \neq 0!) \quad (2.17)$$

This holds for all paths $a^\gamma(\lambda)$, even those that do not extremize $\tau_A{}^B$!

An equation that holds, whether or not the quantities obey any differential equation, is called an <u>identity</u>. Here it is simply an algebraic identity.

VI. PROPER TIME PARAMETRIZATION

The reparametrization freedom is a green light to simplifying the differential equation (2.9) on page 2.10: For any worldline $a^\alpha(\lambda)$ introduce the proper time increment

$$d\tau = \sqrt{-g_{\alpha\beta}\left(a^\gamma(\lambda)\right) \frac{d a^\alpha(\lambda)}{d\lambda} \frac{d a^\beta(\lambda)}{d\lambda}}\, d\lambda \qquad (2.18)$$

The derivatives $\frac{d\tau}{d\lambda}$ and $\frac{d\lambda}{d\tau}$ are well-defined. Consequently, $\tau(\lambda)$ is a monotonic function and so is its inverse $\lambda(\tau)$. Using proper time τ as the new parameter, introduce

$$a^\gamma(\lambda(\tau)) \equiv x^\gamma(\tau) . \qquad (2.19)$$

Thus

2.19

$$\frac{dx^\gamma(\tau)}{d\tau} = \frac{da^\gamma(\lambda)}{d\lambda}\frac{d\lambda}{d\tau} = \frac{1}{\sqrt{\quad}}\frac{da^\gamma}{d\lambda} \equiv \dot{x}^\gamma(\tau) \qquad (2.20)$$

where $\sqrt{\quad}$ is the non-zero square root expression in Eq.(2.18).

In terms of proper time as the new curve parameter, the extremum condition as expressed by Eq. (2.8) and (2.9) on page 2.10 simplifies enormously:

$$0 = f_\gamma(\lambda) = \frac{1}{2}\frac{d}{d\tau}\left(g_{\gamma\nu}\frac{dx^\nu}{d\tau}\right) + \frac{1}{2}\frac{d}{d\tau}\left(g_{\mu\gamma}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

or*

$$0 = g_{\gamma\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\left(g_{\gamma\nu,\mu} + g_{\gamma\mu,\nu} - g_{\mu\nu,\gamma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad (2.21)$$

*\footnote{ Here and elsewhere $g_{\gamma\nu,\mu}$ is short hand for the partial derivative

$$g_{\gamma\nu,\mu} \equiv \frac{\partial g_{\gamma\nu}}{\partial x^\mu} .\}$$

Streamline this differential equation further by introducing the inverse metric $g^{\alpha\gamma}$:

$$g^{\alpha\gamma} g_{\gamma\sigma} = \delta^{\alpha}_{\sigma} . \qquad (2.20)$$

Apply it to Eq. (2.21) and obtain

$$\boxed{\begin{array}{l} 0 = \dfrac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} \\ \hline 0 = \ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \end{array}} \qquad (2.22)$$

,

where $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\gamma}\left(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\alpha}\right)$ (2.23) is the "Chriffel symbol of the 2nd kind."**

** \footnote{ By contrast, the "Christoffel symbols of the 1st kind" are the ones in Eq. (2.21),

$$\Gamma_{\gamma\mu\nu} = \frac{1}{2}\left(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma}\right),$$

before we introduced the inverse metric. These symbols are important in that they mathematize the metric compatibility of the law of parallel transport.

Indeed, add to the above symbol the one with γ and ν interchanged:

$$\Gamma_{\nu\mu\gamma} = \frac{1}{2}(g_{\gamma\nu\mu,\gamma} + g_{\gamma\nu,\mu} - g_{\mu\gamma,\nu}).$$

Their sum is

$$\frac{\partial g_{\gamma\nu}}{\partial x^\mu} = \Gamma_{\gamma,\mu\nu} + \Gamma_{\nu\mu\gamma}$$

$$= g_{\gamma\alpha}\Gamma^{\alpha}_{\mu\nu} + g_{\alpha\nu}\Gamma^{\alpha}_{\mu\gamma} ,$$

$$0 = \frac{\partial g_{\gamma\nu}}{\partial x^\mu} - g_{\gamma\alpha}\Gamma^{\alpha}_{\nu\mu} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\gamma} \equiv g_{\gamma\nu;\mu}$$

The right hand side are the components of the covariant derivative of the metric tensor. That they vanish expresses the fact the metric is <u>covariantly constant</u>. This is the condition that the law as expressed by the $\Gamma^{\alpha}_{\mu\nu}$ in Eq. (2.23) on page 2.20 is

(a) compatible with the same metric that went into the extremization as stated on page 2.5, and

(b) has zero torsion (because $\Gamma^{\alpha}_{\mu\nu}=\Gamma^{\alpha}_{\nu\mu}$)}

The line of reasoning leading to the equation for a geodesic lead to two conclusions:

① The principle of extremal proper time implies a unique torsionless parallel transport which is compatible with the metric:

$$\int\sqrt{-g_{\mu\nu}\,dx^{\mu}dx^{\nu}} = \text{extr.}\begin{cases}\Longrightarrow \Gamma^{\alpha}_{\mu\nu}=\frac{1}{2}g^{\alpha\gamma}\{g_{\gamma\mu,\nu}+g_{\gamma\nu,\mu}-g_{\mu\nu,\gamma}\}\\ \Longrightarrow \Gamma^{\alpha}_{\mu\nu}=\Gamma^{\alpha}_{\nu\mu}\ (\text{i.e. torsion}=0)\end{cases}$$

② The geodesics of curved spacetime coincide with the worldlines of extremal proper time.

VII. A GENERAL CONSTANT OF MOTION

By differentiating the squared magnitude $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ w.r.t. τ, one can verify that

$$f_{\gamma}(\lambda)\frac{dx^{\gamma}}{d\tau}=0 \Rightarrow \frac{d}{d\tau}\left(g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)=0$$

Thus $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=const$ (=-1 for any time-like curve). $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ is always an integral of motion; it expresses the constancy of the magnitude of the unit tangent $\mathbf{u}=\frac{dx^{\mu}}{d\tau}\frac{\partial}{\partial x^{\mu}}$:

$$\mathbf{u}\cdot\mathbf{u}=g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=const.$$

for any curve, even if it is not a geodesic.

LECTURE 4

Geometrization of Newtonian Mechanics

I. Free Particle in a Rotating Frame

II. Free Body Motion vs. Geodesic Motion

III. Free Body Motion in an Accelerated Frame

IV. Geometrization of Gravitation } consigned to Lecture 5

In MTW read the caption to Figure 1.7

In H. Goldstein's "Classical Mechanics" read Rate of change of a vector, which is
Section 4.8-4.9 in the 1st (1959) edition, pages 132-135
Section 4.9-4.10 in the 3rd (2000) edition, pages 171-176
(with C. Poole and J. Safko as coauthors) [PDF copy is available over the internet]

The process of geometrizing (= putting into geometrical form) gravity is an inductive process. It asks and answers the question "Why?" This question is at the center of the law of causality*.

* \footnote{This fundamental law is concretized, identified, defined, and characterized in L. Peikoff's "OBJECTVISM: THE PHILOSOPHY OF AYN RAND" pages 12-17.}

There are several ingredients in that inductive process. Two of them are the motions of a free particle as observed

(i) in a <u>rotating</u> frame

(ii) in an <u>accelerated</u> frame

I. Motion of Bodies Relative to an Inertial Frame

However, the point of departure for this process is an inertial (= "free float") frame. It is defined by Newton's 1st Law of motion: Bodies in uniform motion remain in their states of constant velocities along straight lines, unless their states are changed by forces impelled on them.

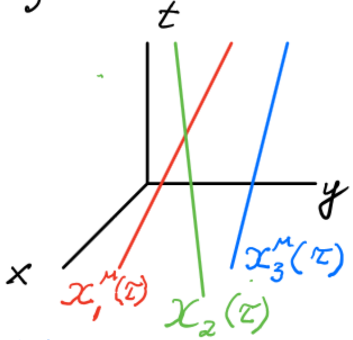


Figure 4.1 In an inertial reference frame all free particle motions are observed to be straight lines. Newton's 1st Law of motion is used to identify an inertial frame by this observed fact.

II. Motion of Bodies Relative to a Rotating Frame

(4.3)

Q: What is the motion of each of these bodies as observed relative to a rotating frame of reference?

A: The observed motion is non-uniform in that it is characterized by Coriolis and centrifugal acceleration; from the geometrical perspetive their components are those Christoffel symbols Γ^{i}_{jk} which are non-zero.

Indeed, consider a rigid frame of reference which rotates with angular velocity $\vec{\omega}$ relative to the fixed stars. A frame is said to be rigid if all of its basis vectors rotate around an axis with the same angular velocity. A vector is said to be attached rigidly to such a frame if its components relative to the frame's basis are independent of time.

Consider a vector, say $\vec{G}$, which is rigidly attached to this frame. This vector will rotate relative to the static inertial

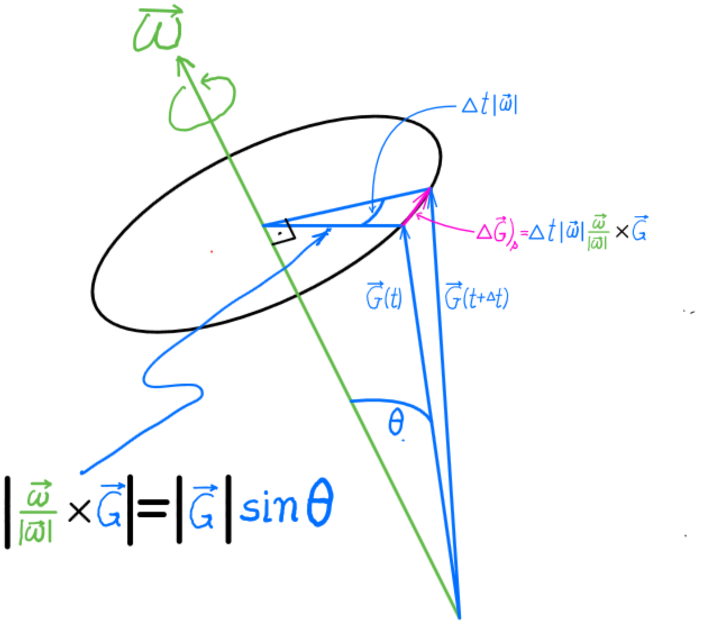


Figure 4.2 Location vector $\vec{G}(t)$ rotates around the rotation axis $\frac{\vec{\omega}}{|\vec{\omega}|}$ at angular rate $|\vec{\omega}|$. During Δt the rotation of $\vec{G}$ is expressed by the vectorial displacement $\Delta\vec{G}$. This vector is

$$\Delta\vec{G} = \Delta t\,\vec{\omega}\times\vec{G}.$$

frame, namely relative to the fixed stars. As depicted is Figure 4.2, during a time interval Δt this rotating vector

$\vec{G}$ will have changed by the amount ④.5

$$\Delta\vec{G})_s = \Delta t \ \vec{\omega}\times\vec{G},$$

which is perpendicular to both $\vec{\omega}$ and $\vec{G}$. The vector $\Delta\vec{G})_s$ expresses an infinitesimal change (relative to the fixed stars) in $\vec{G}$ due to an infinitesimal rotation by an amout $\Delta t|\vec{\omega}|$ around the direction $\vec{\omega}$.

However, consider the circumstance where the vector $\vec{G}$ is not rigidly attached to the rotating frame. But instead changes by the amount $\Delta\vec{G})_{rot}$ in the rotating frame during the time interval Δt. In this case the relation between the change $\Delta\vec{G})_s$ relative to the static ("fixed stars") frame and the change $\Delta\vec{G})_{rot}$ relative to the rotating frame is

4.6

$$\Delta\vec G\Big)_s = \Delta\vec G\Big)_{rot} + \Delta t\,\vec\omega\times G$$

Thus
$$\boxed{\frac{d\vec G}{dt}\Big)_s = \frac{d\vec G}{dt}\Big)_{rot} + \vec\omega\times\vec G} \qquad (4.1)$$

Apply this kinematic relation to the position vector $\vec R(t)$:

$$\frac{d\vec R}{dt}\Big)_s = \frac{d\vec R}{dt}\Big)_{rot} + \vec\omega\times\vec R,$$

and then again to the velocity vector $\vec v_s = \frac{d\vec R}{dt}\Big)_s$.

One finds that for a free particle

$$0 = \frac{d}{dt}\left[\frac{d\vec R}{dt}\Big)_s\right]\Bigg)_s = \frac{d}{dt}\left[\frac{d\vec R}{dt}\Big)_{rot} + \vec\omega\times\vec R\right]\Bigg)_{rot} + \vec\omega\times\left[\frac{d\vec R}{dt}\Big)_{rot} + \omega\times\vec R\right] \qquad (4.2)$$

$$\vec a_{rot} \qquad \vec v_{rot} \qquad \vec v_{rot}$$

$$\vec a_{rot} = \frac{d^2x^i}{dt^2}e^*_i(t) \qquad \vec v_{rot} = \frac{dx^i}{dt}e^*_i(t)$$

↑ rotating frame basis vectors

Our focus of attention is a rotating frame of CONSTANT time-independent angular velocity $\vec{\omega}$.

Thus for a freely moving body of mass m, its acceleration and hence the forces impelling it as observed relative to a rotating frame is

$$\left\{ m\,\frac{d^2x^i}{dt^2} = -m2\underbrace{\left[\vec{\omega}\times\vec{v}_{rot}\right]^i}_{\substack{\text{Coriolis}\\ \text{force}}} - m\underbrace{\left[\vec{\omega}\times(\vec{\omega}\times\vec{R})\right]^i}_{\substack{\text{Centrifugal}\\ \text{force}\\ \|\\ -m(\vec{\omega}\cdot\vec{R})\vec{\omega}+m\,\omega^2\vec{R}}} \right\} e_i^*(t) \qquad (4.3)$$

\footnote{ The "Centrifugal force" is called that because $\vec{\omega}\times(\vec{\omega}\times\vec{R})$ points perpendicularly away from the axis of rotation and has magnitude $\vec{\omega}^2 R \sin\theta$. This is simply $\vec{\omega}^2$. (radius of the tip of $\vec{R}=\vec{G}$ away from the rotation axis) in Figure 4.2.

The Coriolis and the centrifugal forces are inertial pseudo forces. "Inertial" because they are proportional to the mass. "Pseudo" because of the non-inertial nature of that rotating frame.

III. Free Body vs. Geodesic Motion

Compare the 1st component (relative to the rotation basis $\{e_i^*\}$) of the free particle motion with the $\mu=1$ component of a geodesic

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0\,, \qquad (4.4)$$

namely,

$$0=\frac{d^2x^1}{dt^2}+2\left(\omega_2\frac{dx^3}{dt}-\omega_3\frac{dx^2}{dt}\right)+\vec{\omega}\cdot\vec{R}\,\omega_1-\vec{\omega}\cdot\vec{\omega}\,x^1 \qquad (4.5)$$

with

$$0=\frac{d^2x^1}{d\tau^2}+2\,\Gamma^1_{0k}\frac{dt}{d\tau}\frac{dx^k}{d\tau}+\Gamma^1_{00}\frac{dt}{d\tau}\frac{dt}{d\tau}+\Gamma^1_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} \qquad (4.6)$$

In the non-relativistic approximation, the body's proper time τ equals the time t in the rotating frame:

$$\frac{dt}{d\tau}=1.$$

It is understood that time t is measured in units of distance travelled by light.

This implies that

$$t = c\,t_{\text{conventional}}$$

$$\vec{\omega} = \frac{\vec{\omega}_{\text{conventional}}}{c}$$

and that in the non-relativistic limit

$$\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = \frac{1}{c^2}\frac{dx^j}{dt_{\text{conv}}}\;\frac{dx^k}{dt_{\text{conv}}} \ll 1,$$

which therefore is neglegibly small in Eq.(4.6).

In the asymptotic non-relativic limit, with an appropriate choice of the Γ's, the differential Eq.(4.6) is the same as Eq.(4.5) for all functions $x^1(t)$, $x^2(t)$, and $x^3(t)$. The same observations apply to Eq. (4.3) and (4.4) when $i=2,3$ and $\mu=2,3$ respectively.

Consequently, the Γ-terms in Eq.(4.6) are related to those in Eq.(4.5) by the following equations:

a) $$\begin{bmatrix} \Gamma^1_{01} & \Gamma^1_{02} & \Gamma^1_{03} \\ \Gamma^2_{01} & \Gamma^2_{02} & \Gamma^2_{03} \\ \Gamma^3_{01} & \Gamma^3_{02} & \Gamma^3_{03} \end{bmatrix} \begin{bmatrix} \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix}$$

or $\boxed{\Gamma^i{}_{0k} = \epsilon_{ijk}\,\omega^j}$ $\left.\begin{matrix} i \\ k \end{matrix}\right\} = 1,2,3$ (4.10)

b) $$\boxed{\Gamma^i{}_{00}\underbrace{\frac{dt}{d\tau}}_{1}\underbrace{\frac{dt}{d\tau}}_{1} = \vec{\omega}\cdot\vec{R}\,\omega_i - \vec{\omega}\cdot\vec{\omega}\,x^i} \qquad i = 1,2,3$$

c) $$\boxed{\Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \Gamma^i{}_{jk}\frac{dx^j}{dt_{conv}}\frac{dx^k}{dt_{conv}}\frac{1}{c^2}\underset{\substack{\uparrow \\ \text{non-rel.} \\ \text{approx'n}}}{\approx} 0}$$

IV. Conclusions:

a) Coriolis acceleration $\neq 0 \Rightarrow \Gamma^i{}_{0k} \neq 0$

b) Centrifugal acceleration $\neq 0 \Rightarrow \Gamma^i{}_{00} \neq 0$

c) $\left.\begin{matrix}\text{In non-relativistic} \\ \text{mechanics } \frac{dx^j}{dt_{conv}}\frac{dx^k}{dt_{conv}}\frac{1}{c^2} \ll 1\end{matrix}\right\} \Rightarrow \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \approx 0$

Lecture 5

Geometrization of Newtonian Gravity

I. Geometrization of free body motion in an accelerated frame

II. Geometrization of gravity

1. The Equivalence Principle
2. Gravity geometrized via mathematized inertial motion

III. Looking ahead

For grasping what is equivalent in the "Equivalence Principle," read

(i) the caption to the Eötvös experiment on page 26 of A JOURNEY INTO GRAVITY AND SPACETIME by John A. Wheeler (ISBN 0-7167-5016-3 and/or

(ii) Box 38.2 in MTW

(iii) Einstein's 1913 1st attempt at a theory of gravition in "Physical Foundations of a Theory of Gravition" [available on the internet].

I. Geometrization of Free Body Motion in an Accelerated Frame

As before, consider the motion of a free particle in an inertial reference frame. Because of Newton's 1st Law its world line is one which is straight relative to that frame's rectilinear x-t coordinate system.

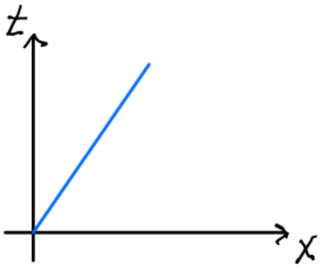


Figure 5.1 Free particle world line observed in an inertial frame of reference, relative to which

$$\frac{d^2x}{dt^2}=0$$

and

$$x(t)=v_0 t. \qquad (5.1)$$

Newton's 1st Law does not apply relative to a uniformly accelerated reference frame. As

5.2

depicted in Figure 5.2, the free particle's 1-d spatial trajectory is observed to make a sharp U-turn relative to such a frame.

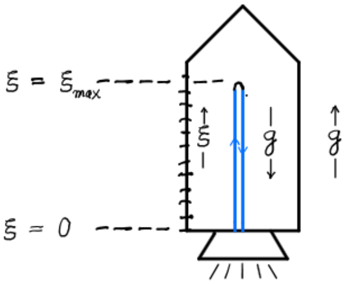


Figure 5.2 Motion of a freely moving particle as observed in a frame with acceleration of magnitude g upward.

Wordlines which are observed to be straight relatve to an inertial frame of reference are observed to be curved relative to an accelerated frame such as the one depicted in Figure 5.2.

The equation motion and its solution* are

$$\frac{d^2x}{dt^2}=0 \qquad (5,2a)$$

and

(5.3)

*\footnote{ $C(t) = \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} v_0 t \\ t \end{pmatrix}$ } $x(t) = v_0 t,$ (5.2b)

relative to the inertial frame; by contrast, relative to the accelerated frame they are

$$\frac{d^2\xi}{d\tau^2} = -g \qquad (5.3a)$$

and

$$\xi(\tau) = v_0 \tau - \frac{1}{2} g \tau^2 \qquad (5.3b)$$

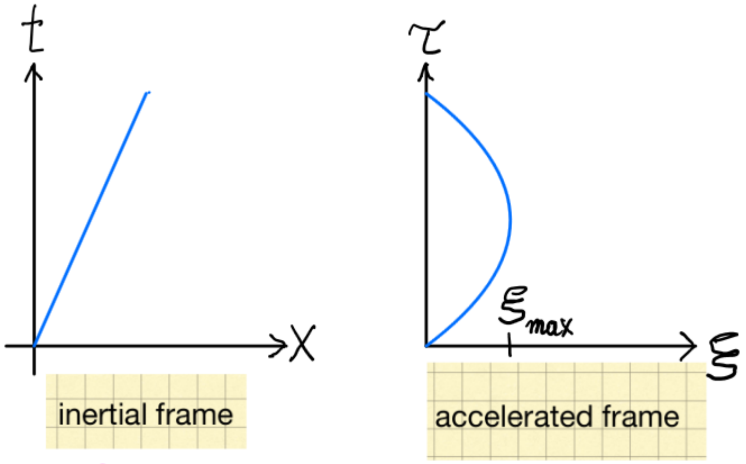
Each of these two frames is a platform for measurements performed on the worldline of a free particle. These frames are mathematized by the (x, t) and the (ξ, τ) coordinate charts (a.k.a. coordinates systems) respectively. The coordinates are the standards of measurement for each frame. The transition map* (a.k.a. coordinate transformations) between the two charts is

*\footnote{ $\varphi_{in} \circ \varphi_{ac}^{-1}$ }

$$\begin{aligned} x &= \xi + \frac{1}{2} g \tau^2 \\ t &= \tau \end{aligned} \qquad (5.4)$$

(5.4)

This transformation relates the free particle world line represention relative to the inertial frame to that relative to the accelerated frame. Figure 5.3 depicts these two representions of one and the same motion of a particle.



\begin{figure}

Figure 5.3 The transformation Eq. (5.4),

$$\begin{pmatrix}\xi\\ \tau\end{pmatrix} \rightsquigarrow \begin{pmatrix} x=\xi+\frac{1}{2}g\tau^{2}\\ t=\tau\end{pmatrix}$$

maps the domain of the accelerated frame with its curvilinear (ξ, τ) coordinatization into the inertial frame with its rectilinear (x, t) coordinatization:

$$(\xi, \tau) \rightsquigarrow (x, t) = \left(v_0 \tau + \tfrac{1}{2} g \tau^2 ;\ \tau\right) \qquad (5.5)$$

Furthermore, it maps the free-particle equation of motion

$$\frac{d^2 x}{dt^2} = 0$$

and its straight world line solution*

$$x(t) = v_0 t$$

in the inertial frame onto the particle equation of motion

$$\frac{d^2 \xi}{d\tau^2} = -g$$

and its solution*

$$\xi(\tau) = v_0 \tau - \tfrac{1}{2} g \tau^2$$

in the accelerated frame.

*\footnote{To be math'lly precise $C_{in}(t) = \begin{pmatrix} C^0(t) \\ C^1(t) \end{pmatrix} = \begin{pmatrix} t \\ v_0 t \end{pmatrix}$}

*\footnote{To be mathematically precise $\left(\varphi_{acc} \circ \varphi_{in}^{-1} \circ C_{in}\right)(\tau) = C_{acc}(\tau) = \begin{pmatrix} C^0_{acc}(\tau) \\ C^1_{acc}(\tau \end{pmatrix} = \begin{pmatrix} \tau \\ v_0 \tau - \frac{1}{2} g \tau^2 \end{pmatrix}$ }

In summary, the transformation Eq.(5.5) maps the two representations of the particle world line, (5.1) and (5.2), onto one another:

\end{figure}

Thus for a freely moving body of mass m, its acceleration and hence the force impelling it as observed relative to an accelerating frame is

$$m\frac{d^2\xi}{d\tau^2} = -mg \qquad (5.4)$$

This force, like the Coriolis and centrifugal force, is an inertial pseudo force. "inertial" because it is proportional to the mass; "pseudo" because it comes from the non-inertial (here, accelerated) frame of reference.

(Lecture 4)

Equation (5.4), just like Eq. (4.3), lendsitself to being geometrized by comparing it with the equation for a geodesic in spacetime.

This is achieved by comparing

(a) the free-body equation of motion measured in an accelerated frame and mathematized by the second derivative of Eq. (5.2),

$$\frac{d^2\xi}{d\tau^2}=-g \qquad (5.5)$$

with

(b) the $\mu=1$ component of the geodesic eq'n

$$\frac{d^2x^\mu}{d\tau^2}=-\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

by setting $x^1=\xi$. In the asymptotic non-relativistic (low velocities: $\left(\frac{dx^1}{d\tau}\right)^2\ll 1$)) limit

$\frac{dx^0}{d\tau}=1$ and hence $\tau=c\,t_{conv}$.

Consequently,

$$\frac{d^2\xi}{d\tau^2} = -\Gamma^1_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

becomes

$$\frac{d^2\xi}{d\tau^2} = -\Gamma^1_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \text{neglegible terms.} \qquad (5.6)$$

Comparison with Eq. (5.5) leads to.

$$\boxed{\Gamma^1_{00} = g = \frac{g_{conv}}{c^2}} \qquad (5.7)$$

and

$$\Gamma^0_{00} = \Gamma^0_{01} = 0$$

because $\frac{d}{d\tau}\left(\frac{dx^0}{d\tau}\right) \approx 0.$

II. Geometrization of Gravity

1. Q: *Must one formulate gravity in geometrical terms?*

A: *The following is an observed fact:* the inertial pseudo force on a particle moving freely in a uniformly accelerated frame is (5,9) indistinguishable from the force on a particle in a uniform gravitational field. This fact is mathematized by the statement that \footnote{Recall that the acceleration in conventional unit, $g_{conv}\left[\frac{length}{(time)^2}\right]$ is related g in standard units by

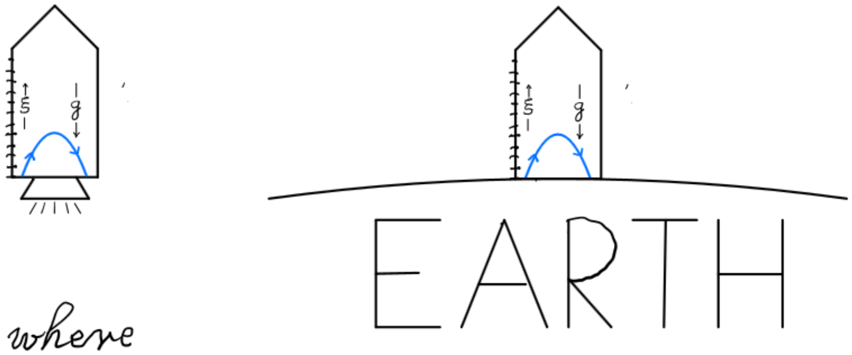
$$g=\frac{g_{conv}}{c^2}\left[\frac{1}{length}\right].$$

Furthermore, the dimensionless grav'l pot'l ϕ is related to the conventional Newtonian grav'l pot'l ϕ_{conv} by $\phi=\frac{\phi_{grav}}{c^2}$.}

$$\boxed{m_{inert}\, g_{conv} = m_{grav'l}\left(-\vec{\nabla}\phi_{grav'l}\right)^{i=1}} \qquad (5,8)$$

or

$$\boxed{m_{inertial}\, g = m_{grav'l}\left(-\vec{\nabla}\phi\right)^{i=1}}$$



where

$$m_{inertial} = m_{gravitational}$$

Taken from *A Journey Into Gravity And Spacetime* by John A. Wheeler

EÖTVÖS'S EXPERIMENT

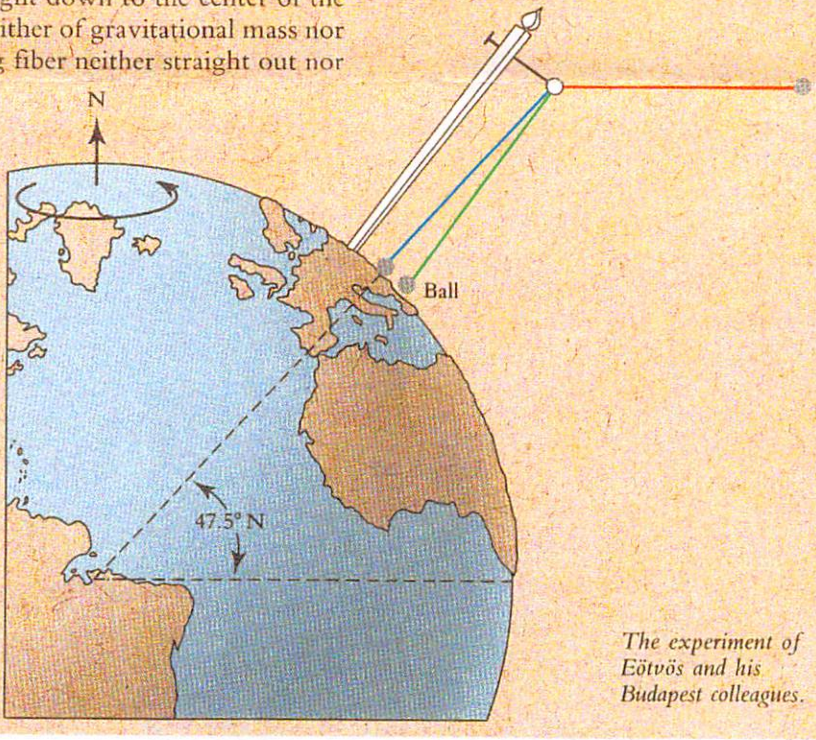
To measure the travel time of a falling body more precisely demands more time. So realized Baron von Eötvös, who therefore devised an experiment of a new kind that offered unlimited time for measurement. It focused on the central issue: Is there for any substance any such distinction, as old writers assumed, between its "gravitational mass," on which the center of the Earth is conceived to pull, and its "inertial mass"? The "inertial mass"—today simply "mass"—resists being set in motion or, if already endowed with a velocity, resists any change in the magnitude or the direction of that velocity.

Imagine an object endowed with inertial mass but no gravitational mass, hanging at the end of a fiber and carried around and around in a circle by the spin of the Earth. With no gravity to hold it down, it will pull the fiber straight out from the axis (red line in the diagram). On the other hand, an object that has no inertial mass at all will not be thrown outward from the axis of spin of the Earth. Instead—if it has any gravitational mass—it will tug the supporting fiber to a position where it points straight down to the center of the Earth (blue line). An object deprived neither of gravitational mass nor of inertial mass will pull the supporting fiber neither straight out nor straight down, but instead to an angle of uprise (green line). Does the ratio of inertial to gravitational mass differ from one substance to another? Then the angle of uprise will differ, too. And for measuring it there's all the time in the world!

This beautiful idea Eötvös put into action. He and his colleagues, in experiments extending over some thirty years, on a variety of substances, were able to establish that there is an angle of uprise: that angle is greatest at latitudes 45°N and 45°S, and amounts there to a tenth of a degree. They found to an accuracy of 5 parts in 10^9 that the angle of uprise was identical for every substance tested.

Baron Roland von Eötvös

Born July 27, 1848, Budapest. Died April 8, 1919, Budapest.



The experiment of Eötvös and his Budapest colleagues.

for all kinds of material particles.

(Eötvös experiment)
accuracy as of 2017: 1 part in 10^{15}

2. The line of reasoning leading to the conclusion

"Geometrized free-body motion + Equivalence principle = Geometrized gravity"

is a four-step process

(i) The gravitational force field is conservative implies

$$\overrightarrow{(\text{grav'l force})} = m_{grav}\,(-)\,\vec{\nabla}\phi_{grav} \,,$$

where ϕ_{grav} is the Newtonian gravitational potential.

\footnote{ Example: For a spherical body of mass M

$$\phi_{grav}(x,y,z) = -\frac{GM}{r} = -GM\frac{1}{\sqrt{x^2+y^2+z^2}}\ \}$$

(ii) Apply the equivalence principle to the one-dimensional geometrized motion, Eq. (5.5)-(5.8) on pages 5.3-5.5:

$$m_{in}\left(\frac{d^2\xi}{dt^2_{conv}}\frac{1}{c^2}\overset{(5.7)}{\overset{\downarrow}{\doteq}}-\frac{g_{conv}}{c^2}\overset{(5.6)}{\overset{\downarrow}{\doteq}}-\Gamma^1_{00}\right)$$

$$\overset{(5.8)}{\overset{\downarrow}{\doteq}} m_{grav}\left(-\frac{1}{c^2}(-)\left(\vec{\nabla}\phi_{grav}\right)^i\right) \qquad (5.8)$$

★ \footnote{Recall that

see below

$$\Gamma^i_{00}=\frac{1}{2}\sum_{\nu=0}^{3}g^{i\nu}\left(g_{\nu 0,0}+g_{\nu 0,0}-g_{00,\nu}\right)$$

a) Focus on time-independent acceleration. Thus

$$g_{\nu 0,0}=g_{00,0}=0$$

b) Use rectilinear coordinates. Thus

$$g^{ij}=\delta^{ij} \qquad \begin{matrix} i \\ j \end{matrix}\Big\}=1,2,3$$

c) Consequently,

$$\Gamma^i_{00}=-\frac{1}{2}g_{00,i}=-\frac{1}{2}\left(\vec{\nabla}g_{00}\right)^i$$

(iii) Introducing★ Γ^{i}_{00} into Eq.(5.8) (5,12)
yields

$$\frac{1}{2}\left(\vec{\nabla} g_{00}\right)^{i} = \frac{m_{grav}}{m_{inert}} \frac{-1}{c^2}\left(\vec{\nabla}\phi_{grav}\right)^{i}$$

Using the Eötvös fact that

$$\frac{m_{grav}}{m_{inert}} = 1$$

results in

$$\vec{\nabla} g_{00} = -\frac{2}{c^2}\left(\vec{\nabla}\phi_{grav}\right)$$

and

$$g_{00} = const - 2\frac{\phi_{grav}}{c^2}$$

(iv) Impose the observed boundary condition that in the absence of gravitation (i.e. $\phi_{grav}=0$) the spacetime metric is flat, which means there exist a global coordinat system such that

$$-d\tau^2 = g_{00}\,dt^2 + \sum_i \sum_j dx^i dx^j$$
$$= -dt^2 + dx^2 + dy^2 + dz^2,$$

Thus, $g_{oo} = -1$ whenever $\phi_{grav} = 0$.

It follows that

☞ $$\boxed{g_{00} = -1 - 2\frac{\phi_{grav.}}{c^2} = -1 - \frac{2}{c^2}\begin{pmatrix}\text{Newtonian} \\ \text{gravitational} \\ \text{potential}\end{pmatrix}}$$

CONCLUSIONS

1. Gravitation is to be mathematized in terms of geometrical concepts. The gravitational potential is the Newtonian limit of a geometrical formulation in terms of the metric tensor

2. The five step line of reasoning implies that fundamentally

$$\begin{pmatrix}\text{set of generalized} \\ \text{gravitational} \\ \text{potentials}\end{pmatrix} = \begin{pmatrix}\text{set of components} \\ \text{of a metric} \\ \text{tensor field}\end{pmatrix}$$

or more briefly

"gravitation = geometry" (properly understood)

3.

Newton's 1st Law geometrized relative to rotating & acc'd frames → $\left\{\begin{matrix}\text{inertial}\\ \text{forces}\end{matrix}\right\} = \left\{\Gamma^{i}_{\alpha\beta}\right\}$

↓ Equivalence Principle → $\Gamma^{i}_{00} = \frac{1}{c^2}\left(\vec{\nabla}\phi_{grav}\right)^{i}$

Gravity $\neq 0$ $\Longrightarrow$ Metric tensor is not flat

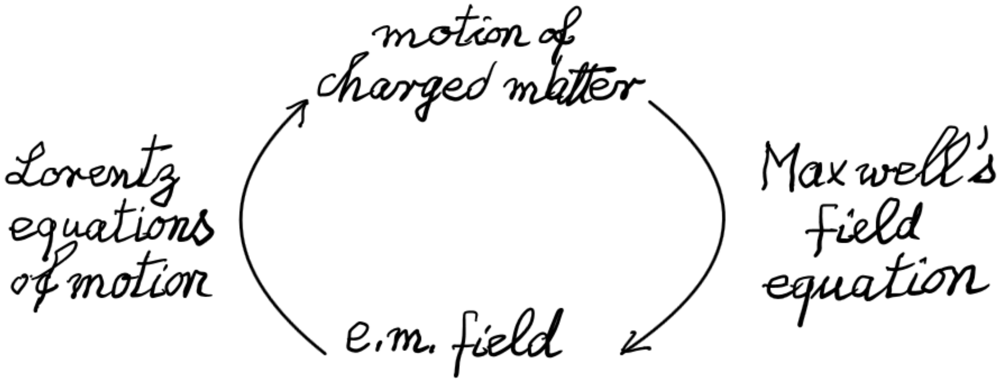
4. What is the geometrized generalization of Newton's gravitational field equations

$$\nabla^2 \phi = 4\pi\rho \, ?$$

Here ρ is the mass density

III. Looking ahead we have the

1. Mathematical Structure of Eletromagnetism



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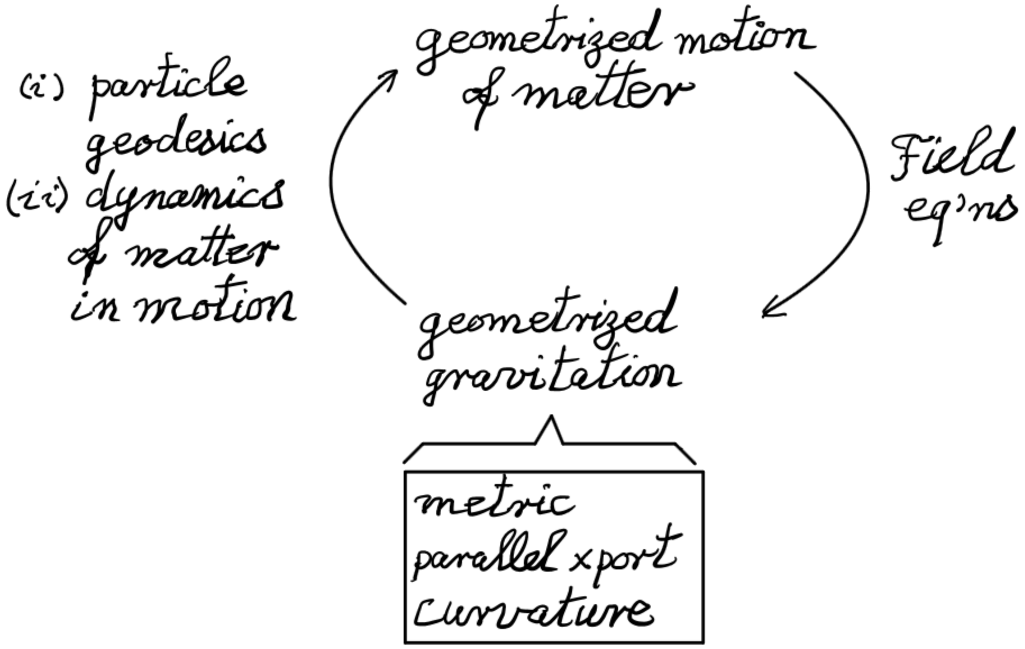
2. Mathematical Structure of Gravitation



Doc. 16.

Physical Foundations of a Theory of Gravitation[1]

by Albert Einstein

[*Naturforschende Gesellschaft in Zürich. Vierteljahrsschrift* 58 (1914): 284—290]

By the word "mass" of a body one denotes two things that are very different according to their definitions: on the one hand, the inertial resistance of the body and, on the other hand, the characteristic constant that is the determining factor for the effect of the gravitational field on the body. It is one of the most remarkable empirical facts of physics that these two masses, the *inertial* and the *gravitational*, agree exactly with each other as regards their magnitude. This agreement was proved most exactly by Eötvös's experiments. A body on the surface of the Earth is acted upon by two generally differently directed forces, which together constitute the apparent gravity of the body: one of these forces, the gravitation proper, depends on the gravitational mass, while the other, the centrifugal force, depends on the inertial mass. By experiments with the torsion balance, Eötvös established that the ratio of these two forces is independent of the nature of the material; in that way he proved the agreement of the two masses of a body with an accuracy that rules out deviations [2] of the relative magnitude of 10^{-7}.

This empirical law can also be expressed in the following way. In a gravitational field all bodies fall with the same acceleration. This suggests the view that, with regard to its influence on mechanical and other physical processes, a gravitational field may be replaced by a state of acceleration of the reference body (coordinate system). This conception does not follow with necessity from the experiments mentioned, but it is of great heuristic interest all the same. For, since the course of physical processes relative to an accelerated reference system can be determined theoretically, this *equivalence hypothesis* permits us to predict the influence of a [3] gravitational field on physical processes of every kind. The experimental test of the conclusions so reached must then show whether the underlying hypothesis was correct.

In the way indicated, one comes to the conclusion that the speed with which a physical process occurs in a gravitational field is greater the greater the gravitational potential at the location where the physical system in question is situated. For that reason, the spectral lines of solar light should, for example, experience a small shift toward the red end of the spectrum as compared with the corresponding spectral lines of terrestrial light sources, namely, a shift of about two millionths of the wavelength. [4] A further consequence of this equivalence hypothesis is the bending of light rays in a gravitational field, which amounts to 0.84 seconds of arc for a light ray passing [5] near the sun and is thus not inaccessible to experimental test. This bending of light rays implies that the velocity of light is not constant, but depends, instead, on the location. This forces us to generalize the theory of space and time, known as the theory of relativity, since the latter was based on the assumption of the constancy of the velocity of light.

According to the familiar theory of relativity, an isolated material point moves rectilinearly and uniformly according to the equation

$$\delta(\int ds) = 0,$$

where

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2,$$

and c denotes the (constant) velocity of light. The equivalence hypothesis permits the conclusion that in a *static* gravitational field (of special kind) a material point moves according to the above equation, in which now, however, c is a function of location and is determined by the gravitational potential. From this special case of the gravitational field, one can arrive at a general case by passing to moving coordinate [6] systems by means of coordinate transformation.[2] In this way one recognizes that the only sufficiently encompassing invariant-theoretical generalization of the indicated law of motion consists in assuming that the "line element ds" has the form

$$ds^2 = \sum_{ik} g_{ik}dx_idx_k, \qquad (i,k = 1,2,3,4)$$

where the g_{ik} are functions of x_1, x_2, x_3, and x_4, while the first three coordinates characterize the position, and the last one the time, and the equation of motion is again to have the form

$$\delta(\int ds) = 0.$$

If one considers that in this view, instead of the customary line element of the original theory of relativity,

$$ds^2 = \sum_i dx_i^2$$

one has the more general

---

[1] Based on a lecture delivered on 9 September 1913 at the annual meeting of the [1] Schweizer Naturforschende Gesellschaft in Frauenfeld.

[2] We postulate that we arrive at an equally justified description of the process if we refer it to an appropriately moving coordinate system; in that way we abide by the basic idea of the theory of relativity.

$$ds^2 = \sum_{ik} g_{ik}\,dx_i\,dx_k$$

as the absolute invariant (scalar), then one sees at once how one attains a generalization of the theory of relativity that encompasses gravitation on the basis of the equivalence hypothesis. While in the original theory of relativity the independence of the physical equations from the special choice of the reference system is based on the postulation of the fundamental invariant $ds^2 = \sum_i dx_i^2$, we are concerned with constructing a theory in which the most general line element of the form

$$ds^2 = \sum_{ik} g_{ik}\,dx_i\,dx_k$$

plays the role of the fundamental invariant. The concepts of vector analysis needed for that purpose are provided by the method of the absolute differential calculus, which will be explained in the lecture by Grossmann which is to come next. [7]

It follows from the idea outlined above that the ten quantities g_{ik} characterize the gravitational field; they replace the scalar gravitational potential φ of Newtonian gravitation theory, and form the second-rank fundamental covariant tensor of the gravitational field. The fundamental physical significance of these quantities g_{ik} consists, i.a., in the fact that they determine the behavior of measuring rods and clocks.

The method of the absolute differential calculus allows us to generalize the systems of equations of any physical process, as they occur in the original theory of relativity, in such a way that they fit into the scheme of the new theory. The components g_{ik} of the gravitational field always appear in these equations. The physical meaning of this is that the equations provide information about the influence of the gravitational field on processes in the region under study. The previously indicated law of motion of the material point may serve as the simplest example of this kind. Otherwise, we shall confine ourselves to the formulation of the most general law known to physics, namely, the law that corresponds to the momentum and energy conservation law in the original theory of relativity. As is well known, one has there a symmetric tensor $T_{\mu\nu}$, the components of which, the stress components, yield the components of the momentum, and the components of energy flux density and energy density. These quantities can be specified for phenomena in any domain. The laws of momentum and energy conservation are contained in the equations

$$\sum_\nu \frac{\partial T_{\sigma\nu}}{\partial x_\nu} = 0, \qquad (\nu,\sigma = 1,2,3,4) \tag{1}$$

since by integrating with respect to the spatial coordinates over the whole system, one can obtain from these equations the conservation equations

$$\frac{d}{dt}\left(\int T_{\sigma 4}\,d\tau\right) = 0, \tag{1a}$$

where $d\tau$ denotes the three-dimensional volume element.

In the general theory, the following equations correspond to equations (1):

$$\sum_\nu \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_\nu} = \frac{1}{2}\sum_{\mu\nu\tau} \frac{\partial g_{\mu\nu}}{\partial x_\nu}\gamma_{\mu\tau}\mathfrak{T}_{\sigma\nu} \qquad (\sigma = 1,2,3,4) \tag{2}$$ [8]

Here

$$\mathfrak{T}_{\sigma\nu} = \sqrt{-g}\cdot\sum_\mu g_{\sigma\mu}\theta_{\mu\nu},$$

where g is the determinant $|g_{ik}|$, and $\gamma_{\mu\tau}$ is the subdeterminant adjoint to $g_{\mu\tau}$ divided by this determinant; $\Theta_{\mu\nu}$ is the symmetrical second-rank contravariant tensor that characterizes the behavior of energy in the domain of phenomena under consideration. The quantities $\mathfrak{T}_{\sigma\nu}$ have the same physical meaning here as the quantity $T_{\sigma\nu}$ in the original theory of relativity; the stress-energy components of the gravitational field are not contained in them.

The right-hand side of equations (2) vanishes if the quantities $g_{\mu\nu}$ are constant, i.e., if no gravitational field is present. In that case, equation (2) reduces to equation (1) and can therefore be brought into the form (1a); in other words: the material process satisfies the conservation laws all by itself. If, on the contrary, the $g_{\mu\nu}$ are variable, i.e., if a gravitational field is present, then the right-hand side of equations (2) expresses the energetic influence of the gravitational field on the material process. It is clear that no conservation laws can be deduced from equation (2) in that case, because the stress-energy components of the material process cannot satisfy any conservation laws all by themselves, without the components of the gravitational field.

The method sketched up to this point shows how the equation systems of physics can be obtained when the influence of a given gravitational field on the processes is taken into account. But this does not solve the main problem of the theory of gravitation, since the latter consists in determining the quantities g_{ik} when the field-generating material processes (including the electrical ones) are to be considered as given. In other words, the generalization of Poisson's equation

$$\Delta\varphi = 4\pi k\rho \tag{3}$$

is sought.

On the one hand, the proportionality of energy and inertial mass that is obtained from the ordinary theory of relativity, and, on the other hand, the empirical proportionality of inertial and gravitational mass lead necessarily to the view that the same quantities that determine the energetic behavior of a system must also determine the gravitational effects of the system. From this we conclude that tensor $\mathfrak{T}_{\mu\nu}$ must appear in equations of gravitation we are seeking, in lieu of the density ρ of equation

(3). We are therefore looking for equations that express the equality of two tensors, one of which is the given tensor $\mathfrak{T}_{\mu\nu}$, while the other comes from the fundamental tensor $g_{\mu\nu}$ through differential operations.

[9] It has now turned out that the conservation laws of momentum and energy make possible the derivation of these equations. It has already been emphasized above that the material process alone cannot satisfy the conservation laws; but we must demand that the conservation laws be satisfied for the material process and the gravitational field *together*. According to the arguments presented above, this means that there must exist four equations of the form

$$\sum_{\nu} \frac{\partial}{\partial x_\nu}(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}) = 0. \qquad (\sigma = 1,2,3,4) \tag{4}$$

Here the $\mathfrak{t}_{\sigma\nu}$ characterize the stress-energy components of the gravitational field in a manner analogous to the way in which the quantities $\mathfrak{T}_{\sigma\nu}$ characterize those of the material process. In particular, the quantities $\mathfrak{T}_{\sigma\nu}$ and $\mathfrak{t}_{\sigma\nu}$ must have the same invariant-theoretical character. It turned out to be possible to show by means of a general argument that the equations that completely determine the gravitational field [10] cannot be covariant with respect to arbitrary substitutions. This fundamental discovery is especially noteworthy because all other physical equations, such as, e.g., equations (2), possess general covariance. In accordance with this general result, the postulated equations (4) are also covariant only with respect to *linear* substitutions, but are not [11] so with respect to arbitrary substitutions. Hence, we will have to demand covariance only with respect to linear transformations from the gravitation equations that we are seeking. It has turned out that one is led to completely determined equations if one adds to these considerations the demand that when these equations are applied to the relevant special case and an approximate solution is sought, they must yield Poisson's [12] equation (3). Using the way indicated, one obtains the following equations:

$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\, g_{\sigma\mu}\, \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = \varkappa(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}); \qquad (\sigma, \nu = 1, 2, 3, 4) \tag{5}$$

Here

$$-2\varkappa \cdot \mathfrak{t}_{\sigma\nu} = \sqrt{-g}\left(\sum_{\beta\tau\varrho} \gamma_{\beta\nu} \frac{\partial g_{\tau\varrho}}{\partial x_\sigma} \frac{\partial \gamma_{\tau\varrho}}{\partial x_\beta} - \frac{1}{2} \sum_{\alpha\beta\tau\varrho} \delta_{\sigma\nu}\, \gamma_{\alpha\beta} \frac{\partial g_{\tau\varrho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\varrho}}{\partial x_\beta}\right); \tag{6}$$

κ is a universal constant that corresponds to the gravitational constants; $\delta_{\sigma\nu}$ is 1 or 0, depending on whether σ and ν are different or equal.

One can see from the system of equations (5), which corresponds to equation (3), that along with the stress-energy components $\mathfrak{T}_{\sigma\nu}$ of the material process, those of the gravitational field (namely, $\mathfrak{t}_{\sigma\nu}$) appear as an equivalent field-inducing cause, a circumstance that obviously must be demanded; for the gravitational effect of a system may not depend on the *physical nature* of the system's field-producing energy.

Since only linear substitutions are admissible, certain one-, two-, and three-dimensional manifolds are privileged, which may be designated as straight lines, planes, and linear spaces. [13]

The theory sketched here overcomes an epistemological defect that attaches not only to the original theory of relativity, but also to Galilean mechanics, and that was especially stressed by E. Mach. It is obvious that one cannot ascribe an absolute [14] meaning to the concept of acceleration of a material point, no more so than one can ascribe it to the concept of velocity. Acceleration can only be defined as relative acceleration of a point with respect to other bodies. This circumstance makes it seem senseless to simply ascribe to a body a resistance to an acceleration (inertial resistance of the body in the sense of classical mechanics); instead, it will have to be demanded that the occurrence of an inertial resistance be linked to the relative acceleration of the body under consideration with respect to other bodies. It must be demanded that the inertial resistance of a body could be increased by having unaccelerated inertial masses arranged in its vicinity; and this increase of the inertial resistance must disappear again if these masses accelerate along with the body. It turns out that this behavior of inertial resistance, which we may call *relativity of inertia*, actually follows from equations (5). This circumstance constitutes one of the [15] strongest pillars of the theory sketched.

LECTURE 6

MATTER and MOTION

I. Important thinkers

II. Momentum
  1. Newtonian
  2. Relativistic

Read Chapter 2 in the 1st Edition of
SPACETIME PHYSICS
by Taylor & Wheeler

OR Chapter 7 in the 2nd Edition of
SPACETIME PHYSICS
by Taylor and Wheeler

Chapter 6 in A JOURNEY INTO
GRAVITY AND SPACETIME
by J.A. Wheeler

In order to mathematize gravitation one must mathematize the motion of bodies, more generally of matter. This is because gravitation leaves its perceptible imprints in the form of the motion of matter. This is to be contrasted with electromagnetism whose imprints of (charged) matter motion is entirely different.

Subsequently we shall mathematize (by means of the Einstein field equations) gravitation as a response to matter in motion.

I. Change, Matter, Motion, and its Causes

The first thinkers to identify the concepts of matter and of motion where the Greek philosophers, even before Socrates (Thales, Heraclitus, Parmenides,

(6.2)

Zeno, Democritus, ...) and then most importantly Aristotle.* He reconciled Parmenides's and Heraclitus's seemingly irreconcilable conclusions about the phenomenon of change. He* pointed out that change presuppes the law of identity, namely: a thing is what it is; its characteristics constitute its identity.

Using his law of identity he* showed there are four causal factors involved in change: 1. the material cause, 2. the formal cause, 3. the efficient cause, 4. the final cause. Ayn Rand (1905-1982) identified the basis of causality in the law of identity by observing that the law of causality is the law of identity applied to action.

*\ footnote { An exposition and explanation of Aristotle's fundamental work on change and causality — which I found to be second to none in clarity combined with pithiness can be found by "googling" "Peikoff Aristotle metaphysics." }

The concepts of change and motion were put into mathematical form by Galileo, Kepler, and most definitively by Newton with his laws of dynamical motion. They are the platform from which all future developments where launched. His familiar fundamental dynamical laws are three in number.

1. Every object in a state of uniform motion tends to remain in that unless a force is applied to it.
2. $\overrightarrow{\text{Force}} = \frac{d}{dt}(\overrightarrow{\text{momentum}})$
3. For every action there is an opposite and equal reaction.

These three laws are integrated into an organic whole by the law of the conservation of momentum in a collision process.

Using Special Relativity, Einstein not only generalized Newton's dynamical laws of motion to relativistic velocities, but he also geometrized them.

He did this by introducing the integrating concept of the energy-momentum (better known as "momenergy") four-vector into the dynamics of relativistically moving matter.

II. Momentum

1. Arrive at the definition of momentum by focusing on low velocity collisions by means of the following Newtonian line of reasoning. Thus ask the question:

What quantity common to all collision processes remains the same before and after any such process?

Answer:

a) Consider the following spacetime

process involving the collion of two particles.

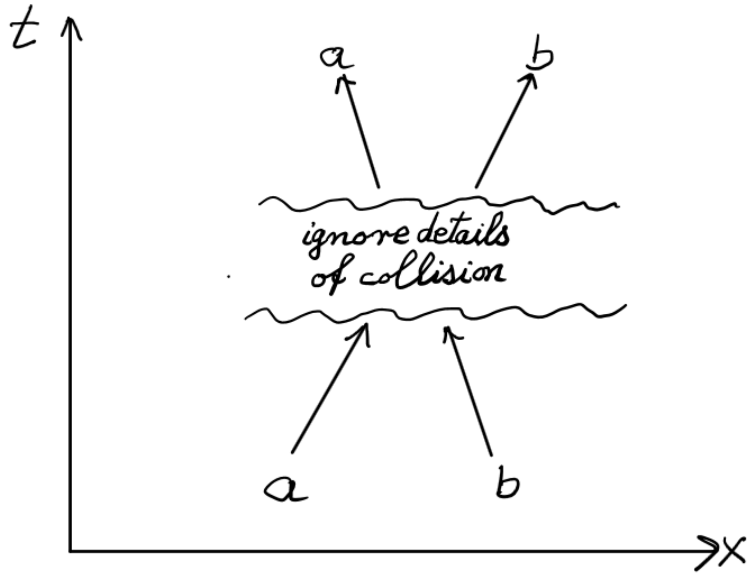


Figure 4.1 Collision between particles a and b.

b) Consider the quantity

$$(\text{mass}) \times (\text{velocity}) \equiv \text{Momentum}$$

c) Focus on Newton's 2nd Law

$$\frac{d}{dt}(\text{Momentum of } a) = F_{b\,\text{on}\,a}$$

d) Take notice of Newton's 3rd Law

$$F_{b\,on\,a} = -F_{a\,on\,b}$$

e) Apply c) & d) to all particles and conclude after integration that

$$\sum_{\substack{\text{particles}\\ a,\,b}} \left(\text{Momentum before}\right) = \sum_{\substack{\text{particles}\\ a,\,b}} \left(\text{Momentum after}\right)$$

2. Generalize the concept of momentum to relativistic:

A. Consider (mass)(velocity) and then label

$$m\,\frac{d\vec{v}}{dt} = \overrightarrow{\text{momentum}}$$

as "momentum". However, this does not lead to any conceptual economy. This is because such an extension of the Newtonian definition does not lead to the expression for any conserved quantity before and after when examined in different inertial frames.

Hence focus attention on those quantities that do exhibit conservation.

B. Definition of relativistic momentum as identified by Tolman & Ehrenfest.

(FYI: A definition is the condensation of a vast body of observations - and stands or falls with the truth or falsehood of the observations.*)

*\footnote{See Chapter 5 ("Definitions") in 2nd Edition of "INTRODUCTION TO OBJECTIVIST EPISTEMOLOGY" by Ayn Rand}

Theorem (Definition of momentum)

GIVEN: a) Principle of Relativity
b) Isotropy of space
c) Symmetry

d) There exists a unique momentum vector associated with a particle having a given velocity (6,7)

e) The momentum is conserved during a collision

f) The correspondence at low velocities with Newton's definition must not be violated

CONCLUSION:

$$\boxed{\text{momentum} = m\frac{\vec{\beta}}{\sqrt{1-\beta^2}};\ \vec{\beta} = \text{velocity}}$$

Comment about the proof:

1. Look at every symmetric collision from the point of view of <u>two</u> reference frames
2. Look for a quantity conserved in <u>both</u> frames to find the momentum-velocity relationship.

Proof in six steps:

1. Q: What is the relation between
$\vec{p}$ = momentum, and
$\vec{v}$ = velocity of the particle?

A: $$\boxed{\vec{p} = f(v)\,\vec{v}}$$

Why? (i) isotropy of space
(ii) $\vec{p}$ is unique
(iii) $f(v) = f\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right)$

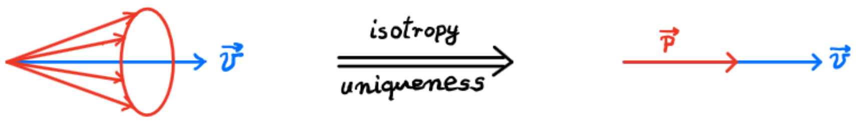


Figure 6.1 Momentum and velocity are necessarily collinear.

2. Principle of Relativity implies $f(v)$ is the <u>same</u> function in <u>all</u> inertial reference frames

3. To determine f, consider the elastic

collision between two identical particles viewed in to inertial frames.

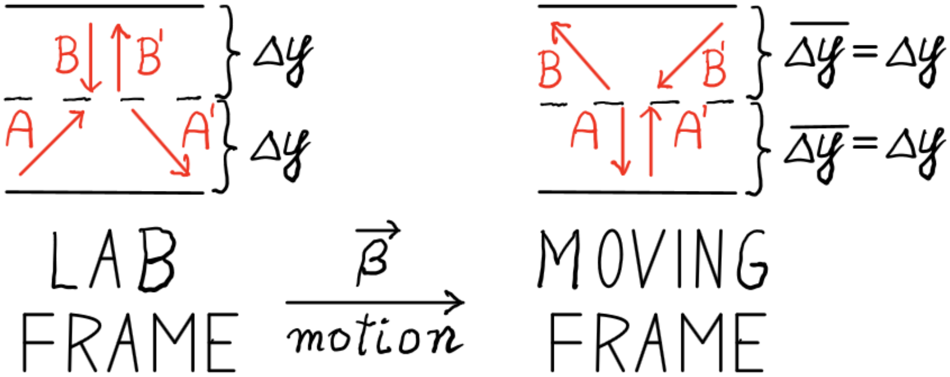


Figure 6.2 Collision observed in two different inertial frames.

For this type of a collision scenario one has

- particles are identical
- Principle of Relativity
- isotropy of space

$\Rightarrow$ the two pictures are symetrical

- symmetry $\Longrightarrow$ pictures are <u>congruent</u>!

Consequently,

$\vec{A}'$ in LAB $= -\vec{B}$ in MOVING FRAME

$\vec{B}'$ in LAB $= -\vec{A}$ in MOVING FRAME

superfluous & of no consequence

(6.10)

4. (i) In the MOVING FRAME the components of A are

A's velocity $= \frac{\overline{\Delta y}}{\overline{\Delta t}}$, where $\overline{\Delta t}$ = time for A to move from bottom to the point of collision

A's four-vector $= (\overline{\Delta t}, 0, \overline{\Delta y}, 0)$.

(ii) In the LAB FRAME the components of A are:

A's four-vector $= (\Delta t, \Delta x, \Delta y, 0)$.

$$= \left(\frac{\overline{\Delta t}}{\sqrt{1-\beta^2}}, \frac{\beta \overline{\Delta t}}{\sqrt{1-\beta^2}}, \overline{\Delta y}, 0\right)$$

A's spatial velocity $= \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, 0\right)$

$$= \left(\beta, \frac{\overline{\Delta y}}{\overline{\Delta t}}\sqrt{1-\beta^2}, 0\right)$$

$$\equiv \left(\beta, \mu\sqrt{1-\beta^2}, 0\right)$$

where $\mu = \frac{\overline{\Delta y}}{\overline{\Delta t}}$

5.) Conservation of momentum in LAB FRAME:

(i) Along the y-direction*

$$\sum P_y)_{\text{before}} = \sum P_y)_{\text{after}}$$

Referring to the l.h.s. of Figure 6.2, find that this equality becomes

$$\underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\times\mu\sqrt{1-\beta^2}}_{A}\ \underbrace{-f\left(\sqrt{0+\mu^2}\right)\mu}_{B} =$$

$$= \underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\times(-)\mu\sqrt{1-\beta^2}}_{A'}\ \underbrace{+f\left(\sqrt{0+\mu^2}\right)\mu}_{B'}$$

The resulting functional identity is

$$f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right) = \frac{f(\mu)}{\sqrt{1-\beta^2}}$$

Take limit $\mu \to 0$ and obtain

$$\boxed{f(\beta) = \frac{f(0)}{\sqrt{1-\beta^2}}}$$

* \footnote{What new information is implied by the corresponding conservation statement $\sum p_x)_{\text{before}} = \sum p_x)_{\text{after}}$?}

6.) The value of $f(0)$ is obtained from the Newtonian correspondence limit.

Asymptotically one has

$$f(v)\, v_y \longrightarrow m\, v_y$$

$$1 = \frac{p_y}{p_y} = \lim_{v \to 0} \frac{m v_y}{f(v) v_y} = \frac{m}{f(0)}$$

$$\therefore \quad f(\beta) = \frac{m}{\sqrt{1-\beta^2}}$$

Conclusion

$$\boxed{\overrightarrow{\text{momentum}} = \frac{m}{\sqrt{1-\beta^2}}\vec{\beta}}$$

Comment:

If $f(v)$ is independent of v then one implicitly assumes the Newtonian (non-relativistic) approximation.

LECTURE 7

I. Conservation of Energy

via Momentum Conservation + the Principle of Relativity

II. The Momenergy 4-vector

Read Sections 7.1-7.7
in SPACETIME PHYSICS (2nd Edition)
by Taylor & Wheeler

Also very good is
Chapter 2 (MOMENTUM and ENERGY)
in the 1st Edition of SPACETIME PHYSICS

I. Momentum, Energy, and their Conservation

(7.1)

In Relativity momentum and energy form an organic whole, namely momenergy. This concept is one in which momentum and energy are inextricably intertwined.

1. Momentum in Relativity

Collisions of particles is the observational basis for momentum and it conseration in Newtonian dynamics as well as its generalization to relativistic dynamics. In this context the 3-d relativistic momentum vector is

$$\overrightarrow{\text{momentum}} = \frac{m}{\sqrt{1-\beta^2}}\vec{\beta} \qquad (7,1)$$

2. Enery in Relativity

In Newtonian mechanics

momentum conservation and energy conservation are distinct physical principles. In relativistic mechanics, by contrast, momentum conservation <u>implies</u> energy conservation:

$$\text{tot. mom.} \equiv \sum_j \frac{m_j \vec{\beta}_j}{\sqrt{1-\beta_j^2}}\Bigg|_{\text{before}} = \sum_i \frac{m_i \vec{\beta}_i}{\sqrt{1-\beta_i^2}}\Bigg|_{\text{after}} \Longrightarrow \text{energy conservation}$$

The line of reasoning leading to this conclusion draws on two facts.

1. The unit tangent to the world line of a particle is a 4-vector with components

$$\{u^\alpha\} = \left\{\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right\} \equiv \left\{\frac{dt}{d\tau}, \vec{u}\right\} \qquad (7.2)$$

whose Lorentz transformation property is known:

$$\boxed{\{u^\alpha\} \overset{\Lambda^{\bar\alpha}{}_{\alpha}}{\rightsquigarrow} \{\bar{u}^{\bar\alpha}\}.} \qquad (7.3)$$

2. The spatial 3-vector is proportional

to the relativistic 3-momentum (7.3)

$$\vec{u} \propto \frac{m\vec{\beta}}{\sqrt{1-\beta^2}}$$

Indeed, in light of

$$(d\tau)^2 = dt^2 - dx^2 - dy^2 - dz^2$$

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \underbrace{\left(\frac{dx}{d\tau}\right)^2}_{\left(\frac{dx}{dt}\frac{dt}{d\tau}\right)^2 \equiv \beta_x^2\left(\frac{dt}{d\tau}\right)^2} - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2$$

Consequently,

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}, \quad \frac{d\vec{x}}{d\tau} = \frac{\vec{\beta}}{\sqrt{1-\beta^2}}$$

and Eq. (7.2) becomes

$$\{u^\alpha\} \equiv \{u^0; \vec{u}\} = \left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\vec{\beta}}{\sqrt{1-\beta^2}}\right\}. \qquad (7.4)$$

Compare the spatial components $\{u^x; u^y; u^z\} \equiv \vec{u}$ of Eq. (7.4) with (7.1) and find

$$\boxed{\vec{p} = m\vec{u}.} \qquad (7.5)$$

The relativistic 3-momentum is proportional, component by component, to $\vec{u}$, the spatial part of the 4-velocity with the mass m serving as the proportionality constant.

The spatial part

$$\vec{u} = \frac{\vec{\beta}}{\sqrt{1-\beta^2}} \qquad (7.6)$$

of the 4-velocity is the key that unlocks the door to the relativistic law of energy conservation.

Indeed, apply Eq. (7.5) to the conservation of momentum in two inertial frames in relative motion.

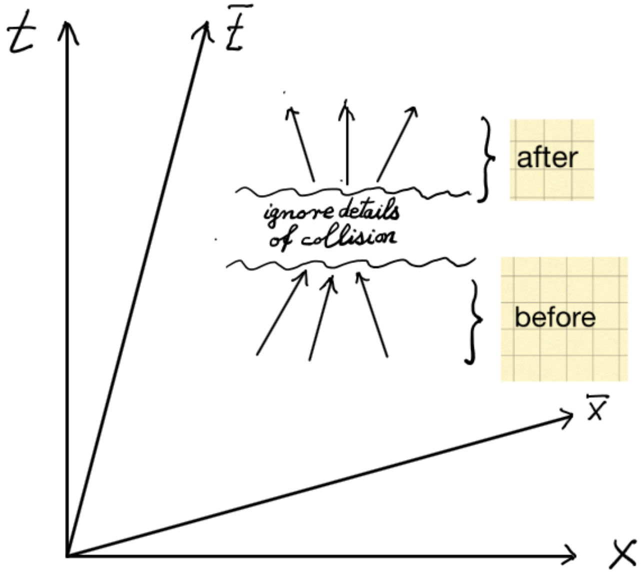


Figure 4.1 Collision between particles observed in two inertial reference frames. In the (x,t)-frame, during a collision total momentum is neither created nor destroyed; it is conserved. The total momentum is the same before and after the collision:

$$0=\sum_i \vec{p}_{(i)}\Big|_{after}-\sum_j \vec{p}_{(j)}\Big|_{before}$$

Because this is a vectorial equality, it holds for each of its components separately.

The theorem on the next page makes this application explicit.

Theorem (Conservation of Energy)

GIVEN:

(a) The Principle of Relativity

(b) Conservation of spatial momentum in a collision of particles

CONCLUSION:

Total energy of particles is conserved

proof (in 4 steps):

1. Focus on a collision process in two inertial frames

Let $\{u^\mu\}$ refer to the 4-velocity of a particle.

2. Apply the transformation law to the momentum of <u>each</u> particle

$$p_x = m u_x = m(\bar{u}_x \, ch\theta + \bar{u}_t \, sh\theta)$$

$$= \bar{p}_x \, ch\theta + m \frac{1}{\sqrt{1-\beta^2}} sh\theta$$

3. Apply the momentum conservation principle to the particle momenta in each inertial reference frame

$$0 = \sum_i p_{x(i)}\Big|_{after} - \sum_j p_{x(j)}\Big|_{before}$$

$$= \underbrace{\left(\sum_i \bar{p}_{x(i)}\Big|_{after} - \sum_j \bar{p}_{x(j)}\Big|_{before}\right)}_{\substack{\parallel \\ 0}} ch\theta$$

$$+ \left(\sum_i \frac{m_i}{\sqrt{1-\beta_i^2}}\Big|_{after} - \sum_j \frac{m_j}{\sqrt{1-\beta_j^2}}\Big|_{before}\right) sh\theta$$

Thus

$$\boxed{\sum_i \frac{m_i}{\sqrt{1-\beta_i^2}} \text{ is conserved.}}$$

4. Go to the Newtonian non-relativistic approximation ($\beta^2 \cong \frac{v^2}{c^2} \ll 1$) and find

$$\frac{mc^2}{\sqrt{1-\beta^2}} = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)$$

$$\simeq \underbrace{m c^2}_{\substack{\text{rest}\\ \text{mass}\\ \text{energy}}} + \underbrace{\tfrac{1}{2} m v^2 + \cdots}_{\substack{\text{kinetic}\\ \text{energy}}} = \begin{pmatrix}\text{mass-energy}\\ \text{of the particle}\end{pmatrix} \tag{7.8}$$

SUMMARY

Momentum conservation + P. of R.

$$\Rightarrow \boxed{\sum_i (\text{mass-energy})_i \equiv \text{Total mass-energy} \atop \text{is conserved}}$$

II. The Energy-Momentum 4-vector

Keeping in mind that a definition is the condensation of a vast body observations – and stands or falls with the truth or falsehoods of these observations, arrive at the following

Definition

a) $$\begin{pmatrix}\text{momenergy}\\ \text{for each}\\ \text{particle}\end{pmatrix} = \mathbf{p} : \{p^\alpha\} \equiv \left\{ \frac{m}{\sqrt{1-\beta^2}}, \frac{m\vec{\beta}}{\sqrt{1-\beta^2}} \right\} \qquad (7.7)$$

b) $$\begin{pmatrix}\text{magnitude of}\\ \text{momenergy}\end{pmatrix}^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$$

$$= m^2 = (\text{rest mass})^2$$

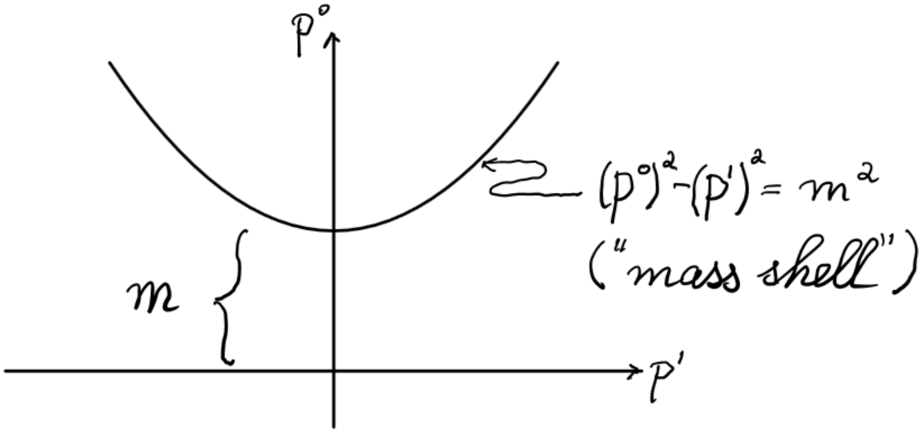


Figure 7.2 The momergy of a particle lies on its (rest) mass-shell.

c) Kinetic Energy

A particle whose observed momenergy components are given by Eq. (7.7) on page 7.9 has kinetic energy

$$K.E. = \frac{m}{\sqrt{1-\beta^2}} - m = \frac{1}{2} m \beta^2 + \frac{3}{8} m \beta^4 + \frac{5}{16} m \beta^6 + \cdots$$

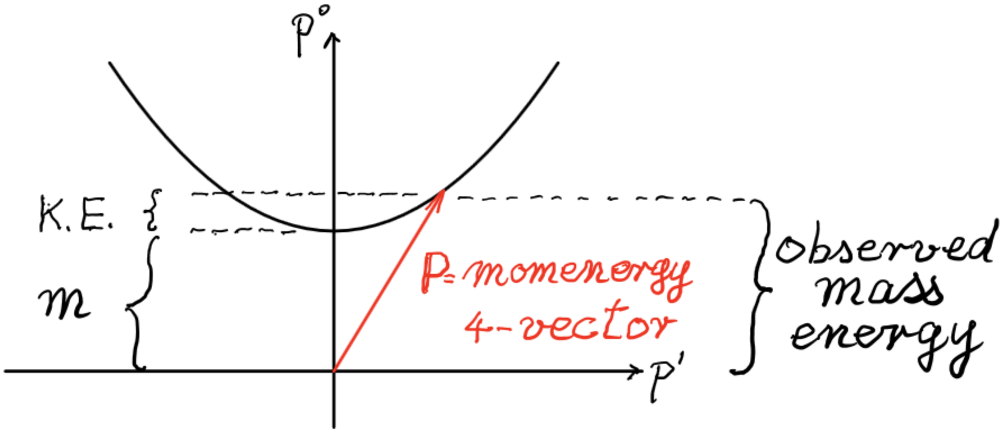


Figure 7.3 A particle with momergy 4-vector $\mathbf{P}$ (an element in the 4-d vector space of relativistic 4-momenta) and rest mass m has observed mass-energy

$$\frac{m}{\sqrt{1-\beta^2}} = m + K.E.$$

\footnote{ Every application of the law of momenergy conservation in a collision process is a statement about a polygon built of 4-momenta in the vector space of relativistic 4-momenta

$$\boxed{\sum_{i=1}^{N} \mathbb{P}_i\Big|_{\text{before}} = \mathbb{P}_{tot}\Big|_{\text{before}} = \mathbb{P}'_{tot}\Big|_{\text{after}} = \sum_{j=1}^{M} \mathbb{P}'_j\Big|_{\text{after}}}$$

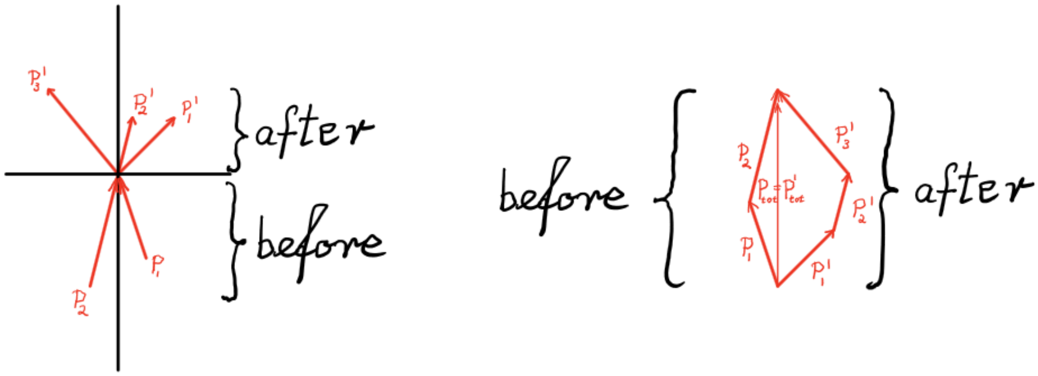


Figure 7.4 Collision event geometrized by a closed polygon in the vector space of 4-momenta

}

Lecture 8

Particle density-flux

A. COMOVING frame vs. LAB frame

B. 3-volumes in spacetime

In MTW know Box 4.4 (the math of $^*J$)
Box 5.2 (the physics of $^*J$)
Box 15.2 (the charge density-flux 3-form)

I. "Particle" as a Fundamental Concept in Physics

In physics a particle is a contextully small entity of a specific nature and having specific attributes (mass, momenergy, spin, charge, color, etc.). It is the building block fundamental to understand the physical world.

Matter, in whatever state of motion, controls gravitation. To understand gravitation requires an understanding of matter in motion. But matter is an aggregate of particles. Thus, to mathematize (and hence to understand) gravitation one must mathematize (an aggregate of) particles in motion, including relativistic motion.

I. Comoving Volume

This process starts by considering an aggregate of particles in a region of spacetime large enough so that one can talk about density, flux, pressure, etc., but then small enough so that these particles can be said to have the same velocity in a given volume element.

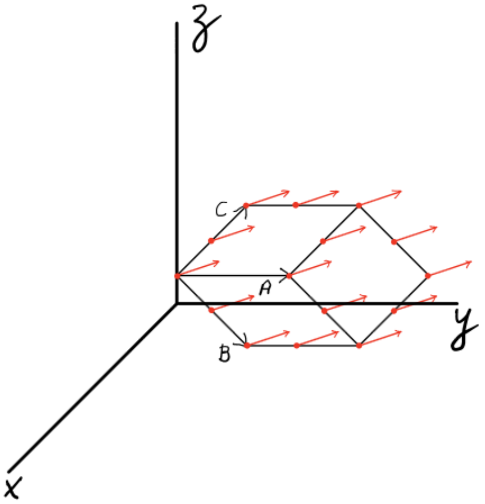


Figure 8.1 A uniform aggregate of particles having the same velocity throughout an element of volume as well as on its boundary.

The common properties of this aggregate is observed relative to two reference frames, a LAB frame and the COMOVING frame. The latter is determined entirely by the particles.

The particles have zero spatial velocity in the COMOVING frame $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$. Their common 4-velocity is

$$\mathbf{u} = u^\mu \frac{\partial}{\partial x^\mu} = 1\,\frac{\partial}{\partial x^0} + 0\frac{\partial}{\partial x^1} + 0\frac{\partial}{\partial x^2} + 0\frac{\partial}{\partial x^3} \equiv \frac{d}{d\tau}$$

By contrast, relative to the LAB frame their 4-velocity* is

$$\mathbf{u} = u^{\bar\mu}\frac{\partial}{\partial \bar x^{\bar\mu}} = u^{\bar 0}\frac{\partial}{\partial \bar x^{\bar 0}} + u^{\bar 1}\frac{\partial}{\partial \bar x^{\bar 1}} + u^{\bar 2}\frac{\partial}{\partial \bar x^{\bar 2}} + u^{\bar 3}\frac{\partial}{\partial \bar x^{\bar 3}} \equiv \frac{d}{d\tau}$$

The element of 3-volume is spanned by the three space-like vectors

$$A \equiv 0\cdot\frac{\partial}{\partial t} + \Delta x\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z} \equiv A^\alpha \frac{\partial}{\partial x^\alpha}$$

$$B \equiv 0\cdot\frac{\partial}{\partial t} + 0\cdot\frac{\partial}{\partial x} + \Delta y\cdot\frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z} \equiv B^\alpha \frac{\partial}{\partial x^\alpha}$$

$$C \equiv 0\cdot\frac{\partial}{\partial t} + 0\cdot\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \Delta z\frac{\partial}{\partial z} \equiv C^\alpha \frac{\partial}{\partial x^\alpha}$$

* \footnote{ The "bar" over an index is a reminder that the components are relative to the basis

$$\{\bar{e}_{\bar{\mu}}\} = \{\frac{\partial}{\partial \bar{x}^{\bar{\mu}}}\}$$

for the LAB frame. The basis elements without such a bar,

$$\{e_{\mu}\} = \{\frac{\partial}{\partial x^{\mu}}\},$$

are for the COMOVING frame.}

As depicted in Figure 8.1, their components are attached to respective pairs of particles in the COMOVING frame with its orthonormal comoving basis $\{e_\mu \frac{\partial}{\partial x^\mu}\}$. Consequently, the element of proper (= comoving) volume is

$$\Delta x \, \Delta y \, \Delta z = \det \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \Delta x & 0 & 0 \\ 0 & 0 & \Delta y & 0 \\ 0 & 0 & 0 & \Delta z \end{vmatrix}$$

$$= \det \begin{vmatrix} u^0 & u^1 & u^2 & u^3 \\ A^0 & A^1 & A^2 & A^3 \\ B^0 & B^1 & B^2 & B^3 \\ C^0 & C^1 & C^2 & C^3 \end{vmatrix}$$

$$= u^\mu \epsilon_{\mu\alpha\beta\gamma} \langle dx^\alpha, A \rangle \langle dx^\beta, B \rangle \langle dx^\gamma, C \rangle$$

$$= u^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} (A, B, C) \quad (8.1)$$

= comoving volume spanned by $\{A, B, C\}$

Comment 8.1

As exhibited by Eq.(8.1), this volume element (which is depicted in Figure 8.2 on page 8.8) is presented in a form which is frame-invariant.

II. Comoving Particle Density

Consider the density of particles in the volume element $\Delta x \Delta y \Delta z$ spanned by $\{A, B, C\}$,

$$N = \frac{\#}{\Delta x \Delta y \Delta z} = \frac{\begin{pmatrix}\text{number of} \\ \text{particles}\end{pmatrix}}{\begin{pmatrix}\text{comoving} \\ \text{volume}\end{pmatrix}}$$

The number of particles in the comoving volume element is a frame invariant; it is independent of an observer's reference frame. Thus

$$\begin{aligned} \# &= N \Delta x \Delta y \Delta z \\ &= N u^{\mu} \epsilon_{\mu |\alpha\beta\gamma|} \, dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma} (A, B, C) \\ &= N \bar{u}^{\bar{\mu}} \epsilon_{\bar{\mu} |\bar{\alpha}\bar{\beta}\bar{\gamma}|} \, d\bar{x}^{\bar{\alpha}} \wedge d\bar{x}^{\bar{\beta}} \wedge d\bar{x}^{\bar{\gamma}} (A, B, C) \end{aligned}$$

Relative to <u>any</u> basis $\{e_{\sigma'}\}$ and its dual $\{\omega^{\sigma}\}$,

$$\omega^{\sigma}(e_{\sigma'}) \equiv \langle \omega^{\sigma}, e_{\sigma'} \rangle = \delta^{\sigma}{}_{\sigma'},$$

(be it coordinate induced, orthonormal, oblique, etc) the frame invariance of the particle count is mathematized by the statement

$$\# = N\, u^{\mu'}\, \epsilon_{\mu'|\alpha'\beta'\gamma'|}\, \omega^{\alpha'} \wedge \omega^{\beta'} \wedge \omega^{\gamma'} (A, B, C). \qquad (8.2)$$

This statement says that # depends on the spanning vectors in the manner of a <u>trilinear function</u>. Given the tensors N, $u = u^{\mu'} e_{\mu'}$, and $\epsilon_{\mu'\alpha'\beta'\gamma'}\ \omega^{\alpha'} \wedge \omega^{\beta'} \wedge \omega^{\gamma'}$ in Eq. (8.2), there must be some chosen triad of vectors (A, B, C) (such as the one on page 8.3), but there may be any such chosen triad.

Using this "some but any" principle[*] one arrives at the concept $^*S$, the <u>density-flux 3-form.</u>

It is defined by the expression

$$\boxed{N u^{\mu} \epsilon_{\mu\alpha\beta\gamma} \frac{\omega^{\alpha} \wedge \omega^{\beta} \wedge \omega^{\gamma}}{3!} \equiv {}^*S} \qquad (8.3)$$

*\footnote{This principle was identified and used by Ayn Rand, "Introduction to Objectivist Epistemology", in her theory of concept formation in the chapter "Abstraction From Abstraction". It, among others, deals with the process of widening and narrowing a concept}

More explicitly, the concept of the density-flux 3-form $^*\mathfrak{S}$ is a trilinear map, a scalar-valued tensor of rank $\binom{0}{3}$. These and other observations yield the following more explicit

Definition (Density-flux 3 form)

$$^*\mathfrak{S}: \quad V\times V\times V \longrightarrow R^1 \quad (\#\text{ of particles})$$

$$(A,B,C) \rightsquigarrow {}^*\mathfrak{S}(A,B,C) = N\, u^\mu\, \epsilon_{\mu|\alpha\beta\gamma|}\, \omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\,(A,B,C).$$

One of the volume spanning vectors A, B, and C can be time-like. This is highlighted by the qualitative but broader definition

$$^*\mathfrak{S} = \frac{(\#\text{ of particles})}{(\text{as-yet-unspecified 3-volume})}.$$

It implies that the element of volume, besides being purely <u>space-like</u>, can also have one of its spanning vectors be time-like. Under this circumstance the element of volume is said to be <u>time-like</u>.

Both cases play a key role in the motion of matter from the spacetime perspective.

Case (1)

The spacetime 3-volume is spanned by three spacelike vectors

In the COMOVING frame one has

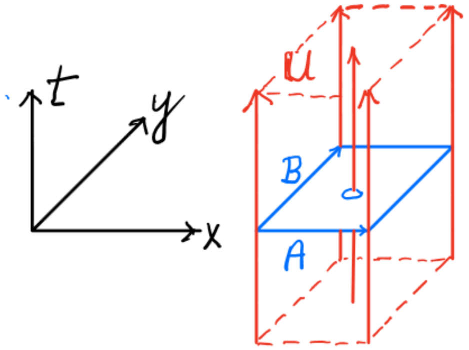


Figure 8.2 Comoving frame representation of finite world tube filled with particle world lines, all with 4-velocity u, passing through the 2-d rendition of the 3-d spatial volume element spanned by A, B, and C.

$$\# = {}^{*}S(A,B,C) = N u^{\mu}\epsilon_{\mu\alpha\beta\gamma}A^{\alpha}B^{\beta}C^{\gamma}$$

$$= \underbrace{N\,u^{0}}_{}\underbrace{\epsilon_{0\alpha\beta\gamma}A^{\alpha}B^{\beta}C^{\gamma}}_{\text{COMOVING volume}}$$

$$S^{0} = \frac{\#}{(\text{COMOVING volume})} = \begin{pmatrix}\text{comoving}\\ \text{particle}\\ \text{density}\end{pmatrix}$$

In the LAB frame consider the triad of space-like vectors

$$\bar{A} \equiv 0\cdot\frac{\partial}{\partial\bar{t}} + \Delta\bar{x}\frac{\partial}{\partial\bar{x}} + 0\cdot\frac{\partial}{\partial\bar{y}} + 0\cdot\frac{\partial}{\partial\bar{z}} \equiv \bar{A}^{\bar{\alpha}}\frac{\partial}{\partial\bar{x}^{\bar{\alpha}}}$$

$$\bar B \equiv 0\cdot\frac{\partial}{\partial \bar t} + 0\cdot\frac{\partial}{\partial \bar x} + \Delta\bar y\,\frac{\partial}{\partial \bar y} + 0\cdot\frac{\partial}{\partial \bar z} \equiv \bar B^{\bar\alpha}\frac{\partial}{\partial \bar x^{\bar\alpha}} \qquad (8.4) \qquad (8.9)$$

$$\bar C \equiv 0\cdot\frac{\partial}{\partial \bar t} + 0\cdot\frac{\partial}{\partial \bar x} + 0\,\frac{\partial}{\partial \bar y} + \Delta\bar z\,\frac{\partial}{\partial \bar z} \equiv \bar C^{\bar\alpha}\frac{\partial}{\partial \bar x^{\bar\alpha}}$$

Each one is a spatial displacement connecting a pair of events (a) simultaneous in the LAB and (b) on the boundary of the particle world tube.

Consequently, the number of world lines intercepted by $u^{\mu}\epsilon_{\mu\bar\alpha\bar\beta\bar\gamma}\bar A^{\bar\alpha}\bar B^{\bar\beta}\bar C^{\bar\gamma}$, i.e. the number of particles observed in volume element spanned by $\bar A$, $\bar B$, and $\bar C$, is the same as before, but the components are relative to the LAB basis $\{\bar e_{\bar\mu} = \frac{\partial}{\partial \bar x^{\bar\mu}}\}$.

Consequently,

$$\begin{aligned}
\# &= N u^{\bar\mu}\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\bar A^{\bar\alpha}\bar B^{\bar\beta}\bar C^{\bar\gamma} \\
&= N u^{\bar 0}\epsilon_{\bar 0\bar\alpha\bar\beta\bar\gamma}\bar A^{\bar\alpha}\bar B^{\bar\beta}\bar C^{\gamma} \\
&= \underbrace{N u^{\bar 0}}\,\underbrace{\epsilon_{\bar 0\bar 1\bar 2\bar 3}\,\overline{\Delta x}\,\overline{\Delta y}\,\overline{\Delta z}}_{\text{LAB volume}}
\end{aligned}$$

$$S^{\bar 0} = \frac{\#}{(\text{LAB volume})} = \frac{\text{particle density}}{(\text{in LAB frame})} \qquad (8.5)$$

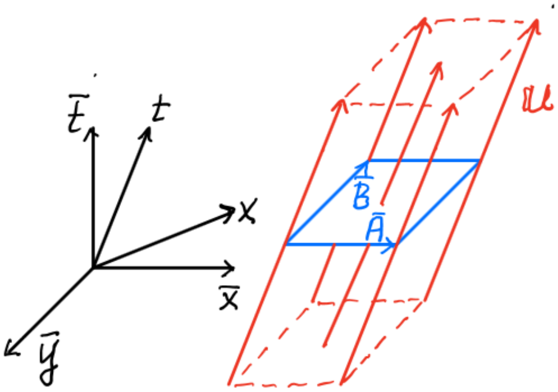


Figure 8.3 World tube filled with straight world lines of type $\mathcal{U}$ that pass through the LAB volume element $\Delta\bar{x}\,\Delta\bar{y}\,\Delta\bar{z}$ spanned by space-like vectors $\bar{A}$, $\bar{B}$, and $\bar{C}$.

Case (2)

The spacetime 3-volume is spanned by one time-like vector ($\bar{\bar{A}}$) and two space-like vectors ($\bar{\bar{B}}, \bar{\bar{C}}$). As a linear combination of the LAB basis $\{\vec{e}_{\bar{\mu}}\}=\{\frac{\partial}{\partial \bar{x}^{\bar{\mu}}}\}$ these vectors are

$$\bar{\bar{A}} \equiv \Delta\bar{\bar{t}} \cdot \frac{\partial}{\partial \bar{t}} + 0 \cdot \frac{\partial}{\partial \bar{x}} + 0 \frac{\partial}{\partial \bar{y}} + 0 \cdot \frac{\partial}{\partial \bar{z}} \equiv \bar{\bar{A}}^{\bar{\alpha}} \frac{\partial}{\partial \bar{x}^{\bar{\alpha}}}$$

$$\bar{\bar{B}} \equiv 0 \cdot \frac{\partial}{\partial \bar{t}} + 0 \cdot \frac{\partial}{\partial \bar{x}} + \Delta\bar{\bar{y}} \frac{\partial}{\partial \bar{y}} + 0 \cdot \frac{\partial}{\partial \bar{z}} \equiv \bar{\bar{B}}^{\bar{\alpha}} \frac{\partial}{\partial \bar{x}^{\bar{\alpha}}} \qquad (8.6)$$

$$\bar{\bar{C}} \equiv 0 \cdot \frac{\partial}{\partial \bar{t}} + 0 \cdot \frac{\partial}{\partial \bar{x}} + 0 \frac{\partial}{\partial \bar{y}} + \Delta\bar{\bar{z}} \frac{\partial}{\partial \bar{z}} \equiv \bar{\bar{C}}^{\bar{\alpha}} \frac{\partial}{\partial \bar{x}^{\bar{\alpha}}}$$

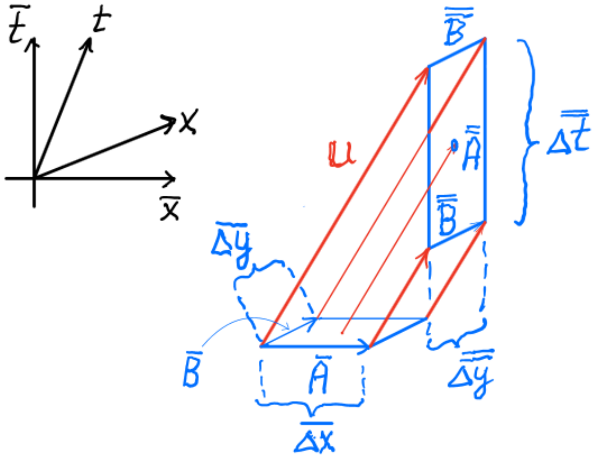


Figure 8.4 The 4-d world tube, which is bounded by 3-d spatial volume element $\overline{\Delta x}\,\overline{\Delta y}\,\overline{\Delta z}$ and by 3-d time-like volume element $\overline{\overline{\Delta t}}\,\overline{\overline{\Delta y}}\,\overline{\overline{\Delta z}}$, is filled with straight world lines of type u. They pass through the LAB volume element $\overline{\Delta x}\,\overline{\Delta y}\,\overline{\Delta z}$ spanned by space-like vectors $\bar A$, $\bar B$, and $\bar C$ and through the time-like volume element spanned by space-like vectors $\bar{\bar B}$ and $\bar{\bar C}$ and by time-like vector $\bar{\bar A}$

The number of particles observed in the time-like 3 volume spanned by $\bar{\bar A}$, $\bar{\bar B}$, and $\bar{\bar C}$ is given by the statement that

$$
\begin{aligned}
\# &= {}^{*}S(\bar{\bar A}, \bar{\bar B}, \bar{\bar C}) \\
&= u^{\mu}\epsilon_{\mu\alpha\beta\gamma}\bar{\bar A}^{\alpha}\bar{\bar B}^{\beta}\bar{\bar C}^{\gamma}. \qquad (8.7)
\end{aligned}
$$

This is an example of a statement which is \emph{objective}. This is because it combines #, a metaphysical feature of the world, with a conceptual method of <u>processing</u> the observed data about #, namely by using that data to evaluate the r.h.s. of Eq.(8.6). The act of gathering the observed data and processing it is always the same regardless of the inertial reference frame where the action was done.

Relative to the LAB frame, whose standard of measurements is the LAB basis $\{\bar{e}_{\bar{\mu}} = \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}\}$, the evaluation of *S yields

$$\# = {}^{*}S(\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}) = u^{\bar{\mu}} \epsilon_{\bar{\mu}|\bar{\alpha}\bar{\beta}\bar{\gamma}|} \bar{\bar{A}}^{\bar{\alpha}} \bar{\bar{B}}^{\bar{\beta}} \bar{\bar{C}}^{\bar{\gamma}}$$

$$= \underbrace{N u^{\bar{1}}}_{\Big\downarrow} \epsilon_{\bar{1}\bar{0}\bar{2}\bar{3}} \underbrace{\bar{\bar{\Delta t}}}_{\bar{\bar{A}}\text{'s lab time window}} \underbrace{\bar{\bar{\Delta y}}\, \overrightarrow{\Delta z}}_{\text{lab area spanned by } \bar{\bar{B}} \text{ and } \bar{\bar{C}}}$$

$$S^{\bar{1}} = \frac{\#}{(\text{time})(\overrightarrow{\text{area}})^{1}} = \frac{\#}{(\text{time})(\vec{B} \times \vec{C})^{1}} \qquad (8.8)$$

Summary

1. Based on Eq. (8.5) on page 8.9, the evaluation

$$^{*}\mathbf{S} = S^{\mu}\, \epsilon_{\mu|\alpha\beta\gamma|}\; dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}$$

on the element of volume spanned by $\bar{A}$, $\bar{B}$, and $\bar{C}$ yields

$$S^{0} = \frac{(\text{particles})}{(\text{volume})} = \text{particle density}$$

Based on Eq. (8.8) on page 8.12, the evaluation of $^{*}\mathbf{S}$ on the element of volume spanned by $\bar{\bar{A}}$, $\bar{\bar{B}}$, and $\bar{\bar{C}}$ yields

$$\left.\begin{matrix} S^{1} \\ S^{2} \\ S^{3} \end{matrix}\right\} S^{i} = \frac{\text{particles}}{(\text{time})\,(\overrightarrow{\text{area}})^{i}} = \begin{matrix}\text{particle flux into} \\ \text{the } i^{th} \text{ direction}\end{matrix}$$

2. The vector formed from the proper particle density N and the particle 4-velocity $\boldsymbol{u}$,

$$N\boldsymbol{u} \equiv \mathbf{S} = S^{\mu}\boldsymbol{e}_{\mu}$$

is called the <u>particle current 4-vector.</u>

Lecture 9

The particle density-flux 3 form:

I. Physical properties

II. Algebraic properties ⎫ Consigned to Lecture 10
III. Geometrical properties ⎭

In MTW know Box 4.4
Box 5.1
Fig 5.1
Box 5.2

The particle density-flux 3-form

$$ {}^*\mathcal{S} = \underbrace{N u^\mu}_{\equiv\ S^\mu} \underbrace{\epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}}_{\Sigma_\mu} \qquad (9.1) $$

mathematizes the physical spacetime properties of a continuous medium of particles. This included the types of media found in extremely relativistic astrophysical environments as well as those driven by ultra-intense laser radiation.

I. Physical Properties of the Particle Density-flux 3-form

The 3-form ${}^*\mathcal{S}$ is the result of the mental integration ("unification") of four physical attributes of particles in motion:

(i) their common 4-velocity

$$ \boxed{\mathbf{u} = u^\mu \frac{\partial}{\partial x^\mu} \equiv u^\mu e_\mu} \qquad (9.2) $$

and their associated particle current

4-vector

$$\boxed{N u^\mu e_\mu = S^\mu \frac{\partial}{\partial x^\mu} \equiv \mathbf{S}} \qquad (9.3)$$

(ii) their numerical particle count (particle "number") in the 3-volume spanned by three freely chosen 4-d vectors A, B, and C, is

$$\# = N u^\mu \epsilon_{\mu\alpha\beta\gamma} \; dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \, (A, B, C)$$

$$= N u^\mu \, \Sigma_\mu (A, B, C).$$

Being a scalar, this is a frame invariant, i.e. independent of the inertial frame for evaluating the scalar.

Comment 9.1 (Calculation)

If all three vectors are space-like, the evaluation yields the particle density.

(a) Relative to the COMOVING instantaneous inertal frame, where $u^\mu \frac{\partial}{\partial x^\mu} = 1 \frac{\partial}{\partial x^0} + 0 \frac{\partial}{\partial x^1} + 0 \frac{\partial}{\partial x^2} + 0 \frac{\partial}{\partial x^3}$, the density is obtained from

$$\# = N u^\mu \epsilon_{\mu\alpha\beta\gamma} \; dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \, (A, B, C)$$

$$= N \det \begin{vmatrix} u^0 & 0 & 0 & 0 \\ 0 & A^1 & A^2 & A^3 \\ 0 & B^1 & B^2 & B^3 \\ 0 & C^1 & C^2 & C^3 \end{vmatrix}$$

$$= N u^0 (\vec{A}\cdot\vec{B}\times\vec{C}).$$

Thus,

$$\boxed{S^0 = N = \frac{\#}{(\vec{A}\cdot\vec{B}\times\vec{C})} = \frac{\#}{\left(\substack{\text{COMOVING}\\ \text{VOLUME}}\right)} = \left(\substack{\text{comoving density}\\ \text{of particles}}\right).} \qquad (9.4)$$

(b) Relative to the LAB frame, where

$$u^{\bar\mu}\frac{\partial}{\partial x^{\bar\mu}} = u^{\bar 0}\frac{\partial}{\partial x^{\bar 0}} + u^{\bar 1}\frac{\partial}{\partial x^{\bar 1}} + u^{\bar 2}\frac{\partial}{\partial x^{\bar 2}} + u^{\bar 3}\frac{\partial}{\partial x^{\bar 3}}$$

and the three chosen space-like vector are $\bar{A}$, $\bar{B}$, and $\bar{C}$ with their tips and tails <u>simultaneous</u> in the LAB, the evaluation yields

$$\# = N u^{\bar\mu}\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\; d\bar{x}^{\bar\alpha}\wedge d\bar{x}^{\bar\beta}\wedge d\bar{x}^{\bar\gamma}/3!\,(\bar{A},\bar{B},\bar{C})$$

$$= N\,\det\begin{vmatrix} u^{\bar 0} & u^{\bar 1} & u^{\bar 2} & u^{\bar 3} \\ 0 & \vec{\bar{A}}^{\bar 1} & \vec{\bar{A}}^{\bar 2} & \vec{\bar{A}}^{\bar 3} \\ 0 & \vec{\bar{B}}^{\bar 1} & \vec{\bar{B}}^{\bar 2} & \vec{\bar{B}}^{\bar 3} \\ 0 & \vec{\bar{C}}^{\bar 1} & \vec{\bar{C}}^{\bar 2} & \vec{\bar{C}}^{\bar 3} \end{vmatrix}$$

$$= N u^{\bar 0}(\vec{\bar{A}}\cdot\vec{\bar{B}}\times\vec{\bar{C}})$$

$$= N u^{\bar 0}(\text{LAB volume}).$$

Thus,

$$\boxed{S^{\bar 0} = N u^{\bar 0} = \frac{\#}{(\text{LAB volume})} \equiv \frac{\text{particle density}}{\text{in LAB frame}}} \qquad (9,5)$$

Comment 9.2

The particle 4-velocity

$$u = u^{\bar\mu}\frac{\partial}{\partial\bar{x}^{\bar\mu}} = u^{\mu}\frac{\partial}{\partial x^{\mu}}$$

is one and the same in both frames. However, in the LAB frame one has

$$\{u^{\bar\mu}\}=\left\{\frac{1}{\sqrt{1-\bar\beta^2}}\ \frac{\vec{\bar\beta}}{\sqrt{1-\bar\beta^2}}\right\}.$$

Thus, in the LAB frame

$$S^{\bar 0}\equiv N u^{\bar 0}\geqslant N = N u^{0}\equiv S^{0}.$$

This means that the observed density of moving matter is larger than the comoving density.

Comment 9.3

If one of the three chosen 4-vectors $\bar{\bar A}$, $\bar B$, and $\bar C$ is time like, while the other two are space-like with tip and tail events simultaneous in the LAB frame, then

$$\# = N u^{\bar\mu}\,\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\ d\bar x^{\bar\alpha}\wedge d\bar x^{\bar\beta}\wedge d\bar x^{\bar\gamma}/3!\,(\bar{\bar A},\bar B,\bar C)$$

$$= N\,det\begin{vmatrix} u^{\bar 0} & u^{\bar 1} & u^{\bar 2} & u^{\bar 3}\\ -\Delta\bar t & 0 & 0 & 0\\ 0 & \vec{\bar B}^{\bar 1} & \vec{\bar B}^{\bar 2} & \vec{\bar B}^{\bar 3}\\ 0 & \vec{\bar C}^{\bar 1} & \vec{\bar C}^{\bar 2} & \vec{\bar C}^{\bar 3}\end{vmatrix}$$

$$= N\,\Delta\bar t\left[u^{\bar 1}(\vec{\bar B}\times\vec{\bar C})^{\bar 1}+u^{\bar 2}(\vec{\bar B}\times\vec{\bar C})^{\bar 2}\ \ u^{\bar 3}(\vec{\bar B}\times\vec{\bar C})^{\bar 3}\right]$$

$$= N\,\Delta\bar t\left[\vec u\cdot\vec{\bar B}\times\vec{\bar C}\right]$$

This # is the number of particles passing through the area $\vec{\bar B}\times\vec{\bar C}$ during the LAB time $\Delta\bar t$. On a per unit time basis one has

$$\boxed{\frac{\#}{\Delta\bar t}=N u^{\bar 1}(\vec{\bar B}\times\vec{\bar C})^{\bar 1}+N u^{\bar 2}(\vec{\bar B}\times\vec{\bar C})^{\bar 2}+N u^{\bar 3}(\vec{\bar B}\times\vec{\bar C})^{\bar 3}=\frac{(\text{particle current})}{(\text{through area }\vec{\bar B}\times\vec{\bar C})}}$$

The three parts contributing to this current are due to the three flux components:

$$\boxed{\begin{aligned} S^{\bar i} = N u^{\bar i} &= \frac{\#}{\Delta \bar t \; (\vec{\vec B} \times \vec{\vec C})^{\bar i}} = \frac{\text{number of particles}}{\begin{pmatrix}\text{LAB}\\ \text{time}\end{pmatrix}\begin{pmatrix}\text{area whose}\\ \text{normal points into}\\ \text{the } i^{th} \text{ direction}\end{pmatrix}} \\ &= \begin{pmatrix}\bar i^{th} \text{ component}\\ \text{of the particle flux}\end{pmatrix} \qquad \bar i = 1,2,3 \end{aligned}} \qquad (9.6)$$

Having identifies the four physical attributes via the boxed Eqs. (9.1)-(9.6), integrate the LAB particle density, Eq. (9.5) and the LAB flux components, Eq. (9.6) into the spacetime framework of the single concept, the <u>particle current 4-vector</u>

$$\mathbf{S} = N\mathbf{u} = N u^{\bar\mu} \frac{\partial}{\partial x^{\bar\mu}} : \quad \boxed{\begin{aligned} N u^{\bar 0} &= \frac{\text{particles}}{\text{volume}} \\ N u^{\bar 1} &= \frac{\text{particles}}{(\text{time})(\text{area})^{\bar 1}} \\ N u^{\bar 2} &= \frac{\text{particles}}{(\text{time})(\overrightarrow{\text{area}})^{\bar 2}} \\ N u^{\bar 3} &= \frac{\text{particles}}{(\text{time})(\overrightarrow{\text{area}})^{\bar 3}} \end{aligned}}$$

Lecture 10

The Density-Flux 3-form

I. Algebraic properties

II. Geometrical properties

In MTW grasp
the ideas in
Box 4.4
Box 5.1
Fig 5.1
Box 5.2

I. Particle Conservation

The spacetime history of particles is geometrized by their world lines. Particles which do not go out of existence have world lines that do not terminate.

An aggregate of non-colliding particles having the same observed 4-velocity in a local spacetime region form a world tube which is filled with the world lines of these particles.

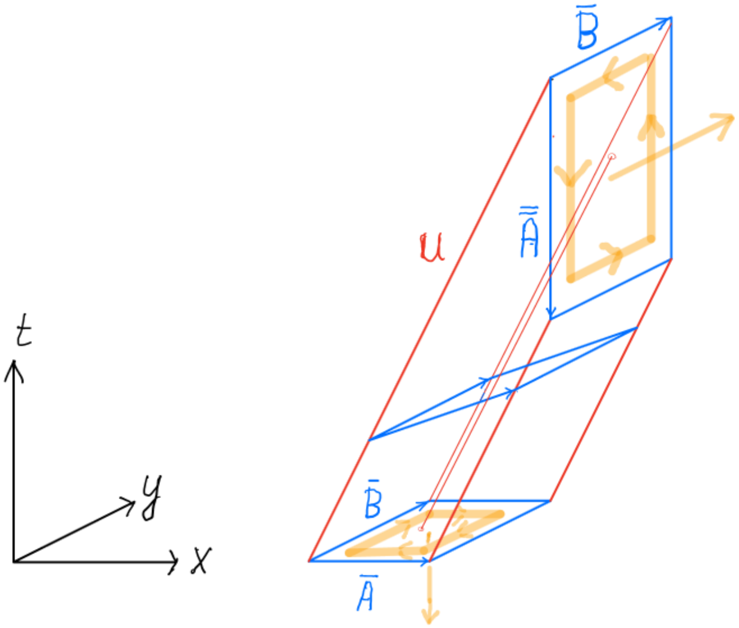


Figure 10.1 World tube composed type u particle world lines with 3-d cross sections $(\bar{A}, \bar{B}, \bar{C})$ and $(\bar{\bar{A}}, \bar{B}, \bar{C})$ which respectively are space like and time like elements of 3-volume. These volume elements contain the same number of particles. This is because the particle world lines do not terminate as they evolve from $(\bar{A}, \bar{B}, \bar{C})$ to $(\bar{\bar{A}}, \bar{B}, \bar{C})$. This is a geometrical statement of particles not being destroyed (or created).

The number observed particles is geometrized by the number of worldlines that cut through sections across the world tube. As depicted in Figure 10.1, a cross section is spanned by three 4-vectors such as $\bar{A}$, $\bar{B}$, and $\bar{C}$ or $\bar{\bar{A}}$, $\bar{B}$, and $\bar{C}$. A typical cross section such as $(\bar{A}, \bar{B}, \bar{C})$ or (A, B, C) consists of three space like vectors. They span a spatial element of volume which contains the number of particles

$$^{*}S(\bar{A},\bar{B},\bar{C}) = \# = {}^{*}S(A,B,C) \qquad (10.1a)$$

$$N u^{\bar{\mu}} \epsilon_{\bar{\mu}\bar{\alpha}\bar{\beta}\bar{\gamma}}\, d\bar{x}^{\bar{\alpha}} \wedge d\bar{x}^{\bar{\beta}} \wedge d\bar{x}^{\bar{\gamma}} (\bar{A},\bar{B},\bar{C}) = \# = N u^{\mu} \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha} \wedge dx^{\beta} \wedge d\bar{x}^{\gamma} (A,B,C). \qquad (10.1b)$$

On the other hand, a time-like cross section $(\bar{\bar{A}},\bar{B},\bar{C})$ where one of its spanning vectors, $\bar{\bar{A}}$, is time-like, contains the number of particles

$$^{*}S(\bar{\bar{A}},\bar{B},\bar{C}) = \# = N u^{\bar{\mu}} \epsilon_{\bar{\mu}\bar{\alpha}\bar{\beta}\bar{\gamma}}\, d\bar{x}^{\bar{\alpha}} \wedge d\bar{x}^{\bar{\beta}} \wedge d\bar{x}^{\bar{\gamma}} (\bar{\bar{A}},\bar{B},\bar{C}). \qquad (10.2)$$

They flow across the spatial opening of area $\vec{\bar{B}} \times \vec{\bar{C}}$ during the time interval $\Delta\bar{t}$ of the time like 4-vector

$$\bar{\bar{A}} = -\Delta\bar{t}\, \frac{\partial}{\partial \bar{x}^{\bar{0}}} + 0\, \frac{\partial}{\partial \bar{x}^{\bar{1}}} + 0\, \frac{\partial}{\partial \bar{x}^{\bar{2}}} + 0\, \frac{\partial}{\partial \bar{x}^{\bar{3}}}$$

There is no creation nor destruction of the type u particles that make up the type u world tube depicted in Figure 10.1. Consequently, the particle # in Eqs. (10.1) and (10.2) are the same:

(10,4)

$$\boxed{N u^{\bar{\mu}} \epsilon_{\bar{\mu}\bar{\alpha}\bar{\beta}\bar{\gamma}}\, d\bar{x}^{\bar{\alpha}} \wedge d\bar{x}^{\bar{\beta}} \wedge d\bar{x}^{\bar{\gamma}} (\bar{A}, \bar{B}, \bar{C}) = \# = N u^{\bar{\mu}} \epsilon_{\bar{\mu}\bar{\alpha}\bar{\beta}\bar{\gamma}}\, d\bar{x}^{\bar{\alpha}} \wedge d\bar{x}^{\bar{\beta}} \wedge d\bar{x}^{\bar{\gamma}} (\bar{\bar{A}}, \bar{B}, \bar{C})} \qquad (10.3)$$

Equation (10.3) is a statement of the <u>law of particle conservation</u> in the 4-d domain inside the world tube depicted by Figure 10.1.

II. Differential Law of Particle Conservation

The local law of particle conservation depicted in Figure 10.1 also holds globally as depicted in Figure 10.2. There the type u particle world tube connecting the initial with the final 3-volumes consists of world lines of whatever Nature dictates.

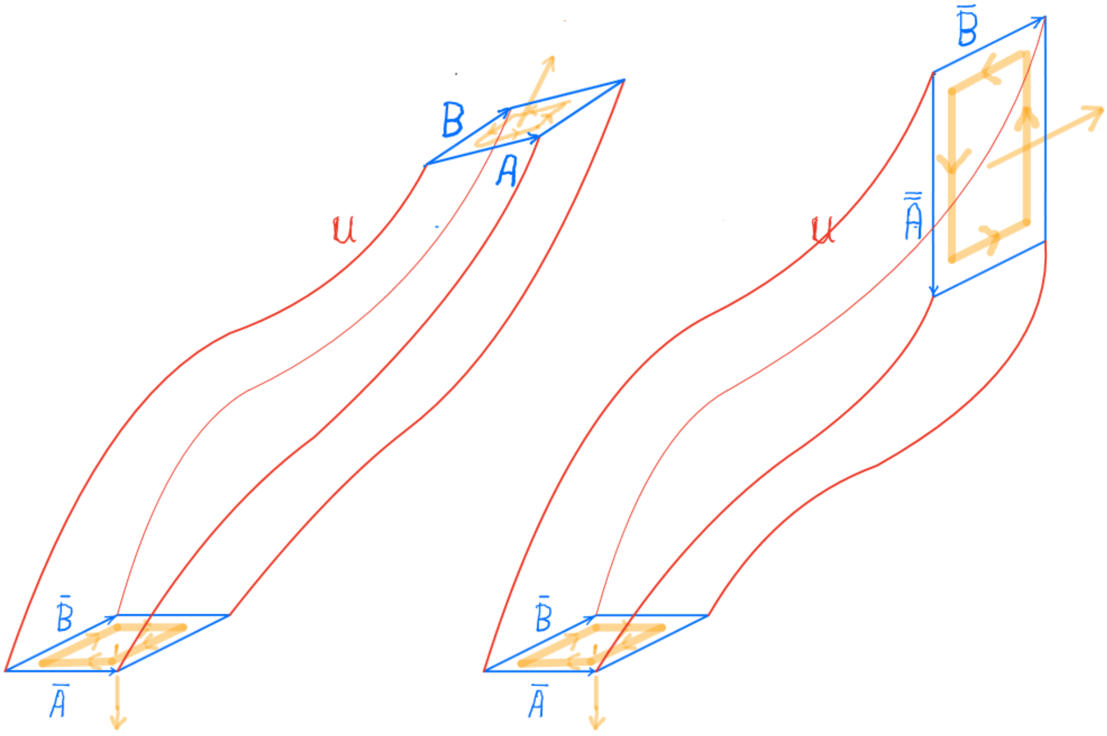


Figure 10.2 World tubes with the common purely space like initial cross section $(\bar{A}, \bar{B}, \bar{C})$, but with different final cross sections; namely, (i) (A, B, C), which is pure space like as depicted in panel (a), and (ii) $(\bar{\bar{A}}, \bar{B}, \bar{C})$, which is time like because $\bar{\bar{A}}$, a time like vector depicted in panel (b), is one of the three that span the volume.

The line of reasoning leading to this conclu-conclusion is a **four-step process.**

Step 1. ("The volume vector")

Introduce in 4-D spacetime what in 3-D Euclidean space is the "bivector" or cross-product:

a) In 3-D space

$$(\vec{A}\times\vec{B})_j = \epsilon_{jk\ell}A^kB^\ell \qquad \text{(covector components)}$$

$$\begin{aligned}\vec{A}\times\vec{B} &= \vec{e}_i\, g^{ij}\epsilon_{jk\ell}A^kB^\ell \\ &= \vec{e}_i\,\epsilon^i{}_{k\ell}\,A^kB^\ell \\ &= \vec{e}_i\,\epsilon^i{}_{k\ell}\,dx^k\wedge dx^\ell/2!\,(\vec{A}\,\vec{B}) \qquad (10.4\}\end{aligned}$$

The vectorial differential 2-form

$$\boxed{{}^{(2)}\vec{\underset{\approx}{\Sigma}} \equiv \vec{e}_i\,{}^{(2)}\underset{\approx}{\Sigma}{}^i \equiv \vec{e}_i\,\epsilon^i{}_{k\ell}\,dx^k\wedge dx^\ell/2!} \qquad (10.5)$$

is a tensor field of rank $\{{1 \atop 2}\}$

It has the property that it is constant under parallel transport into any direction:

$$\boxed{d\left({}^{(2)}\vec{\underset{\approx}{\Sigma}}\right)\equiv d\left(e_i\, g^{ij}\epsilon_{jk\ell}\,dx^k\wedge dx^\ell\right)=0} \qquad (10.6)$$

b) In 4-D space *

$$\star(A\wedge B\wedge C)_\nu = \epsilon_{\nu\alpha\beta\gamma}A^\alpha B^\beta C^\gamma \qquad (10.7\}$$

$$\star(A\wedge B\wedge C) = e_\nu\, g^{\nu\sigma}\epsilon_{\sigma\alpha\beta\gamma}\, A^\alpha B^\beta C^\gamma \qquad \boxed{10.6}$$

$$= e_\nu\, \epsilon^{\nu}{}_{\alpha\beta\gamma}\, A^\alpha B^\beta C^\gamma \qquad (10.8\}$$

$$= e_\nu\, \epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!\,(A,B,C)$$

Definition ("Volume vector")

The vectorial 3-form

$$\boxed{{}^{(3)}\vec{\underset{\sim}{\Sigma}} \equiv e_\nu\, {}^{(3)}\underset{\sim}{\Sigma}^{\nu} \equiv e_\nu\, \epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!} \qquad (10.9)$$

is called the <u>3-volume vector in 4 dimensions.</u>

- It is a tensor field of rank $\binom{1}{3}$.
- It is the vector perpendicular to the volume spanned by three as-yet-unspecified 4-D vectors.
- Its magnitude is a measure of the spanned volume.
- It has the property that it is constant under parallel transport into any direction in 4-D:

$$\boxed{d\left({}^{(3)}\vec{\underset{\sim}{\Sigma}}\right) = d\left(e_\nu\, g^{\nu\sigma}\epsilon_{\sigma\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!\right) = 0}$$

$$(10.10)$$

Step 2. (The matter-volume decomposition)

Recalling the particle 4-current

$$\mathbf{S} = N\mathbf{u} = \underbrace{Nu^\mu}_{S^\mu} e_\mu \quad ,$$

reformulate the scalar density-flux 3-form, Eq.(10.1) as the inner product 3-form*

$$^{*}\mathbf{S} = \mathbf{S}\cdot^{(3)}\underset{\approx}{\vec{\Sigma}}.$$

---

*\footnote{The mapping

$$\bigstar : V\wedge V\wedge V \xrightarrow{\bigstar} V,$$

defined by Eq.(10.8), is the same as MTW's Eq.(15.15), except that their's is

$$\bigstar(A\wedge B\wedge C) = A^\alpha B^\beta C^\gamma \epsilon_{\alpha\beta\gamma}{}^{\nu} e_\nu ,$$

which differs from ours only by a change in sign.

$*$

\footnote{

In MTW's Box 5.4 the charge density-flux 3-form $^*J$ is factored into the product

$$^*\mathbb{J} = J^{\mu}\, d^3\Sigma_{\mu} = \mathbb{J}\cdot d^3\boldsymbol{\Sigma}$$

$$= J^{\mu}\,\epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!$$

This factorization holds with or without any preexisting metric tensor field $g = g_{\mu\nu}\, dx^{\mu}\otimes dx^{\nu}$; it depends only on the existence of the orientation Levi-Civita tensor $\epsilon = \epsilon_{\mu\alpha\beta\gamma}\, dx^{\mu}\wedge dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}$. However, if a metric is known or given, one can replace $d^3\Sigma_{\mu}$ with $d^3\Sigma^{\nu} = \epsilon^{\nu}{}_{\alpha\beta\gamma}$ and introduce the vectorial 3-form

$$e_{\nu}\, d^3\underset{\sim}{\Sigma}^{\nu} = e_{\nu}\,\epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3! \equiv {}^3\underset{\sim}{\vec{\Sigma}}$$

Its essential property is its invariance under parallel transport into any direction:

$$d\left(e_{\nu}\, d^3\underset{\sim}{\Sigma}^{\nu}\right) = 0.$$

Introduce this translation invariant vector(ial) measure into the charge density-flux 3-form:

$$^*\mathbb{J} = J^{\mu}\,\epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!$$

$$= \underbrace{J^{\mu} e_{\mu}}_{=\ \vec{J}} \cdot \underbrace{e_{\nu}\,\epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}}_{e_{\nu}\, d^3\underset{\sim}{\Sigma}}$$

Thus, by projecting the 4-current $\vec{J} = J^{\mu} e_{\mu}$ onto the vector(ial) measure ${}^3\underset{\sim}{\vec{\Sigma}} = e_{\nu}\, d^3\underset{\sim}{\Sigma}$ one obtain the amount of charge contained in the volume element to be specified in terms of some triad of spanning vectors.}

Indeed, a notational computation based on the boxed definition, Eq.(10.9) on page 10.5, yields

$$
\begin{aligned}
{}^{*}\boldsymbol{S} &= N u^{\mu}\ \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3! \\
&= S^{\mu}\ g_{\mu\nu}\underbrace{\epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!}_{\equiv\ {}^{(3)}\underset{\sim}{\Sigma}^{\nu}} \\
&= S^{\mu} e_{\mu}\cdot e_{\nu}\, {}^{(3)}\underset{\sim}{\Sigma}^{\nu} \\
&= \boldsymbol{S}\cdot {}^{(3)}\Sigma .
\end{aligned}
$$

$$
\boxed{{}^{*}\boldsymbol{S} = (e_{\sigma} S^{\sigma})\cdot(e_{\mu}\, {}^{(3)}\underset{\sim}{\Sigma}^{\mu})} \qquad (10.11)
$$

This inner product decomposition of the particle density-flux 3-form mathematizes the conceptual separation between (i) the nature of matter (here its four-current **S**) and (ii) the geometrical space (here its 3-volume ${}^{(3)}\Sigma$) available for its occupation.

Step 3.

Take exterior derivative d of Eq. (10.11) and find

$$
\begin{aligned}
d\,{}^*\mathfrak{S} &= d\left[S^{\mu} \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma} \right] \\
&= d\left[S^{\mu} \delta^{\nu}_{\mu} \epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma} \right] \\
&= d\left[S^{\mu} g_{\mu\nu} g^{\nu\sigma} \epsilon_{\sigma\alpha\beta\gamma}\, dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma} \right] \\
&= d\left[S^{\mu} e_{\mu} \cdot e_{\nu} \epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma} \right] \\
&= d\left[\mathfrak{S} \cdot {}^{(3)}\Sigma \right] \qquad (10.12)
\end{aligned}
$$

The exterior derivative of this product* is

$$d\,{}^*\mathfrak{S} = d(S^{\sigma} e_{\sigma}) \wedge\cdot (e_{\nu} {}^{(3)}\underset{\sim}{\Sigma}^{\nu}) + S^{\mu} e_{\mu} \cdot d(e_{\nu} {}^{(3)}\underset{\sim}{\Sigma}^{\nu}) \qquad (10.13)$$

\footnote{
Exterior product and interior product are commutative operations, i.e. "$\wedge\cdot$" = "$\cdot\wedge$". This is because in exterior algebra the coefficients may also be those of a vector (or tensor) field besides those of a mere scalar field. Thus "$\wedge\cdot\vec{v}$" = "$\cdot\vec{v}\wedge$". The operation "$\wedge$" and "$\cdot$" are freely interchangeable.}

The second term vanishes because vectorial 3-volume form is constant, $d({}^{(3)}\Sigma) \equiv d(e_\nu {}^{(3)}\underset{\sim}{\Sigma}^\nu) = 0$. To evaluate the first term, start with (i), the fact that the differential of the vector $\mathbf{S} = e_\sigma S^\sigma$ is

$$\begin{aligned} d(e_\sigma S^\sigma) &= e_\sigma\, dS^\sigma + S^\sigma d e_\sigma \\ &= e_\sigma \frac{\partial S^\sigma}{\partial x^\mu} dx^\mu + e_\nu S^\sigma \Gamma^\nu_{\sigma\mu} dx^\mu \\ &= e_\sigma \left(S^\sigma_{,\mu} + S^\nu \Gamma^\sigma_{\nu\mu} \right) dx^\mu \\ &\equiv e_\sigma S^\sigma_{;\mu} dx^\mu \, . \end{aligned} \qquad (10.14)$$

Here $S^\sigma_{;\mu}$ are the component of the covariant derivative of the particle 4-current S.

Then (ii), take advantage the wedge product of $dx^\mu \wedge e_\nu {}^{(3)}\underset{\sim}{\Sigma}^\nu$ simplifies considerarably:

$$\begin{aligned} dx^\mu \wedge e_\nu {}^{(3)}\underset{\sim}{\Sigma}^\nu &= e_\nu \, dx^\mu \wedge \epsilon^\nu{}_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \\ &= e_\nu \, g^{\nu\rho} \epsilon_{\rho\alpha\beta\gamma} \, dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \\ &= e_\nu \, g^{\nu\rho} \delta^\mu_\rho \sqrt{-g} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \qquad (10.15)$$

Applying Eqs. (10.14)-(10.15) to (10.13)

$$\begin{aligned} d({}^*S) &= e_\sigma S^\sigma_{;\mu} \cdot e_\nu \, g^{\nu\rho} \delta^\mu_\rho \sqrt{-g} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= S^\sigma_{;\sigma} \sqrt{-g} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

Both the covariant divergence

$$S^{\sigma}{}_{;\sigma} = \frac{1}{\sqrt{-g}} \frac{\partial (S^{\sigma}\sqrt{-g})}{\partial x^{\sigma}},$$

and the 4-volume element $\sqrt{-g}\; dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ are coordinate frame invariants.

Consequenly,

$$d({}^{*}S) = \frac{\partial (S^{\sigma}\sqrt{-g})}{\partial x^{\sigma}} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$= \begin{pmatrix} \text{\# of particles created in} \\ \text{the invariant spacetime} \\ \text{4-volume } \sqrt{-g}\; dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{pmatrix}$$

Whenever particles are neither created nor destroyed, then such a state of affairs is mathematized by the statement

$$\boxed{\frac{\partial (S^{\sigma}\sqrt{-g})}{\partial x^{\sigma}} = 0} \quad \begin{pmatrix} \text{"Conservation} \\ \text{of particles"} \end{pmatrix}$$

Appendix to Lecture 10 and 26

The vector(ial) measure of an as-yet-to-be specified area is

$$e_\ell d^2 \underset{\sim}{\Sigma}^\ell \equiv d^2 \underset{\sim}{\vec{\Sigma}} \equiv {}^{(2)}\underset{\sim}{\vec{\Sigma}} = e_\ell \in^\ell{}_{ij} \frac{dx^i \wedge dx^j}{2!};$$

We have 1) $e_\ell \in^\ell{}_{ij}\, dx^i \wedge dx^j / 2!\, (\vec{u}, \vec{v}) \equiv \vec{u} \times \vec{v}$

and 2.) $d\left(e_\ell d^2 \Sigma^\ell\right) = 0$

PROOF:

$$d\left(e_\ell d^2 \underset{\sim}{\Sigma}^\ell\right) = d\left(e_\ell \in^\ell{}_{ij} \frac{dx^i \wedge dx^j}{2!}\right) = d\left(e_\ell g^{\ell k} \sqrt{g}\, [k\,i\,j] \frac{dx^i \wedge dx^j}{2!}\right)$$

$$= \left[d e_\ell \; g^{\ell k} + e_\ell\, d g^{\ell k} + e_\ell g^{\ell k} \frac{d\sqrt{g}}{\sqrt{g}}\right] \sqrt{g}\, [k\,i\,j] \wedge \frac{dx^i \wedge dx^j}{2!}$$

$$= \underset{\textcircled{1}}{e_n \Gamma^n{}_{\ell r}\, dx^r g^{\ell k}} + \underset{\textcircled{2}}{e_\ell (-) g^{\ell r} g^{sk}\, d g_{rs}} + \underset{\textcircled{3}}{e_\ell g^{\ell k} \frac{d\sqrt{g}}{\sqrt{g}}}\Big] \sqrt{g}\, [k\,i\,j] \frac{dx^i \wedge dx^j}{2!}$$

Recall that

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\, u^r\right) = u^r{}_{;r} = u^r{}_{,r} + u^s \Gamma^r{}_{sr}$$

$$u^r{}_{,r} + u^r \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^r} = u^r{}_{,r} + u^r \Gamma_r{}^s{}_s \Rightarrow \boxed{\Gamma_r{}^s{}_s = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^r} = \frac{1}{2} g^{sm}\left(\cancel{g_{mr,s}} + g_{ms,r} - \cancel{g_{rs,m}}\right)}$$

Thus

$$d\left(e_\ell d^2 \underset{\sim}{\Sigma}^\ell\right) = e_n \Gamma^n{}_{\ell r}\, g^{\ell k}\, dx^r \wedge \sqrt{g}\, [k\,i\,j] \frac{dx^i \wedge dx^j}{2!} + \textcircled{2} + \textcircled{3}$$

$$\delta^r{}_k \sqrt{g}\; dx^1 \wedge dx^2 \wedge dx^3$$

$$= \underbrace{e_n \Gamma^n{}_{\ell r}\, g^{\ell r} \sqrt{g}\; dx^1 \wedge dx^2 \wedge dx^3}_{\textcircled{1}} + \textcircled{2} + \textcircled{3} = \textcircled{1} + \textcircled{2} + \textcircled{3}$$

$$\textcircled{1} = e_n \tfrac{1}{2} g^{nm}\left(g_{m\ell,r} + g_{mr;\ell} - g_{r\ell,m}\right) g^{\ell r} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= e_n\left(g^{nm} g_{m\ell,r}\, g^{\ell r} - \tfrac{1}{2} g^{nm} g^{\ell r} g_{r\ell,m}\right) \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$\textcircled{2} = e_\ell (-) g^{\ell r} g^{sk}\, d g_{rs}\; \sqrt{g}\, [k\,i\,j]\; dx^i \wedge dx^j / 2!$$

$$= -e_\ell\, g^{\ell r} g^{sk}\, g_{rs,p} \sqrt{g}\, [k\,i\,j]\, dx^p \wedge dx^i \wedge dx^j / 2!$$

$$= -e_\ell\, g^{\ell r} g^{sk}\, g_{rs,p} \sqrt{g}\; \delta^p{}_k\; dx^1 \wedge dx^2 \wedge dx^3$$

$$= -e_\ell\, g^{\ell r} g^{sp}\, g_{rs,p} \sqrt{g}\; dx^1 \wedge dx^2 \wedge dx^3$$

$$= -e_n\, g^{nm} g^{sp}\, g_{ms,p} \sqrt{g}\; dx^1 \wedge dx^2 \wedge dx^3$$

③ $$= e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}}\, d\sqrt{g} \wedge [k\,i\,j]\, dx^i \wedge dx^j/2!$$

$$= e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}}\, \frac{\partial \sqrt{g}}{\partial x^p}\, [k\,i\,j]\, dx^p \wedge dx^i \wedge dx^j/2!$$

$$= e_\ell\, g^{\ell k}\, \frac{1}{2}\, g^{ms}\, g_{ms,p}\, \delta^p_k \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= e_\ell\, g^{\ell k}\, \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 \ ,$$

$$= e_n\, g^{nk}\, \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

① + ② + ③ $$= e_n \left(g^{nm}\, g_{m\ell,r}\, g^{\ell r} - \frac{1}{2}\, g^{nm}\, g^{\ell r}\, g_{r\ell,m} \right) \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$- e_n\, g^{nm}\, g^{sp}\, g_{ms,p} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$+ e_n\, g^{nk}\, \frac{1}{2}\, g^{\ell r}\, g_{\ell r,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= 0$$

Thus, in terms of the original notation one has

$$d\left(e_\ell\, d^2 \underset{\approx}{\Sigma}^\ell\right) = 0$$

$$d\left({}^3\vec{\Sigma}\right) = 0$$

$$d\left(e_\ell \in^\ell{}_{ij}\, dx^i \wedge dx^j\right) = 0$$

Lecture 11

Particle conservation
mathematized

I. Particle world lines in a particle world tube

II. Differential law of particle conservation

III. Coordinate invariant volume

In MTW grasp
the ideas in Box 4.4
Box 5.1
Fig 5.1
Box 5.2

I. Particle World Tube

The spacetime history of particles is geometrized by their <u>world lines</u>. Particles which do not go out of existence have world lines that do not terminate.

An aggregate of non-colliding particles having the same observed 4-velocity in a local spacetime region form a <u>world tube</u> which is filled with the world lines of these particles.

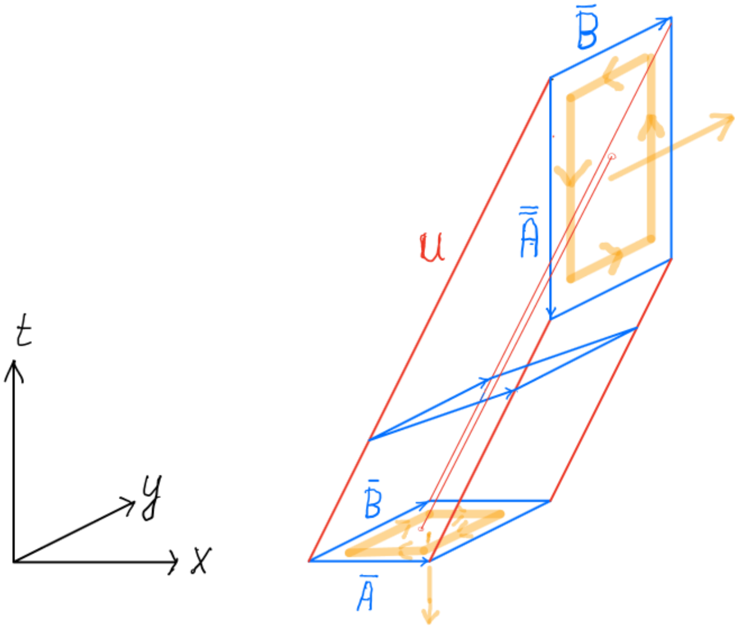


Figure 11.1 World tube composed type u particle world lines with 3-d cross sections $(\bar{A}, \bar{B}, \bar{C})$ and $(\bar{\bar{A}}, \bar{B}, \bar{C})$ which respectively are space like and time like elements of 3-volume. These volume elements contain the same number of particles. This is because the particle world lines do not terminate as they evolve from $(\bar{A}, \bar{B}, \bar{C})$ to $(\bar{\bar{A}}, \bar{B}, \bar{C})$. This is a geometrical statement of particles not being destroyed (or created).

The number observed particles is geometrized by the number of worldlines that cut through sections across the world tube. As depicted in Figure 11.1, a cross section is spanned by three 4-vectors such as $\bar{A}$, $\bar{B}$, and $\bar{C}$ or $\bar{\bar{A}}$, $\bar{B}$, and $\bar{C}$. A typical cross section such as $(\bar{A}, \bar{B}, \bar{C})$ or (A, B, C) consists of three space like vectors. They span a spatial element of volume which contains the number of particles

$$ {}^*\mathbf{S}(\bar A,\bar B,\bar C) = \# = {}^*\mathbf{S}(A,B,C) \qquad (11.1a) $$

$$ N u^{\bar\mu}\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\, d\bar x^{\bar\alpha}\wedge d\bar x^{\bar\beta}\wedge d\bar x^{\bar\gamma}(\bar A,\bar B,\bar C) = \# = N u^{\mu}\epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge d\bar x^{\gamma}(A,B,C). \qquad (11.1b) $$

On the other hand, a time like cross section $(\bar{\bar A}, \bar B, \bar C)$ where one of its spanning vectors, $\bar{\bar A}$, is time like, contains the number of particles

$$ {}^*\mathbf{S}(\bar{\bar A},\bar B,\bar C) = \# = N u^{\bar\mu}\epsilon_{\bar\mu\bar\alpha\bar\beta\bar\gamma}\, d\bar x^{\bar\alpha}\wedge d\bar x^{\bar\beta}\wedge d\bar x^{\bar\gamma}(\bar{\bar A},\bar B,\bar C). \qquad (10.2) $$

They flow across the spatial opening of area $\vec{\bar B}\times\vec{\bar C}$ during the time interval $\Delta\bar t$ of the time like 4-vector

$$ \bar{\bar A} = -\Delta\bar t\,\frac{\partial}{\partial\bar x^{\bar 0}} + 0\,\frac{\partial}{\partial\bar x^{\bar 1}} + 0\,\frac{\partial}{\partial\bar x^{\bar 2}} + 0\,\frac{\partial}{\partial\bar x^{\bar 3}} $$

There is no creation nor destruction of the type u particles that make up the type u world tube depicted in Figure 10.1. Consequently, the particle # in Eqs. (11.1) and (11.2) are the same:

(11,4)

$$\boxed{N u^{\bar\mu} \epsilon_{\bar\mu \bar\alpha \bar\beta \gamma} \, d\bar{x}^{\bar\alpha} \wedge d\bar{x}^{\bar\beta} \wedge d\bar{x}^{\bar\gamma} (\bar{A}, \bar{B}, \bar{C}) = \# = N u^{\bar\mu} \epsilon_{\bar\mu \bar\alpha \bar\beta \gamma} \, d\bar{x}^{\bar\alpha} \wedge d\bar{x}^{\bar\beta} \wedge d\bar{x}^{\bar\gamma} (\bar{\bar{A}}, \bar{B}, \bar{C})}$$

(11.3)

Equation (11.3) is a statement of the <u>law of particle conservation</u> in the 4-d domain inside the world tube depicted by Figure 11.1.

II. Differential Law of Particle Conservation

The local law of particle conservation depicted in Figure 11.1 also holds globally as depicted in Figure 11.2. There the type u particle world tube connecting the initial with the final 3-volumes consists of world lines of whatever Nature dictates.

The global mathematization of particle conservation, as in Figure 11.2, is accomplished by applying the 3-4 version of Green's Theorem to the (local) differential law of particle conservation.

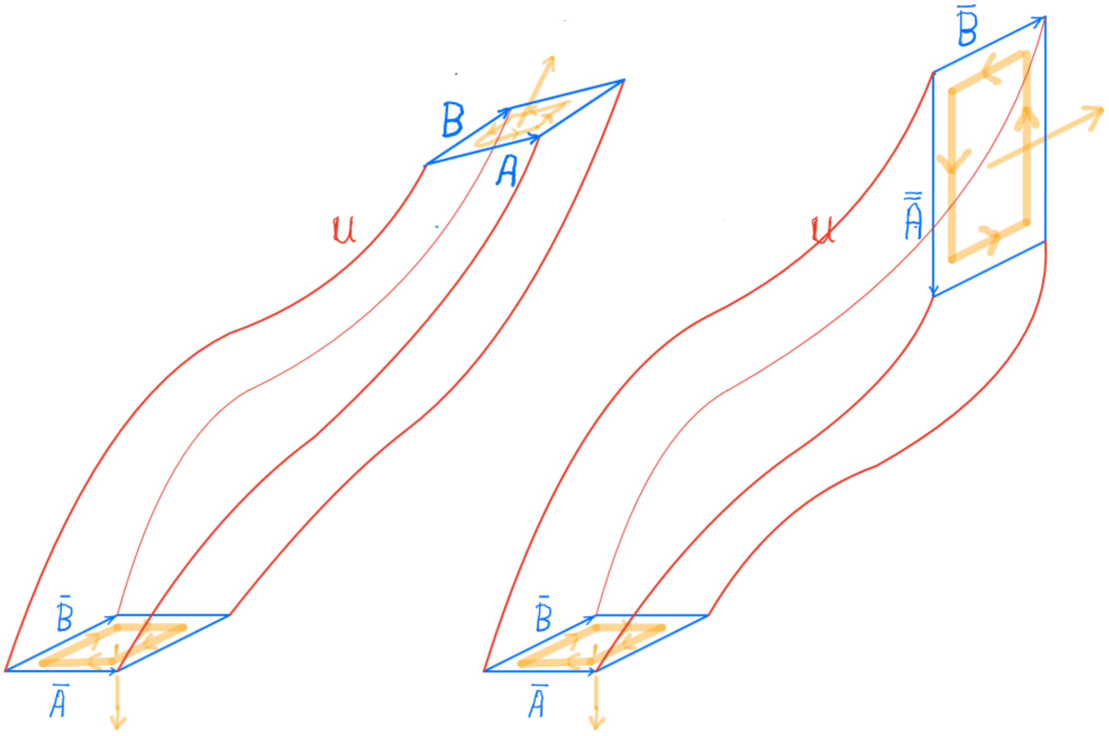


Figure 11.2 World tubes with the common purely space like initial cross section $(\bar{A}, \bar{B}, \bar{C})$, but with different final cross sections; namely, (i) (A, B, C), which is pure space like as depicted in panel (a), and (ii) $(\bar{\bar{A}}, \bar{B}, \bar{C})$, which is time like because $\bar{\bar{A}}$, a time like vector depicted in panel (b), is one of the three that span the volume.

The line of reasoning leading to the local differential law of particle conservation is four-step process.

Step 1. ("The volume vector")

Introduce in 4-D spacetime what in 3-D Euclidean space is the "bivector" or cross-product:

a) In 3-D space

$$(\vec{A}\times\vec{B})_j = \epsilon_{jk\ell}A^k B^\ell \qquad \text{(covector components)}$$

$$\vec{A}\times\vec{B} = \vec{e}_i\, g^{ij}\epsilon_{jk\ell}A^k B^\ell$$

$$= \vec{e}_i\, \epsilon^i{}_{k\ell}A^k B^\ell$$

$$= \vec{e}_i\, \epsilon^i{}_{k\ell}\, dx^k\wedge dx^\ell/2!\,(\vec{A}\ \vec{B}) \qquad (11.4)$$

The vectorial differential 2-form

$$\boxed{{}^{(2)}\Sigma \equiv \vec{e}_i\,{}^{(2)}\underset{\sim}{\Sigma}{}^i \equiv \vec{e}_i\,\epsilon^i{}_{k\ell}\, dx^k\wedge dx^\ell/2!} \qquad (11.5)$$

is a tensor field of rank $\binom{1}{2}$

It has the property that it is constant under parallel transport into any direction:

$$\boxed{d({}^{(2)}\Sigma) \equiv d\left(e_i\, g^{ij}\epsilon_{jk\ell}\, dx^k\wedge dx^\ell\right) = 0} \qquad (11.6)$$

b) In 4-D space *

$$\star(A\wedge B\wedge C)_\nu = \epsilon_{\nu\alpha\beta\gamma}A^\alpha B^\beta C^\gamma \qquad (11.7)$$

$$\star(A\wedge B\wedge C) = e_\nu\, g^{\nu\sigma} \epsilon_{\sigma\alpha\beta\gamma} A^\alpha B^\beta C^\gamma \qquad \text{(11.6)}$$

$$= e_\nu\, \epsilon^\nu{}_{\alpha\beta\gamma} A^\alpha B^\beta C^\gamma \qquad (11.8)$$

$$= e_\nu\, \epsilon^\nu{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!\,(A,B,C)$$

Definition ("Volume vector")
The vectorial 3-form

$$\boxed{{}^{(3)}\Sigma \equiv e_\nu\, {}^{(3)}\underset{\sim}{\Sigma}{}^\nu \equiv e_\nu\, \epsilon^\nu{}_{\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!} \qquad (11.9)$$

is called the <u>3-volume vector in 4 dimensions.</u>

- It is a tensor field of rank $\left\langle{1 \atop 3}\right\rangle$.
- It is the vector perpendicular to the volume spanned by three as-yet-unspecified 4-D vectors.
- Its magnitude is a measure the spanned volume.
- It has the property that it is constant under parallel transport into any direction in 4-D:

$$\boxed{d({}^{(3)}\Sigma) = d\left(e_\nu\, g^{\nu\sigma} \epsilon_{\sigma\alpha\beta\gamma}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma/3!\right) = 0}$$

(11.10)

* \ footnote {The mapping

$$\bigstar : V \wedge V \wedge V \xrightarrow{\bigstar} V,$$

defined by Eq. (11.8), is the same as MTW's Eq. (15.15), except that their's is

$$\bigstar (A \wedge B \wedge C) = A^{\alpha} B^{\beta} C^{\gamma} \, \epsilon_{\alpha\beta\gamma}{}^{\nu} \, e_{\nu} ,$$

which differs from ours only by a change in sign.

Step 2. ("The matter-volume decomposition")

Recalling the particle 4-current

$$\boldsymbol{S} = N\boldsymbol{u} = \underbrace{N u^{\mu}}_{S^{\mu}} e_{\mu} ,$$

reformulate the scalar density-flux 3-form, Eq. (11.1) as the inner product 3-form

$$^{*}\boldsymbol{S} = \boldsymbol{S} \cdot {}^{(3)}\Sigma .$$

Indeed, a notational computation based on the boxed definition, Eq.(11.9) on page 11.5, yields

$$
\begin{aligned}
{}^*\mathbf{S} &= N u^\mu \;\; \epsilon_{\mu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \\
&= S^\mu \;\; g_{\mu\nu} \underbrace{\epsilon^\nu{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!}_{\equiv\; {}^{(3)}\underset{\sim}{\Sigma}^\nu} \\
&= S^\mu e_\mu \cdot e_\nu \, {}^{(3)}\underset{\sim}{\Sigma}^\nu \\
&= \mathbf{S} \cdot {}^{(3)}\Sigma .
\end{aligned}
$$

$$
\boxed{{}^*\mathbf{S} = \left(e_\sigma S^\sigma\right)\cdot\left(e_\mu \, {}^{(3)}\underset{\sim}{\Sigma}^\mu\right)} \qquad (11.11)
$$

This inner product decomposition of the particle density-flux 3-form mathematizes the conceptual separation between (i) the nature of matter (here its four-current **S**) and (ii) the geometrical space (here its 3-volume ${}^{(3)}\Sigma$) available for its occupation.

Step 3. ("The differential law") 11.9

Take exterior derivative d of Eq. (11.11) and find

$$
\begin{aligned}
d\,{}^*\mathfrak{S} &= d\left[S^{\mu} \epsilon_{\mu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right] \\
&= d\left[S^{\mu} \delta^{\nu}_{\mu} \epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right] \\
&= d\left[S^{\mu} g_{\mu\nu} g^{\nu\sigma} \epsilon_{\sigma\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right] \\
&= d\left[S^{\mu} e_{\mu}\cdot e_{\nu} \epsilon^{\nu}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}\right] \\
&= d\left[\mathfrak{S}\cdot {}^{(3)}\Sigma \right] \qquad (11.12)
\end{aligned}
$$

The exterior derivative of this product is*

$$
d\,{}^*\mathfrak{S} = d\left(S^{\sigma} e_{\sigma}\right) \wedge\cdot \left(e_{\nu}\, {}^{(3)}\underset{\sim}{\Sigma}^{\nu}\right) + S^{\mu} e_{\mu}\cdot d\left(e_{\nu}\, {}^{(3)}\underset{\sim}{\Sigma}^{\nu}\right) \qquad (11.13)
$$

\footnote{
Exterior product and interior product are commutative operations, i.e. "$\wedge\cdot$" = "$\cdot\wedge$". This is because in exterior algebra the coefficients may also be those of a vector (or tensor) field besides those of a mere scalar field. Thus "$\wedge\cdot\vec{v}$" = "$\cdot\vec{v}\wedge$". The operation "$\wedge$" and "$\cdot$" are freely interchangeable.}

The second term vanishes because vectorial 3-volume form is constant, $d\left({}^{(3)}\boldsymbol{\Sigma}\right) \equiv d\left(e_\nu {}^{(3)}\underset{\sim}{\Sigma}^{\nu}\right)=0$. To evaluate the first term, start with (i), the fact that the differential of the vector $\mathbf{S}=e_\sigma S^\sigma$ is

$$\begin{aligned} d(e_\sigma S^\sigma) &= e_\sigma\, dS^\sigma + S^\sigma\, d e_\sigma \\ &= e_\sigma \frac{\partial S^\sigma}{\partial x^\mu}\, dx^\mu + e_\nu S^\sigma \Gamma^{\nu}_{\ \sigma\mu}\, dx^\mu \\ &= e_\sigma\left(S^\sigma_{\ ,\mu} + S^\nu \Gamma^{\sigma}_{\ \nu\mu}\right) dx^\mu \\ &\equiv e_\sigma S^\sigma_{\ ;\mu}\, dx^\mu \ . \qquad (11.14) \end{aligned}$$

Here $S^\sigma_{\ ;\mu}$ are the component of the covariant derivative of the particle 4-current S.

Then (ii), take advantage of the fact that the wedge product of $dx^\mu \wedge e_\nu {}^{(3)}\underset{\sim}{\Sigma}^{\nu}$ simplifies considerarably:

$$\begin{aligned} dx^\mu \wedge e_\nu {}^{(3)}\underset{\sim}{\Sigma}^{\nu} &= e_\nu\ dx^\mu \wedge \epsilon^{\nu}_{\ \alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \\ &= e_\nu\ g^{\nu\rho} \epsilon_{\rho\alpha\beta\gamma}\ dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \\ &= e_\nu\ g^{\nu\rho} \delta^{\mu}_{\rho} \sqrt{-g}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

$$(11.15)$$

Applying Eqs. (1.14)-(11.15) to (11.13)

$$\begin{aligned} d({}^*S) &= e_\sigma S^\sigma_{\ ;\mu} \cdot e_\nu\, g^{\nu\rho} \delta^{\mu}_{\rho} \sqrt{-g}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= S^\sigma_{\ ;\sigma}\ \sqrt{-g}\ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

Both the covariant divergence

$$S^{\sigma}{}_{;\sigma} = \frac{1}{\sqrt{-g}}\,\frac{\partial\left(S^{\sigma}\sqrt{-g}\right)}{\partial x^{\sigma}},$$

and the 4-volume element $\sqrt{-g}\; dx^0\wedge dx^1\wedge dx^2\wedge dx^3$ are coordinate frame invariants.

Consequenly,

$$d({}^*S) = \frac{\partial\left(S^{\sigma}\sqrt{-g}\right)}{\partial x^{\sigma}}\, dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \qquad (11.16)$$

$$= \begin{pmatrix} \text{\# of particles created in} \\ \text{the invariant spacetime} \\ \text{4-volume } \sqrt{-g}\; dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \end{pmatrix}$$

Whenever particles are neither created nor destroyed, then such a state of affairs is mathematized by the statement

$$\boxed{\frac{\partial\left(S^{\sigma}\sqrt{-g}\right)}{\partial x^{\sigma}} = 0} \quad \begin{pmatrix} \text{"Conservation} \\ \text{of particles"} \end{pmatrix} \qquad (11.17)$$

III. Coordinate invariant volume

11.12

An element of volume is rooted in two key concepts: that of the Jacobian and that of the Levi-Civita Tensor.

1.) The Jacobian

Consider the 4-volume element which is spanned by the tetrad of 4-vectors

$$\begin{aligned}
\bar{A} &= \Delta \bar{x}^{\bar{0}} \frac{\partial}{\partial \bar{x}^{\bar{0}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{1}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{2}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{3}}} = \bar{A}^{\bar{\alpha}} \frac{\partial}{\partial \bar{x}^{\bar{\alpha}}} \\
\bar{B} &= 0 \frac{\partial}{\partial \bar{x}^{\bar{0}}} + \Delta \bar{x}^{\bar{1}} \frac{\partial}{\partial \bar{x}^{\bar{1}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{2}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{3}}} = \bar{B}^{\bar{\beta}} \frac{\partial}{\partial \bar{x}^{\bar{\beta}}} \\
\bar{C} &= 0 \frac{\partial}{\partial \bar{x}^{\bar{0}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{1}}} + \Delta \bar{x}^{\bar{2}} \frac{\partial}{\partial \bar{x}^{\bar{2}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{3}}} = \bar{C}^{\bar{\gamma}} \frac{\partial}{\partial \bar{x}^{\bar{\gamma}}} \\
\bar{D} &= 0 \frac{\partial}{\partial \bar{x}^{\bar{0}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{1}}} + 0 \frac{\partial}{\partial \bar{x}^{\bar{2}}} + \Delta x^{3} \frac{\partial}{\partial \bar{x}^{\bar{3}}} = \bar{D}^{\delta} \frac{\partial}{\partial \bar{x}^{\bar{\delta}}}
\end{aligned} \qquad (11.18)$$

namely

$$\Delta \bar{x}^{0} \Delta \bar{x}^{\bar{1}} \Delta \bar{x}^{\bar{2}} \Delta \bar{x}^{\bar{3}} = [\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}\,\bar{\delta}]\; \bar{A}^{\bar{\alpha}} \bar{B}^{\bar{\beta}} \bar{C}^{\bar{\gamma}} \bar{D}^{\bar{\delta}} = \det \begin{vmatrix} \Delta \bar{x}^{\bar{0}} & 0 & 0 & 0 \\ 0 & \Delta \bar{x}^{\bar{1}} & 0 & 0 \\ 0 & 0 & \Delta \bar{x}^{\bar{2}} & 0 \\ 0 & 0 & 0 & \Delta \bar{x}^{\bar{3}} \end{vmatrix}$$

Here

$$[\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}\,\bar{\delta}] = \begin{cases} +1 \text{ when } \bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta} \text{ is an even permutation of } 0123 \\ -1 \text{ when } \bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta} \text{ is an odd permutation of } 0123 \\ 0 \text{ when any two indeces repeat} \end{cases}$$

are the components of the Levi-Civita tensor relative to the rectilinear coordinate system $\{\bar{x}^{\bar{\mu}}\}$.

Relative to the curvilinear coordinate system $\{y^{\mu}\}$ this element of 4-volume is

$$\Delta\bar{x}^{\bar{0}}\Delta\bar{x}^{\bar{1}}\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}} = [\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}\,\bar{\delta}]\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\frac{\partial\bar{x}^{\bar{\beta}}}{\partial y^{\nu}}\frac{\partial\bar{x}^{\bar{\gamma}}}{\partial y^{\rho}}\frac{\partial\bar{x}^{\bar{\delta}}}{\partial y^{\sigma}}\frac{\partial y^{\mu}}{\partial\bar{x}^{\bar{0}}}\frac{\partial y^{\nu}}{\partial\bar{x}^{\bar{1}}}\frac{\partial y^{\rho}}{\partial\bar{x}^{\bar{2}}}\frac{\partial y^{\sigma}}{\partial\bar{x}^{\bar{3}}}\Delta\bar{x}^{\bar{0}}\,\Delta\bar{x}^{\bar{1}}\,\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}}$$

$$= \det\left[\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\right][\mu\,\nu\,\rho\,\sigma]\;\frac{\partial y^{\mu}}{\partial\bar{x}^{\bar{0}}}\frac{\partial y^{\nu}}{\partial\bar{x}^{\bar{1}}}\frac{\partial y^{\rho}}{\partial\bar{x}^{\bar{2}}}\frac{\partial y^{\sigma}}{\partial\bar{x}^{\bar{3}}}\Delta\bar{x}^{\bar{0}}\,\Delta\bar{x}^{\bar{1}}\,\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}} \qquad \text{(11.13)}$$

$$\equiv \epsilon_{\mu\nu\rho\sigma}\;dy^{\mu}\wedge dy^{\nu}\wedge dy^{\rho}\wedge dy^{\sigma}/4!\;\left(\frac{\partial}{\partial\bar{x}^{\bar{0}}},\frac{\partial}{\partial\bar{x}^{\bar{1}}},\frac{\partial}{\partial\bar{x}^{\bar{2}}},\frac{\partial}{\partial\bar{x}^{\bar{3}}}\right)\Delta\bar{x}^{\bar{0}}\,\Delta\bar{x}^{\bar{1}}\,\Delta\bar{x}^{\bar{2}}\Delta\bar{x}^{\bar{3}}$$

The determinant, $\det\left[\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\right]$, of the matrix of partial derivatives $\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}$ is the Jacobian of the coordinate transformation $\{y^{\mu}\}\to\{\bar{x}^{\bar{\alpha}}\}$.

2,) The Levi-Civita Tensor.

Furthermore,

$$\epsilon_{\mu\nu\rho\sigma} = \det\left[\frac{\partial\bar{x}^{\bar{\alpha}}}{\partial y^{\mu}}\right][\mu\,\nu\,\rho\,\sigma] \qquad (11.19)$$

are the components of the Levi-Civita tensor

$$\epsilon = \epsilon_{\mu\nu\rho\sigma}\;dy^{\mu}\wedge dy^{\nu}\wedge dy^{\rho}\wedge dy^{\sigma}/4!\;, \qquad (11.20)$$

The defining property of ϵ is that it is the multilinear map which assigns a numerical measure to the 4-volume spanned by an arbitrarily tetrad of vectors A, B, C, D Its additional key property is that it assigns an orientation to such a tetrad of vectors. Thus, the ordered tetrads (A,B,C,D) and (B,A,C,D) are distinguishable by their orientation because

$$\epsilon(A,B,C,D) = -\,\epsilon(B,A,C,D).$$

It is understood, but nevertheless worth pointing out, that the mathematization of the size and orientation of the volume element spanned by a tetrad of 4-D vectors, such as those in Eq.(11.18), does not depend on one's knowledge of their metric properties, even if they have any.

However, if a metric tensor is given, both the Jacobian determinant and the Levi-Civita tensor are determined. For a given metric tensor

$$g = \eta_{\bar\alpha\bar\beta}\; d\bar x^{\bar\alpha} \otimes d\bar x^{\bar\beta} = g_{\mu\nu}\; dy^\mu \otimes dy^\nu \qquad (11.21)$$

the determinant of its components are readily related to the Jacobian determinant $\det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right]$. Indeed, Eq.(11.21) implies

$$g_{\mu\nu} = \frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\, \eta_{\bar\alpha\bar\beta}\, \frac{\partial \bar x^{\bar\beta}}{\partial y^\nu}$$

or in matrix notation

$$\left[g_{\mu\nu}\right] = \left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right]^{tr} \left[\eta_{\bar\alpha\bar\beta}\right] \left[\frac{\partial \bar x^{\bar\beta}}{\partial y^\nu}\right], \qquad \left[\eta_{\bar\alpha\bar\beta}\right] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\det\left[g_{\mu\nu}\right] \equiv g = \det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right](-1)\det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right]$$

or

$$\boxed{\det\left[\frac{\partial \bar x^{\bar\alpha}}{\partial y^\mu}\right] = \sqrt{-g}}$$

(11.15)

Apply this metric-based Jacobian determinant to the Levi-Civita tensor, Eqs. (11.19)-(11.20) and obtain

$$\boxed{\epsilon = \sqrt{-g}\,[\mu\nu\rho\sigma]\; dy^\mu \wedge dy^\nu \wedge dy^\rho \wedge dy^\sigma/4!}$$

where

$$\boxed{\sqrt{-g}\,[\mu\nu\rho\sigma] = \epsilon_{\mu\nu\rho\sigma}}$$

are the components of ϵ relative to the curvilinear coordinate basis $\{dy^\mu\}$.

Lecture 12

The Law of Momenergy Conservation

I. Momenergy For A Mixture Of Particles

II. The momergy density-flux 3-form $^*\mathbb{T}$.

III. Conservation of momenergy via $^*\mathbb{T} \Longrightarrow d^*\mathbb{T}=0$

In MTW read (i.e. grasp) § 5.4 and Box 5.4

In Wheeler's A JOURNEY INTO GRAVITY AND SPACETIME read Chapter 6 on Momenergy. This very readable chapter particularizes and conceptualizes the phyical basis of momenergy and its conservation.

I. Momenergy of a Mixture of Particles

The 19th century atomic theory of matter, as well as its 20th century version, condense the fact that, within the relevant context, matter comes in the form of different kinds of particles, and particles have the attribute of being carriers of momenergy. Hence the Question: "What is the momenergy carried by an aggregate of different particles moving with different 4-velocities?"

Answer: Consider 3-D spacetime element of volume spanned by a triad of 4-D vectors (A, B, C) and populated by different particles having their own respective

rest masses $m_1, m_2, \ldots$
4-velocities $u_1, u_2, \ldots$
momenergies $p_1, p_2, \ldots$
invariant ("proper") densities $N_1, N_2, \ldots$

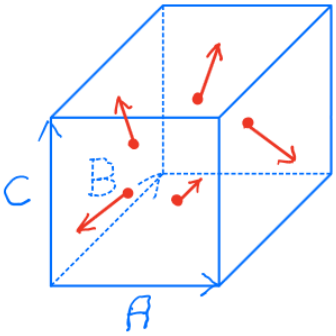


Figure 12.1 Particles with their respective 4-velocities in a 3-D volume element spanned by the triad (A, B, C) of 4-D vectors one of which may be time-like.

The total amount of momenergy of these particles in this (A,B,C)-spanned volume element is the sum of the contributions, $p_a {}^*S_a(A,B,C)$, from each particle species.

$$
\begin{aligned}
{}^*\mathbb{T}(A,B,C) &= \sum_a p_a {}^*S_a(A,B,C) \\
&= \sum_a p_a N_a u_a^\nu \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(A,B,C) \\
&= \mathbf{e}_\mu \underbrace{\sum_a p_a^\mu N_a u_a^\nu}_{T^{\mu\nu}} \overbrace{\mathbf{e}_\nu \cdot \mathbf{e}_\sigma}^{g_{\nu\sigma}} \underbrace{\epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!}_{\mathbf{e}_\sigma {}^{(3)}\underset{\sim}{\Sigma}^\sigma \equiv {}^{(3)}\Sigma^\sigma}(A,B,C)
\end{aligned}
$$

II. The Momenergy Density-Flux 3-form ${}^*\mathbb{T}$

Following the introduction of the 3-volume

$$ {}^{(3)}\underset{\sim}{\Sigma} = \mathbf{e}_\sigma {}^{(3)}\underset{\sim}{\Sigma} = \mathbf{e}_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! $$

in Eq.(11.11), page 11.8 of Lecture 11, find that here, just as there, the density-flux has the same inner product decomposition

$$ {}^*\mathbb{T}(A,B,C) = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu \cdot \mathbf{e}_\sigma {}^{(3)}\underset{\sim}{\Sigma}^\sigma (A,B,C) $$

This holds for all triads of 4-vectors (A, B, C). Consequenly, the <u>momenergy density-flux</u> is mathematized by

$$ \boxed{{}^*\mathbb{T} = \mathbb{T} \cdot {}^{(3)}\Sigma} \qquad (12.1) $$

Here ${}^*\mathbb{T} = e_\mu \sum_a p_a^\mu N_a U_a^\nu \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3!$, (12.2) (12,3)

a tensor of rank $\binom{1}{3}$,

is the momenrgy density-flux, where

$$T^{\mu\nu} = \sum_a p_a^\mu N_a U_a^\nu$$

are the components of the <u>momenergy 4-current</u>

$$\boxed{\mathbb{T} = e_\mu \otimes T^{\mu\nu} e_\nu}, \qquad (12.3)$$

a tensor of rank $\binom{2}{0}$, and

$$^{(3)}\underset{\sim}{\Sigma}^{\sigma} = \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

are the components of the <u>invariant vectorial 3-volume</u> measure

$$\boxed{^{(3)}\Sigma = e_\sigma \, ^{(3)}\underset{\sim}{\Sigma}^{\sigma}}, \qquad (12.4)$$

which is a tensor of rank $\binom{1}{3}$.

The inner product decomposition Eq.(12.1) of the invariant momenergy density-flux (${}^*\mathbb{T}$, Eq.(12.2)) integrates mathematically and invariantly two disjoint concepts,

(i) the momenergy property ($\mathbb{T}$, Eq.(12.3)) of matter, and

(ii) the 3-volume measure ($^{(3)}\Sigma$, Eq.(12.4)) of any 3-d box bounded by a triad (A,B,C) of 4-d vectors.

The 3-volume measure, Eq. (12.4), is compatible with metric-induced parallel transport. This means $^{(3)}\Sigma$ does not change under parallel transinto any direction. This fact is mathematized by the vanishing of its exterior derivative

$$d\left({}^{(3)}\Sigma\right)=0 \qquad (12.5)$$

in the same way as the compatibility between the metric $\mathbf{g}=g_{\mu\nu}\,dx^\mu\otimes dx^\mu$ and parallel transport, namely

$$d(\mathbf{g})=0.$$

III. Conservation of Momenergy

The inner product decomposition of the density-flux of matter is the same for particle number (or charge) $^*\mathbb{S} = \mathbb{S} \cdot {}^{(3)}\Sigma$ and for momenergy

$$^*\mathbb{T} = \mathbb{T} \cdot {}^{(3)}\Sigma .$$

Consequently, the evaluation of $d(^*\mathbb{T})$ parallels that of $d(^*S)$ in Lecture 11:

$$\begin{aligned} d(^*\mathbb{T}) &= d(\mathbb{T} \cdot {}^{(3)}\Sigma) \\ &= \quad d(\mathbb{T}) \quad \wedge \cdot {}^{(3)}\Sigma + \mathbb{T} \cdot \wedge d({}^{(3)}\Sigma) \\ &= d(T^{\tau\sigma} \mathbf{e}_\tau \otimes \mathbf{e}_\sigma) \wedge \cdot {}^{(3)}\Sigma + \quad 0 \qquad (12,6) \end{aligned}$$

The covariant differential of $\mathbb{T}$ follows the same rules* as those of $\mathbb{S}$, Eq. (11.14) in Lecture 11. Thus,

$$d(T^{\tau\sigma} \mathbf{e}_\tau \otimes \mathbf{e}_\sigma) = e_\tau \otimes e_\sigma \, T^{\tau\sigma}{}_{;\mu} \, dx^\mu . \qquad (12,7)$$

Here $T^{\tau\sigma}{}_{;\mu}$ are the components of the covariant derivative of the momenergy 4-current $\mathbb{T}$, a.k.a. the energy-momentum tensor. As in Eq. (11.15), the wedge product $dx^\mu \wedge e_\nu {}^{(3)}\underset{\approx}{\Sigma}$ simplifies considerably:

$$\begin{aligned} dx^\mu \wedge e_\nu {}^{(3)}\Sigma^\nu &= dx^\mu \wedge e_\nu \, \epsilon^\nu{}_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \\ &= e_\nu \, g^{\nu\rho} \, \epsilon_{\rho\alpha\beta\gamma} \, dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \\ &= e_\nu \, g^{\nu\rho} \sqrt{-g} \, \delta^\mu_\rho \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \qquad (12,8) \end{aligned}$$

\footnote{ There are two types of products in (12.6) working with tensor-valued ($^{*}\mathbb{T}$), vector-valued ($^{(3)}\Sigma$), and scalar-valued ($^{*}\mathcal{S}$) three-forms, namely the wedge product "$\wedge$" between differential forms and (ii) the inner product "$\cdot$" between a vector and a vector, or a tensor (i.e. a "bivector") and a vector, or the simple product of a scalar with a vector/tensor.

All these objects (scalars, vectors, tensors) are to be viewed as mere coefficients in the space of linear 3-forms, whose basis is $\{dx^\alpha \wedge dx^\beta \wedge dx^\gamma\}$. These scalar, vector and tensor-valued 3-forms are multilinear maps of rank $\binom{0}{3}$, $\binom{1}{3}$ and $\binom{2}{3}$. An inner product operation between

$\mathbf{T} = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu$ and $^{(3)}\Sigma = \mathbf{e}_\sigma \, {}^{(3)}\underset{\sim}{\Sigma}^{\sigma} \in \binom{1}{3}$ is to be viewed as having an effect only on the coefficient of the 3-form $^{(3)}\underset{\sim}{\Sigma}$ so that

$$^{*}\mathbf{T} = \mathbf{T} \cdot {}^{(3)}\Sigma = \mathbf{e}_\mu T^{\mu\nu} g_{\nu\sigma} \, {}^{(3)}\underset{\sim}{\Sigma} \in \binom{1}{3}.$$

and similarly

$$^{*}\mathcal{S} = \mathcal{S} \cdot {}^{(3)}\Sigma = S^{\nu} g_{\nu\sigma} \, {}^{(3)}\underset{\sim}{\Sigma} \in \binom{0}{3}.$$

Both belong to the space spanned by $\{dx^\alpha \wedge dx^\beta \wedge dx^\gamma\}$, one to the subspace $\binom{1}{3}$, the other to the subspace $\binom{0}{3}$.

Taking the exterior derivative of these 3-forms follows the usual rule $d(f\underset{\approx}{\omega}) = df \wedge \underset{\approx}{\omega} + f\, d\underset{\approx}{\omega}$. Applied to each of the two 3-forms one finds that (10.7)

$$
\begin{aligned}
d\left(\mathbb{T}\cdot {}^{(3)}\Sigma\right) &= \qquad\qquad d\mathbb{T}\cdot\wedge {}^{(3)}\Sigma \qquad + \mathbb{T}\cdot d\, {}^{(3)}\Sigma \\
&= \mathbf{e}_\mu T^{\mu\nu}{}_{;\sigma}\mathbf{e}_\nu\, dx^\sigma \cdot \wedge {}^{(3)}\Sigma \qquad + \mathbb{T}\cdot d\, {}^{(3)}\Sigma \\
&= \mathbf{e}_\mu T^{\mu\nu}{}_{;\sigma}\mathbf{e}_\nu \cdot dx^\sigma \wedge \mathbf{e}_\rho\, {}^{(3)}\underset{\approx}{\Sigma}{}^{\rho} + \mathbb{T}\cdot d\, {}^{(3)}\Sigma
\end{aligned}
$$

For the wedge algebra $\mathbf{e}_\rho$ is to be viewed as a mere coefficient. Consequently, it is uneffected by the wedge product operation dx^σ, and one finds that

$$
= \mathbf{e}_\mu T^{\mu\nu}{}_{;\sigma}\mathbf{e}_\nu \cdot \mathbf{e}_\rho\, dx^\sigma \wedge {}^{(3)}\underset{\approx}{\Sigma}{}^{\rho} + \mathbb{T}\cdot d\, {}^{(3)}\Sigma
$$

is a vector-valued 4-form, an element of $\binom{1}{4}$. }

Combine Eq. (12.7) with Eq.(12.8), insert the result into Eq.(12.5) and obtain

$$
\begin{aligned}
d({}^*\mathbb{T}) &= \mathbf{e}_\tau \otimes \mathbf{e}_\sigma T^{\tau\sigma}{}_{;\mu}\cdot \mathbf{e}_\nu\, g^{\nu\rho}\sqrt{-g}\,\delta^\mu_\rho\, dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \\
&= \mathbf{e}_\tau T^{\tau\sigma}{}_{;\sigma}\sqrt{-g}\, dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \qquad (12.9)
\end{aligned}
$$

$$
= \left(\begin{array}{c}\text{amount of momenergy} \\ \text{created in} \\ \text{the invariant spacetime} \\ \text{4-volume } \sqrt{-g}\, dx^0\wedge dx^1\wedge dx^2\wedge dx^3\end{array}\right) \qquad (12.10)
$$

Whenever momenergy is neither created nor destroyed, then such a state of affairs is mathematized by the statement (12,8)

(12.11)

$$\boxed{T^{\tau\sigma}{}_{;\sigma} = 0} \quad \left(\text{"Conservation of momenergy"}\right)$$

Lecture 13

Stress tensor as the area force relation

In MTW read Chapter 5, starting with Box 5.1 on P131,
then Section § 5.3, p 138

I. Particle-induced Force on an Area.

All matter is composed of particles. Their averaged motion and/or interaction across a small area manifests itself as a force on this area. Moreover, for for small areas (but still large enough to preserve the applicability of the averaging process) the relation between the size of the area and force on it is a linear one.

II. Force-area Relation

Focus your mind on a volume element with its bounding surface areas. Each of them has a (spatial) normal vector, and also has a force acting on it. This force vector acting on a surface element characterized by its normal vector is a type of *stress*. The mathematization process of this circumstance is executed as follows:

A) Elements of Area

Focus your mind on a laboratory coordinate frame coordinatized by (x,y,z) and an element of (triangular) area with vertices at $x=a_1$, $y=a_2$, and $z=a_3$

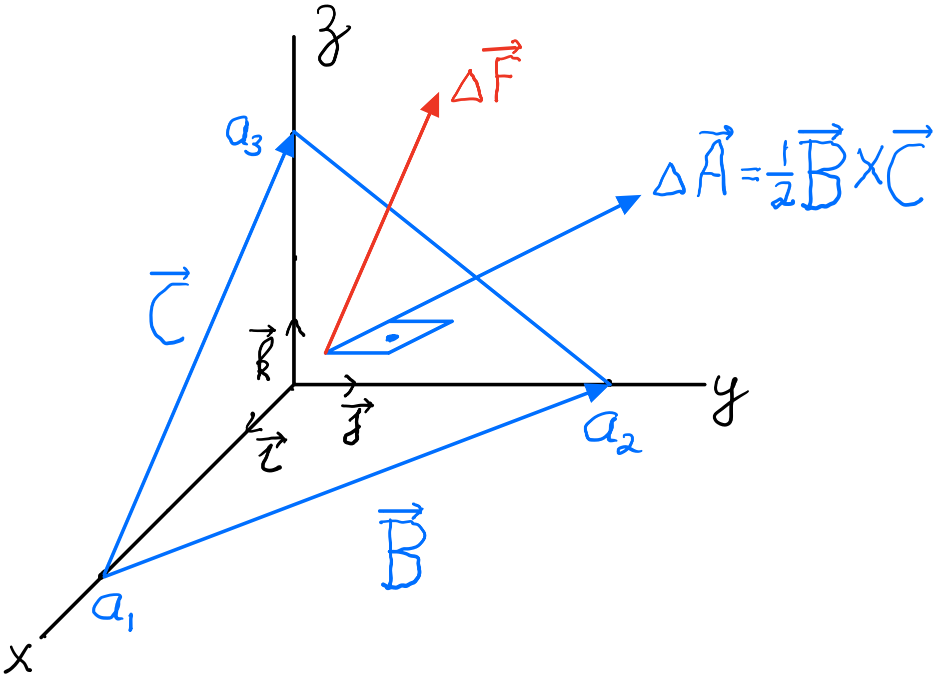


Figure 13.1 Force $\Delta\vec{F}$ acts on area spanned by vectors $\vec{B}$ and $\vec{C}$ and whose normal is $\Delta\vec{A}=\frac{1}{2}\vec{B}\times\vec{C}$.

The area is subtended by the vectors

$$\vec{B}=-a_1\vec{i}+a_2\vec{j}$$
$$\vec{C}=-a_1\vec{i}+a_3\vec{k},$$

and the vector normal to this area is

$$\Delta\vec{A}=\frac{1}{2}\vec{B}\times\vec{C}$$
$$=\frac{1}{2}\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -a_1 & a_2 & 0\\ -a_1 & 0 & a_3\end{vmatrix}$$

$$= \frac{1}{2}\left[\vec{i}\, a_2 a_3 + \vec{j}\, a_3 a_1 + \vec{k}\, a_1 a_2\right]$$

This decomposes the normal vector

$$\Delta\vec{A} = \vec{i}\,\Delta A_x + \vec{j}\,\Delta A_y + \vec{k}\,\Delta A_z$$

into its components

$$\Delta A_x = \frac{1}{2} a_2 a_3$$

$$\Delta A_y = \frac{1}{2} a_3 a_1$$

$$\Delta A_z = \frac{1}{2} a_1 a_2$$

relative to the lab basis $(\vec{i}, \vec{j}, \vec{k})$. These lab components are the projections of $\Delta\vec{A}$ onto the respective coordinate planes.

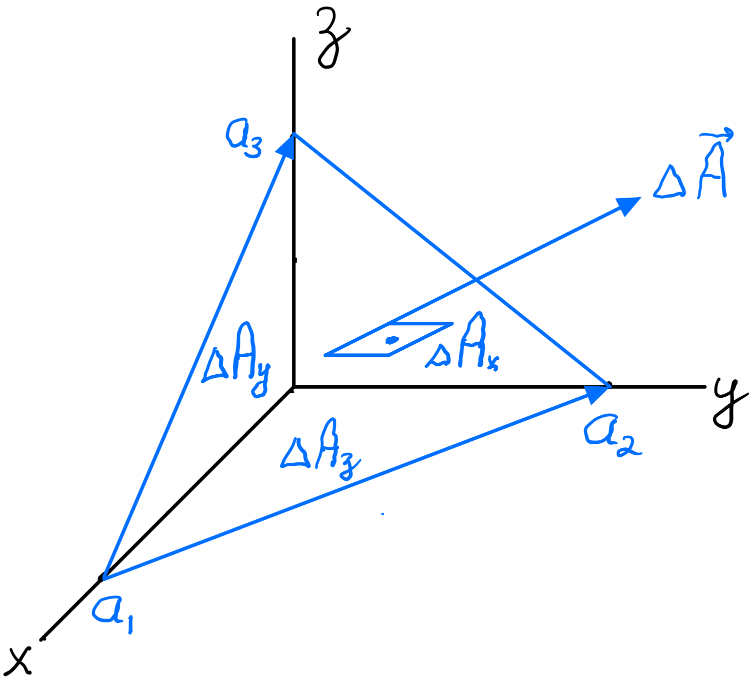


Figure 13.2 Projections of the area, whose normal is $\Delta\vec{A}$, onto the coordinate planes. The sum of the squares of these projections equals the squred magnitude of $\Delta\vec{A}$:

$$(A_x)^2 + (A_y)^2 + (A_z)^2 = \frac{1}{4}\left[a_2^2 a_3^2 + a_3^2 a_1^2 + a_1^2 a_2^2\right] = |\Delta\vec{A}|^2.$$

B.) Stress

The fact that a force field $\vec{F}$ is distributed uniformly over the planar neighborhood which contains $\Delta\vec{A}$ implies that doubling the size of $\Delta\vec{A}$ doubles the size of $\Delta\vec{F}$. In other words, $\Delta\vec{F}$ is a linear function of $\Delta\vec{A}$, i.e. by observing that changing $\Delta\vec{A}$ causes a change in $\Delta\vec{F}$, one says that the causal relationship between is one which is linear.

This linear function is the "<u>stress</u>" to which the matter in the volume element is subjected:

$$\Delta\vec{F} = \vec{i}\,\Delta F^x + \vec{j}\,\Delta F^y + \vec{k}\,\Delta F^z = \text{"stress"}(\Delta\vec{A}),$$

where "stress" is mathematized by the following equations

$$\Delta F^x = T^{xx}\Delta A_x + T^{xy}\Delta A_y + T^{xz}\Delta A_z$$

$$\Delta F^y = T^{yx}\Delta A_x + T^{yy}\Delta A_y + T^{yz}\Delta A_z$$

$$\Delta F^z = T^{zx}\Delta A_x + T^{zy}\Delta A_y + T^{zz}\Delta A_z$$

They comprise the linear causal relationship between vectors $\Delta\vec{A}$ and $\Delta\vec{F}$

Each of the stress components T^{xx}, T^{xy}, etc is measurable. They characterize the stress to which matter in the neighborhood of the origin in Figures 13.1 and 13.2 is subjected to.

For a given volume element of matter these components form a square array

$$\text{"Stress"} = \begin{bmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{bmatrix}. \qquad (13.1)$$

Here each of the diagonal elements refers to a pressure,

$$T^{xx} = \frac{(\text{force into x-direction})}{\begin{pmatrix}\text{unit area pointing} \\ \text{into the x-direction}\end{pmatrix}} = \begin{matrix}\text{"pressure} \\ \text{into the} \\ \text{x-direction"},\end{matrix}$$

while each of the off-diagonal elements is a shear stress

$$T^{xy} = \frac{(\text{force into x-direction})}{\begin{pmatrix}\text{unit area pointing} \\ \text{into the y-direction}\end{pmatrix}} = \begin{matrix}\text{"shear} \\ \text{stress"}\end{matrix}$$

(13.6)

In general,

$$T^{ii} = \text{pressure (no sum)}$$

$$T^{ij} = \text{shear stress} \; (i \neq j)$$

LECTURE 14

Decomposition of
The Momenergy Density-Flux

Components of the stress-energy tensor

- Energy density
- Momentum density
- Energy flux
- Momentum flux

The momenergy density-flux

$$ {}^{*}\mathbb{T} = e_\mu T^{\mu\nu} e_\nu \cdot e_\sigma \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma /3! \qquad (14.1) $$

is the product of a mental integration of

<u>momenergy</u> $\quad p = e_\mu p^\mu ,$

<u>the particle 4-current</u>

$$ S = N u^\nu e_\nu , $$

and the vector-valued measure*

$$ {}^{(3)}\Sigma = e_\sigma {}^{(3)}\underset{\sim}{\Sigma} \equiv e_\sigma \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma /3! , $$

which is a 3-form with vectorial coefficients.**

*\footnote{This vector-valued measure was introduced en passant by MTW with their Eq. (15.15) in order to mathematize the "moment of rotation" of the Einstein field equations.

Modulo a minus sign, they introduce this "measure" by

$$ \star(A \wedge B \wedge C) \equiv e_\sigma \epsilon^{\sigma}{}_{\alpha\beta\gamma} A^\alpha B^\beta C^\gamma $$
$$ = e_\sigma \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma /3! \,(A, B, C) $$

This is a direct generalization of the familiar cross product of a pair of vectors $\vec{A} = A^j \frac{\partial}{\partial x^j}$ and $\vec{B} = B^k \frac{\partial}{\partial x^k}$ in 3-d Euclidean space:

This vector

$$ \star(\vec{A} \wedge \vec{B}) = e_\ell \epsilon^{\ell}{}_{jk} A^j B^k $$
$$ = e_\ell \epsilon^{\ell}{}_{jk}\, dx^j \wedge dx^k /2! \,(\vec{A}, \vec{B}) $$
$$ \equiv {}^{(2)}\Sigma(\vec{A}, \vec{B}) $$

in Euclidean

This is because the curvilinear coordinate basis expansion of the cross product is

$$(\vec{A}\times\vec{B})^{\ell}\frac{\partial}{\partial x^{\ell}} = e_{\ell}\ \frac{\partial(x,y,z)}{\partial(x^1,x^2,x^3)}\begin{vmatrix} g^{\ell 1} & g^{\ell 2} & g^{\ell 3} \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{vmatrix}$$

$$= e_{\ell}\, g^{\ell i}\, \epsilon_{ijk}\, A^j B^k$$

$$= e_i\, \epsilon^{i}{}_{jk}\, dx^j \wedge dx^k\, (\vec{A},\vec{B})$$

$$= \star\, (\vec{A},\vec{B}) \qquad \}$$

**\footnote{As compared to a scalar-valued 3-form, such as $^*S = S^{\nu}\epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!$, which was identified in Lecture 8 by Eq. (8.3), which is a scalar valued measure, and which measures the number of particles in the 3-d volume elements spanned by some triad of four-dimensional vectors (A, B, C),

$$\# = {}^*S^{\mu}\,(A,B,C)$$

$$= S^{\nu}\epsilon_{\nu\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3!\ (A,B,C)$$

$$= \sqrt{-g}\begin{vmatrix} S^0 & S^1 & S^2 & S^3 \\ A^0 & A^1 & A^2 & A^3 \\ B^0 & B^1 & B^2 & B^3 \\ C^0 & C^1 & C^2 & C^3 \end{vmatrix} \quad . \qquad \}$$

Applied to matter composed of a variety of particle species, this product is an organic whole each of whose three factors is a spacetime coordinate frame invariant. Its explicit form is

$$^*\mathbb{T} = e_{\mu}\sum_{a} p^{\mu} N_a u_a^{\nu}\, e_{\nu}\cdot e_{\sigma}\, \epsilon^{\sigma}{}_{\alpha\beta\gamma}\, dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}/3! \qquad (14.2)$$

This spacetime invariant geometrizes four measurable physics concepts: energy density, energy flux, momentum density, and momentum flux. All of them are packed into ${}^*\mathbb{T}$. Their measurability requires their quantitative identification by unpacking ${}^*\mathbb{T}$ so as to exhibit them explicitly.

This unpacking process consists of subjecting ${}^*\mathbb{T}$ in Eq. (14.1) a 3+1, i.e. space plus time, decomposition.

The explicit split of ${}^*\mathbb{T}$ into its space and time components is a matter isolating the time and the spatial terms of the double sum in Eq. (14.1):

$$\mathbb{T} = e_\mu T^{\mu\nu} e_\nu = \left(e_0 T^{00} + e_\ell T^{\ell 0}\right) e_0 + \\ + \left(e_0 T^{0m} + e_\ell T^{\ell m}\right) e_m \quad . \qquad (14.3)$$

Here the latin summation indeces range only over the spatial indeces 1, 2, and 3.

Similarly one has

$$\begin{aligned} {}^{(3)}\Sigma &= e_\sigma \, \epsilon^\sigma{}_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \\ &= \left(e_0 \, \epsilon^0{}_{\alpha\beta\gamma} + e_\ell \, \epsilon^\ell{}_{\alpha\beta\gamma}\right) dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \end{aligned} \qquad (14.4)$$

Applying these decompositions to Eqs. (14.1) and (14.2) decomposes both the geometrical and the physical mathematization of $^*\mathbb{T}$:

$$^*\mathbb{T} = \left(\mathbf{e}_0 T^{00} + \mathbf{e}_m T^{m0}\right)\epsilon_{0\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!$$
$$+\left(\mathbf{e}_0 T^{0\ell} + \mathbf{e}_m T^{m\ell}\right)\epsilon_{\ell\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \qquad (14.5)$$

and

$$^*\mathbb{T} = \left(\mathbf{e}_0 \sum_a p^0 N_a u_a^0 + \mathbf{e}_m \sum_a p^m N_a u_a^0\right)\epsilon_{0\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! +$$
$$+\left(\mathbf{e}_0 \sum_a p^0 N_a u_a^\ell + \mathbf{e}_m \sum_a p^m N_a u_a^\ell\right)\epsilon_{\ell\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3! \qquad (14.6)$$

1.) Density Components of $\mathbb{T}$.

The physical meaning of T^{00} and T^{m0} are recovered from $^*\mathbb{T}$ by evaluating it on the triad of space-like vectors

$$A = \Delta x \frac{\partial}{\partial x^1}\,,\; B = \Delta y \frac{\partial}{\partial x^2}\,,\; C = \Delta z \frac{\partial}{\partial x^3}.$$

From Eqs. (14.5) and (14.6) one finds that

$$\left(\mathbf{e}_0 T^{00} + \mathbf{e}_m T^{m0}\right)\epsilon_{0123}\,\Delta x \Delta y \Delta z = {}^*\mathbb{T}(A,B,C) = \left(\mathbf{e}_0 \sum_a p^0 N_a u_a^0 + \mathbf{e}_m \sum_a p^m N_a u_a^0\right)\epsilon_{0123}\,\Delta x \Delta y \Delta z$$

a) The fact that

$$\sum_a P_a^0 N_a U_a^0\, \Delta x \Delta y \Delta z = \begin{pmatrix}\text{mass-energy}\\ \text{observed in}\\ \text{volume } \Delta x \Delta y \Delta z\end{pmatrix}$$

implies

$$\boxed{T^{00} = \frac{\{\text{mass-energy}\}}{(\text{volume})} = \text{"energy density".}}$$

b) The fact that

$$\sum_a P_a^m N_a U_a^0 \,\Delta x \Delta y \Delta z = \begin{pmatrix}\text{mass-energy} \\ \text{observed in} \\ \text{volume } \Delta x \Delta y \Delta z\end{pmatrix}$$

implies

$$\boxed{e_m T^{m0} = \frac{\{\overrightarrow{\text{momentum}}\}}{(\text{volume})} = \text{"momentum density"}}$$

2.) Flux Components of T.

The physical meaning of $T^{0\ell}$ and $T^{m\ell}$ are recovered from $^*T$ by evaluating it on the triad of vectors one of which is time-like

$$A = -\Delta t \frac{\partial}{\partial x^0}, \; B = \Delta y \frac{\partial}{\partial x^2}, \; C = \Delta z \frac{\partial}{\partial x^3}.$$

From Eqs. (14.5) and (14.6) one finds that

$$(e_0 T^{01} + e_m T^{m1}) \epsilon_{1023}(-) \Delta t \Delta y \Delta z = {}^*T(A,B,C) = \left(e_0 \sum_a p^0 N_a u_a^1 + e_m \sum_a p^m N_a u_a^1\right) \epsilon_{1023}(-) \Delta t \Delta y \Delta z$$

a) The fact that

$$\sum_a P^0_a N_a U^1_\alpha \,\Delta t \Delta y \Delta z = \begin{pmatrix} \text{mass-energy} \\ \text{observed to pass} \\ \text{through area } \Delta y \Delta z \\ \text{during time } \Delta t \end{pmatrix}$$

implies that

$$\boxed{T^{01} = \frac{(\text{mass-energy})}{(\text{time})(\overrightarrow{\text{area}})^x} = \text{"energy flux into the x-direction"} = \text{"power per x-directed area"} = \text{"intensity"}}$$

b) The fact that

$$e_m \sum_a P^m_a N_a U_\alpha \,\Delta t \Delta y \Delta z = \begin{pmatrix} \{\overrightarrow{\text{momentum}}\} \\ \text{observed to pass} \\ \text{through area } \Delta y \Delta z \\ \text{during time } \Delta t \end{pmatrix}$$

implies

$$e_m T^{m1} = \frac{\overrightarrow{\text{momentum}}}{(\text{time})(\overrightarrow{\text{area}})^x} = \text{"Force per x-directed area"}$$

In particular,

T^{11} = pressure along the x-direction

T^{21} = shear stress:

= "y-force per x-area"

= "y-momentum flux into x-direction

3. Summary.

By observing matter, i.e. particles in aggregate characterized by p, N, and u, one finds that the momenergy 4-current

$$\mathbb{T} = e_\mu T^{\mu\nu} e_\nu$$

has components that decompose into

$$[T^{\mu\nu}] = \begin{array}{ll} \left[\begin{array}{c|c} T^{00} & T^{0\ell} \\ \hline T^{m0} & T^{m\ell} \end{array}\right] & \begin{array}{l} \}\ \text{energy} \\ \}\ \text{momentum} \end{array} \\ \begin{array}{cc} \underbrace{\phantom{T^{00}}}_{\text{density}} & \underbrace{\phantom{T^{m\ell}T^{m\ell}}}_{\text{flux}} \end{array} & \end{array}$$

Lecture 15

Conservation of Momenergy

In MTW read (i.e. grasp) § 5.4 and Box 5.4

In Wheeler's A JOURNEY INTO GRAVITY AND SPACETIME read Chapter 6 on Momenergy. This very readable chapter particularizes and conceptualizes the phyical basis of momenergy and its conservation.

Momenergy is conserved not only on the retail level in isolated collision processes (Lecture 7) here and there but also on the wholesale level of trillions of collision in the Mike hydrogen bomb explosion of Novembe 1, 1952, in the barrel of matter at the center of a star, or in the kettle of boiling water.

I. Change in volume content vs. outflow across its faces.

Momenergy conservation is the statement that momenergy is neither created nor destroyed.

Every measurement is based on a standard.

For momenergy this consists of a triad of displacement vectors, say A, B, and C, in spacetime.

If they all are space-like then one has a spatial volume $\Delta x \Delta y \Delta z$ as the standard.

If one of them is time-like then one has each of the three temporal volumes $\Delta t \Delta y \Delta z$, $\Delta t \Delta z \Delta x$, and $\Delta t \Delta x \Delta y$ as the standard; which one depends on which of the areas $\Delta y \Delta z$, $\Delta z \Delta x$, or $\Delta x \Delta y$ are experimentally relevant.

The statement of momenergy conservation necessitates all four of the spacetime volumes.

A cubical domain of volume $\Delta x \Delta y \Delta z$ during the time interval Δt sweeps out the 4-d spacetime volume $\Delta t\Delta x\Delta y\Delta z$. If there is any momenergy created in $\Delta t\Delta x\Delta y\Delta z$, then this is a unique index , a two-part sum of momenergy, each part measurable and accountable and given by the vectorial index

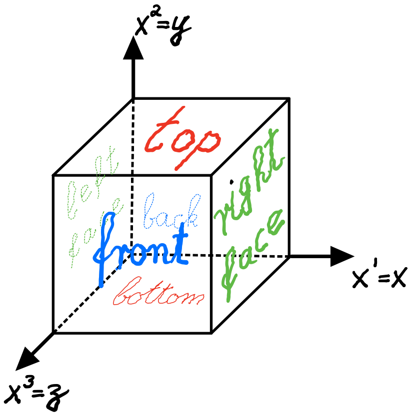
$$Q = \begin{pmatrix} \text{momenergy} \\ \text{created in } \Delta t\Delta x\Delta y\Delta z \end{pmatrix}$$

$$= \begin{pmatrix} \text{change in m.e.} \\ \underline{\text{inside}}\ \Delta x\Delta y\Delta z \\ \text{during time } \Delta t \end{pmatrix} + \begin{pmatrix} \underline{\text{outflow}} \text{ of m.e.} \\ \text{through the sides} \\ \text{of } \Delta x\Delta y\Delta z \text{ during} \end{pmatrix};$$

the explicit form of these two contributions

$$= \begin{pmatrix} \text{m.e. in } \Delta x\Delta y\Delta z \\ \text{at the end, } t+\Delta t, \\ \text{of time interval } \Delta t \end{pmatrix} - \begin{pmatrix} \text{m.e. in } \Delta x\Delta y\Delta z \\ \text{at the beginning, } t, \\ \text{of time interval } \Delta t \end{pmatrix}$$

$$\left.\begin{array}{l} + \begin{pmatrix} \text{flow of m.e. out} \\ \text{of right hand} \\ \text{face of } \Delta x\Delta y\Delta z \\ \text{during } \Delta t \end{pmatrix} + \begin{pmatrix} \text{flow of m.e. out} \\ \text{of left hand} \\ \text{face of } \Delta x\Delta y\Delta z \\ \text{during } \Delta t \end{pmatrix} \\ + \begin{pmatrix} \text{flow of m.e. out} \\ \text{of top} \\ \text{face of } \Delta x\Delta y\Delta z \\ \text{during } \Delta t \end{pmatrix} + \begin{pmatrix} \text{flow of m.e. out} \\ \text{of bottom} \\ \text{face of } \Delta x\Delta y\Delta z \\ \text{during } \Delta t \end{pmatrix} \\ + \begin{pmatrix} \text{flow of m.e. out} \\ \text{of front} \\ \text{face of } \Delta x\Delta y\Delta z \\ \text{during } \Delta t \end{pmatrix} + \begin{pmatrix} \text{flow of m.e. out} \\ \text{of back} \\ \text{face of } \Delta x\Delta y\Delta z \\ \text{during } \Delta t \end{pmatrix} \end{array}\right\}$$



$$\boxed{\text{Conservation of momenergy: } Q=0}$$

Example

1. $\begin{pmatrix}\text{change in m.e.}\\ \text{in } \Delta x \Delta y \Delta z\end{pmatrix} < 0 \iff \underline{\text{net outflow}} > 0$

2. $\begin{pmatrix}\text{change in m.e.}\\ \text{in } \Delta x \Delta y \Delta z\end{pmatrix} > 0 \iff \underbrace{\underline{\text{net outflow}} < 0}_{\text{"inflow"}}$

3. $\begin{pmatrix}\text{change in m.e.}\\ \text{in } \Delta x \Delta y \Delta z\end{pmatrix} = 0 \iff \underline{\text{net outflow}} = 0$

II. Mathematization of change in volume content and of outward flow.

The descriptive momenergy ledger for Q calls for four triads of 4-d vectors. One triad for specifying the interior of the volume element $\Delta x \Delta y \Delta z$ to accommodate the measured density of momenergy, and the other three for specifying the time-like domains $\Delta t\, \Delta y \Delta z$, $\Delta t\, \Delta z \Delta x$, and $\Delta t\, \Delta x \Delta y$ to accommodate the measured flux of momenergy.*

* \footnote{ Recall that

these measured momenergy quantities are condensed into the momenergy density-flux (See Lecture 12)

$$ {}^{*}\mathbb{T} = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu \cdot \mathbf{e}_\sigma \, \epsilon^{\sigma}{}_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3! \qquad (15.1) $$

where

$$ [T^{\mu\nu}] = \left[\begin{array}{c|c} T^{00} & T^{0\ell} \\ \hline T^{m0} & T^{m\ell} \end{array}\right] \begin{array}{l} \}\ \text{energy} \\ \}\ \text{momentum} \end{array} $$

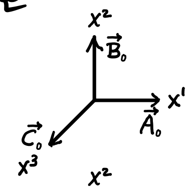
$$ \underbrace{\qquad\quad}_{\text{density}} \; \underbrace{\qquad\qquad\qquad}_{\text{flux}} $$

are the coefficients of its momenergy 4-current

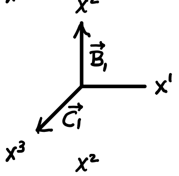
$$ \mathbb{T} = \mathbf{e}_\mu T^{\mu\nu} \mathbf{e}_\nu \, . \qquad \} $$

These four triads of vectors are

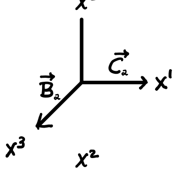
$$ A_0 = \Delta x \frac{\partial}{\partial x^1}, \quad B_0 = \Delta y \frac{\partial}{\partial x^2}, \quad C_0 = \Delta z \frac{\partial}{\partial x^3} $$



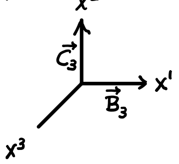
$$ A_1 = \Delta t \frac{\partial}{\partial x^0}, \quad B_1 = \Delta y \frac{\partial}{\partial x^2}, \quad C_1 = \Delta z \frac{\partial}{\partial x^3} $$



$$ A_2 = \Delta t \frac{\partial}{\partial x^0}, \quad B_2 = \Delta z \frac{\partial}{\partial x^3}, \quad C_2 = \Delta x \frac{\partial}{\partial x^1} $$



$$ A_3 = \Delta t \frac{\partial}{\partial x^0}, \quad B_3 = \Delta x \frac{\partial}{\partial x^1}, \quad C_3 = \Delta y \frac{\partial}{\partial x^2} $$



The momenergy density-flux 3-form $^*T$ evaluated* on these vectors quantifies the contribution to the Q index of momenergy created by the volume element's interior $\Delta x \Delta y \Delta z$ during the time interval Δt.

The result is that the momenergy ledger consist of four paired** contributions,

$$Q = e_\mu T^{\mu 0}\sqrt{-g}\,\Big|_{x^0+\Delta t} \Delta x \Delta y \Delta z - e_\mu T^{\mu 0}\sqrt{-g}\,\Big|_{x^0} \Delta x \Delta y \Delta z$$

$$+ e_\mu T^{\mu 1}\sqrt{-g}\,\Big|_{x^1+\Delta x} \Delta y \Delta z \Delta t + e_\mu T^{\mu 1}\sqrt{-g}\,\Big|_{x^1} (-)\Delta y \Delta z \Delta t$$

$$+ e_\mu T^{\mu 2}\sqrt{-g}\,\Big|_{x^2+\Delta y} \Delta z \Delta x \Delta t + e_\mu T^{\mu 2}\sqrt{-g}\,\Big|_{x^2} (-)\Delta z \Delta x \Delta t$$

$$+ e_\mu T^{\mu 3}\sqrt{-g}\,\Big|_{x^3+\Delta z} \Delta x \Delta y \Delta t + e_\mu T^{\mu 3}\sqrt{-g}\,\Big|_{x^3} (-)\Delta x \Delta y \Delta t$$

*\footnote{

for p 15,5

The evaluation of $^*\mathbb{T}$ on each of the triads of vectors follows the same computational line of reasoning. Thus,

$$^*\mathbb{T}(A_0, B_0, C_0) = e_\mu T^{\mu\nu} e_\nu \cdot e_\sigma \epsilon^\sigma{}_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\ (A_0, B_0, C_0)$$

$$= e_\mu T^{\mu\nu} \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\ (A_0, B_0, C_0)$$

$$= e_\mu T^{\mu\nu} \epsilon_{\nu 123}\, \Delta x\, \Delta y\, \Delta z$$

$$= e_\mu T^{\mu 0} \sqrt{-g}\ \Delta x\, \Delta y\, \Delta z,$$

$$^*\mathbb{T}(A_1, B_1, C_1) = e_\mu T^{\mu\nu} \epsilon_{\nu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\ (A_1, B_1, C_1)$$

$$= e_\mu T^{\mu\nu} \epsilon_{\nu 023} (-) \Delta t\, \Delta y\, \Delta z$$

$$= e_\mu T^{\mu 1} \epsilon_{1023} (-) \Delta t\, \Delta y\, \Delta z$$

$$= e_\mu T^{\mu 1} \sqrt{-g}\ \Delta t\, \Delta y\, \Delta z,$$

and similarly for the others. }

**\footnote{ The all-important minus sign in second contribution for the volume elements $\Delta t\, \Delta y\, \Delta z$, $\Delta t\, \Delta z\, \Delta x$, and $\Delta t\, \Delta x\, \Delta y$ is due to the fact that outward normal points into the direction opposite to that of the first contribution. }

or, neglecting higher order terms,

$$
\begin{aligned}
Q = \; & \nabla_0 \left(e_\mu T^{\mu 0} \sqrt{-g} \right) \Delta t \Delta x \Delta y \Delta z \\
& + \nabla_1 \left(e_\mu T^{\mu 1} \sqrt{-g} \right) \Delta x \Delta y \Delta z \Delta t \\
& + \nabla_2 \left(e_\mu T^{\mu 2} \sqrt{-g} \right) \Delta y \Delta z \Delta x \Delta t \\
& + \nabla_3 \left(e_\mu T^{\mu 3} \sqrt{-g} \right) \Delta z \Delta x \Delta y \Delta t \\
= \; & \nabla_\nu \left(e_\mu T^{\mu\nu} \sqrt{-g} \right) \Delta t \Delta x \Delta y \Delta z
\end{aligned}
$$

Based on the properties of the covariant derivative operator ∇ this expressed becomes

$$
\begin{aligned}
& = e_\mu T^{\mu\nu}{}_{;\nu} \sqrt{-g} \; dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 (A_1, A_0, B_0, C_0) \\
& = d({}^*\mathbb{T})(A_1, A_0, B_0, C_0)
\end{aligned}
$$

where $d({}^*\mathbb{T})$ is the exterior derivative of

$$
{}^*\mathbb{T} = e_\mu T^{\mu\nu} \epsilon_{\nu\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma / 3!
$$

Relative to the rectilinear parallel (t, x, y, z)-induced basis one has $\nabla_\nu e_\nu = 0$. Consequently, the created momenergy/4-volume is

$$
\frac{Q}{\Delta t \Delta x \Delta y \Delta z} = e_\mu T^{\mu\nu}{}_{,\nu}
$$

Relative to curvilinear coordinates this momenergy is

$$\frac{Q}{\sqrt{-g}\,\Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3} = e_\mu \, T^{\mu\nu}{}_{;\nu}$$

III. Conservation of momenergy

However, in nature momenergy is neither created nor destroyed :

$$Q = 0.$$

The local mathematization of this fact is the statement

$$\boxed{\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0} \qquad \mu = 0,1,2,3$$

or more generally

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0} \qquad \mu = 0,1,2,3 \;,$$

which is equivalent to

$$\boxed{d\,{}^*\mathbb{T} = 0} \;.$$

Lecture 16

Relativistic Fluid Dynamics

I. Momenergy Tensor of a Perfect Fluid

II. Momenergy of an Electromagnetic Field

III. Conservation of Momenergy

In MTW read and peruse

Ex. 22.1 : Change in comoving volume : Kinematics

§ 22.5 : Matter in motion : Dynamics

Box 5.5 : Momenergy for a perfect fluid

Ex. 3.18 : Momenergy for E.&M

Page 152,155 : Conservation momenergy of charged matter plus e.m. radiation

(16.1)

I. Momenergy tensor of a Perfect Fluid

In its comoving coordinate frame, where its spatial velocity is zero, a perfect fluid is characterized by its three attributes:

- mass-energy density $\rho(x^\alpha)$
- isotropic pressure $p(x^\alpha)$
- 4-velocity with components $\{u^\mu(x^\alpha)\} = \{1, 0, 0, 0\}$

The momenergy tensor components relative to such a frame are

$$[T^{\mu\nu}] = \begin{bmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}$$

$$[T_\mu{}^\nu] = \rho \begin{bmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} + p \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} =$$

$$= \left[\rho \quad u_\mu u^\nu \quad + p\,(\delta_\mu{}^\nu + u_\mu u^\nu)\right]$$

$$T_A{}^B = \rho\, u_A u^B + p\left\{\delta_A{}^B + u_A u^B\right)$$

$$T_a{}^b = p_\perp \delta_a^b$$

Relative to any frame the momenergy tensor components for a perfect fluid are therefore

$$\boxed{T_\mu{}^\nu = (p+\rho)\, u_\mu u^\nu + p\, \delta_\mu{}^\nu} \qquad (16.1)$$

II. Momenergy of an Electromagnetic Field

The electromagnetic field

$$\underset{\sim}{F} = F_{\mu\nu}\, dx^\mu \wedge dx^\nu / 2!$$

has a momenergy tensor whose components are

$$T^{\mu\nu}_{e.m.} = \frac{1}{4\pi} F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \qquad (16.2)$$

The 4-current of a charged fluid with comoving particle density N is a 4-vector whose components are

$$J^\mu = qNu^\mu. \qquad (16.3)$$

Here q is the charge per particle so that qN is the comoving charge density.

It is an exercise in "index gymnastic" (MTW, Ex. 3.18, p 89) to show that, in light of the Maxwell field equations, the divergence of the e.m. momenergy tensor has components given by

$$T^{\mu\nu}_{e.m.} = -F^{\mu\alpha} J_\alpha \qquad (16.4)$$

III. Conservation of Momenergy

It is a fact that total momenergy, that of matter plus that of the e.m. field, is neither created nor destroyed. This is mathematized by the statement

$$(T^{\mu\nu}_{matter} + T^{\mu\nu}_{e.m.})_{;\nu} = 0,$$

or in light of Eqs. (16.3)-(16.4)

$$\boxed{T^{\mu\nu}_{\text{matter};} = -NqF^{\mu}{}_{\alpha}u^{\alpha}} \qquad (16.5)$$

The right hand side mathematizes the amount of momenergy injected into the invariant 4-d spacetime volume $\Delta\tau\, v$.

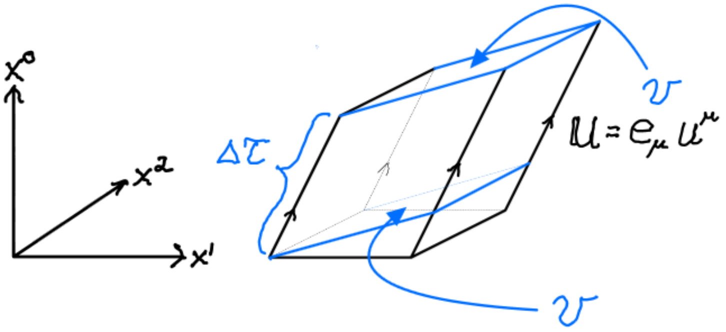


Figure 16.1 : World tube bounded by and filled with congruent particle world lines.
The 4-d basis invariant spacetime volume $\Delta\tau v$ is the product of the elapsed proper time $\Delta\tau$ in a comoving element of fluid whose comoving volume is v.

Indeed, one has

$$N \cdot qF^{\mu}{}_{\alpha}u^{\alpha} = \frac{(\text{particles})}{\left(\begin{matrix}\text{comoving}\\ \text{volume}\end{matrix}\right)} \cdot \frac{\left(\begin{matrix}\text{Lorentz}\\ \text{4-force}\end{matrix}\right)^{\mu}}{(\text{particle})} .$$

But recall that

$$qF^{\mu}{}_{\alpha}u^{\alpha} = \frac{(\Delta\,\text{momenergy})^{\mu}}{(\Delta\tau)} \approx \frac{dp^{\mu}}{d\tau}$$

and

$$N = \frac{\#}{v}$$

Consequently,

$$N \cdot qF^{\mu}{}_{\alpha}u^{\alpha} = \frac{\#}{\mathcal{V}} \times \frac{(\Delta\,\text{momenergy})^{\mu}}{(\Delta\tau)}$$

$$= \frac{\Delta(\text{total momenergy})}{\mathcal{V}\Delta\tau}$$

$$= \frac{(\text{momenergy})^{\mu}}{\begin{pmatrix}\text{invariant}\\ \text{spacetime}\\ \text{volume}\end{pmatrix}}$$

Thus Eq. (16.5) mathematizes the amount of mechanical momenergy created per 4-volume $\Delta\tau\,\mathcal{V}$ by the e.m. field $F^{\mu}{}_{\alpha}$ interacting with the fluid particles each one of which has charge q.

Q: what is the mechanical dynamics of the responding particles?

Lecture 17

Relativistic Hydrodynamics

In MTW read § 22.2 and § 22.3

For our purposes one may initially set the entropy in these sections equal to zero

17.1

The power of the mathematized law of momenergy conservation, when applied to the mechanics of a continuum, is that it furnishes the dynamical equations that govern it. For a perfect fluid whose momenergy tensor, whose components are

$$T^{\mu\nu} = (p+\rho)u^{\mu}u^{\nu} + g^{\mu\nu}p\,, \qquad (17.1)$$

these equation are the relativistic equations of motion for a fluid:

$$0 = T^{\sigma\nu}{}_{;\nu} = (g^{\sigma\mu}T_{\mu}{}^{\nu})_{;\nu} = \underbrace{g^{\sigma\mu}{}_{;\nu}}_{\text{zero}}T_{\mu}{}^{\nu} + g^{\sigma\mu}T_{\mu}{}^{\nu}{}_{;\nu}$$

or

$$0 = T_{\mu}{}^{\nu}{}_{;\nu} = \left[(p+\rho)u_{\mu}u^{\nu} + p\,\delta_{\mu}{}^{\nu}\right]_{;\nu} \qquad (17.2)$$

$$= \left[(p+\rho)u_{\mu}\right]_{;\nu}u^{\nu} + (p+\rho)u_{\mu}u^{\nu}{}_{;\nu} + p_{,\mu} \qquad (17.3)$$

⋮

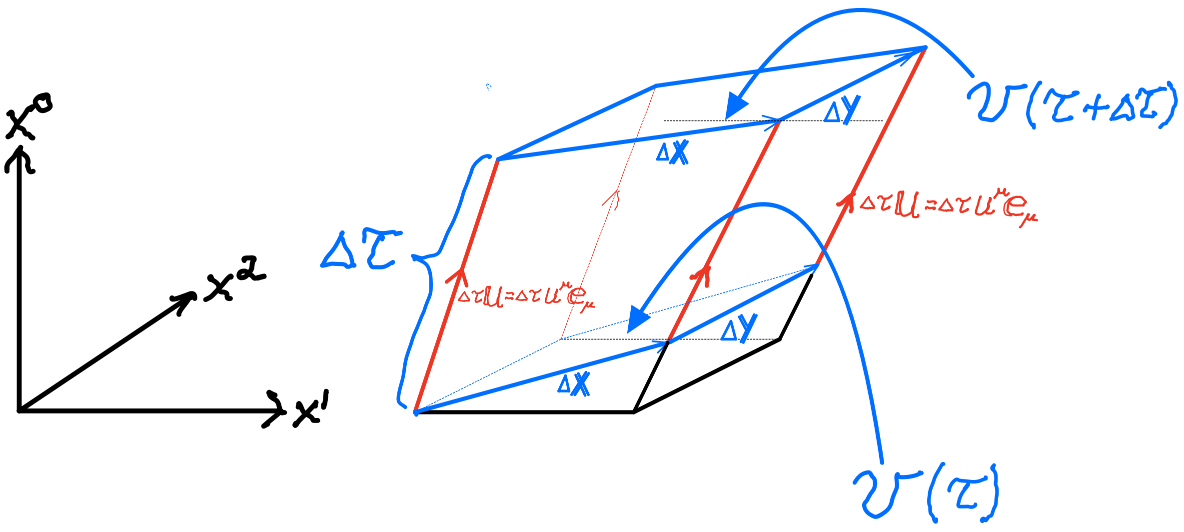


Figure 17.1 World tube whose boundary is generated by the world lines of adjacent particles, each one having its own 4-velocity $\mathbf{u} = \mathbf{e}_\mu u^\mu$. When the 4-velocities of these particles diverge, the comoving volume $\mathcal{V}$ bounded by these particles increases. Over the proper time interval $[\tau, \tau+\Delta\tau]$ this increase is

$$\Delta \mathcal{V} = \mathcal{V}(\tau+\Delta\tau) - \mathcal{V}(\tau),$$

a difference which depends on the divergent nature of the particle 4-velocity vector field $\mathbf{u} = \mathbf{e}_\mu u^\mu$.

This difference is mathematized by the following

Theorem.

GIVEN: A triad of 4-d spatial comoving displacement vectors

$$\Delta \mathbb{X} = \Delta x \frac{\partial}{\partial x} = \Delta X^\alpha \frac{\partial}{\partial x^\alpha}$$

$$\Delta \mathbb{Y} = \Delta y \frac{\partial}{\partial y} = \Delta Y^\beta \frac{\partial}{\partial x^\beta}$$

$$\Delta \mathbb{Z} = \Delta z \frac{\partial}{\partial z} = \Delta Z^\gamma \frac{\partial}{\partial x^\gamma}$$

emanating from a common event on the world line of a reference particle whose 4-velocity is

$$\mathbb{U} = \frac{\partial}{\partial \tau} = u^\mu \frac{\partial}{\partial x^\mu}$$

Here $\{\frac{\partial}{\partial\tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ is the comoving orthonormal basis, while $\{\frac{\partial}{\partial x^\mu} = e_\mu\}_{\mu=0}^{3}$ is a generic coordinate-induced basis.

CONCLUSION

1.) The comoving 3-volume v spanned by this triad is

$$
\begin{aligned}
v &= u^\mu \epsilon_{\mu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma/3!\,(\Delta X, \Delta Y, \Delta Z) \\
&= u^\mu \epsilon_{\mu\alpha\beta\gamma}\, \Delta X^\alpha \Delta Y^\beta \Delta Z^\gamma \\
&= \Delta x\, \Delta y\, \Delta z\,.
\end{aligned}
$$

2.) The increase in this comoving volume during $[\tau, \tau+\Delta\tau]$ is

$$v(\tau+\Delta\tau) - v(\tau) \equiv \Delta v = v\, u^\mu{}_{;\mu}\, \Delta\tau,$$

i.e.

$$\boxed{u^\mu{}_{;\mu} = \frac{1}{v}\frac{dv}{d\tau}} \qquad (17.4)$$

The divergence of the fluid 4-velocity equals the fractional rate of change of a comoving element of fluid volume.

The mathematical proof of this kinematic law of motion is consigned to the appendix of this Lecture 17.

A. Relativistic generalization of Newton's 2nd Law

Introduce the kinematic law Eq. (17.4) into Eq.(17.2).

Recall that v is a scalar function on spacetime and therefore $\frac{dv}{d\tau} = v_{,\alpha} \frac{dx^\alpha}{d\tau}$.

Consequently, the resulting form of the conservation law,

$$0 = v\, T_\mu{}^\nu{}_{;\nu} = v\left[(p+\rho)u_\mu\right]_{;\nu} \frac{dx^\nu}{d\tau} + (p+\rho)u_\mu \frac{dv}{d\tau} + v\, p_{,\mu}$$

simplifies into

$$\left[v\,(p+\rho)u_\mu\right]_{;\nu} \frac{dx^\nu}{d\tau} = -v\,p_{,\mu} \quad . \qquad (17,5)$$

The left hand side is the convective time derivative

$$\frac{D}{d\tau} \equiv \left[\quad\right]_{;\nu} \frac{dx^\nu}{d\tau}$$

of the relativistic momenergy in the 3-volume v. The right hand side is the relativistic buoyancy force; it is a volume force whose cause is the pressure gradient.

Because of these identifications,

$$\frac{D}{d\tau}\left[v\,(p+\rho)\,u_\mu\right] = -v\,\frac{\partial p}{\partial x^\mu} \qquad \mu = 0,1,2,3$$

mathematizes the fact that the rate of change of fluid 4-momentum in a comoving volume is controlled by the 4-d pressure gradient acting on this volume.

B. Energy conservation

Momenergy conservation is the statement that in any comoving volume v during its proper time interval $\Delta\tau$ there is no creation or annihilation of momenergy. The vectorial index of <u>momenergy creation per spacetime 4-volume</u>,

$$\frac{Q}{(\Delta\tau\, v)} = \mathbf{e}_\mu T^{\mu\nu}{}_{;\nu} = \frac{(\text{momenergy})}{\begin{pmatrix}\text{invariant}\\ \text{spacetime}\\ \text{4-volume}\end{pmatrix}}$$

vanishes everywhere at all times! This applies also to the energy observed in the comoving frame. Thus,

$$\mathbf{u}\cdot\mathbf{e}_\mu T^{\mu\nu}{}_{;\nu} = u^\mu T_\mu{}^\nu{}_{;\nu} = 0.$$

Apply this to Eq. (17.2) on page 17.1 and find

$$0 = u^\mu T_\mu{}^\nu{}_{;\nu} = \underbrace{u^\mu (\cancel{p}+\rho)_{,\nu}\, u_\mu}_{=\,-1} u^\nu + \underbrace{u^\mu (p+\rho) u_{\mu;\nu}}_{=\,0} u^\nu + \underbrace{u^\mu (p+\rho)\, u_\mu}_{=\,-1} \underbrace{u^\nu{}_{;\nu}}_{\frac{1}{v}\frac{dv}{d\tau}} + u^\mu \cancel{p_{,\mu}}$$

$$= -\frac{d\rho}{d\tau} - (p+\rho)\frac{1}{v}\frac{dv}{d\tau}$$

Thus $\quad v\dfrac{d\rho}{d\tau} + \rho\dfrac{dv}{d\tau} = -p\dfrac{dv}{d\tau}$

or $\quad \boxed{d(\rho v) = -p\, dv} \qquad (17.6)$

This holds along every comoving volume element v. This equation mathematizes

$$\begin{pmatrix}\text{change in energy}\\ \text{in volume}\end{pmatrix}=\begin{pmatrix}\text{compressional mechanical}\\ \text{work done on the fluid}\\ \text{volume element}\end{pmatrix}$$

This is the 1st law of thermodynamics.

C. Chemical potential

The concept of a chemical potential (a.k.a. enthalpy, a.k.a. injection energy) arises whenever matter exhibits particle conservation, i.e. when the number of particles in a comoving element of volume is a constant,

$$\frac{d(Nv)}{d\tau}=0. \quad (\text{"particle conservation"})$$

Here

$$N=\frac{(\#\text{ of particles})}{\begin{pmatrix}\text{"comoving}\\ \text{volume"}\end{pmatrix}}$$

This law is geometrized in terms of the particle 4-velocity by observing that

$$0=v\frac{dN}{d\tau}+N\frac{dv}{d\tau}$$

$$= v N_{,\nu} u^{\nu} + N u^{\nu}{}_{;\nu}$$

$$\boxed{0 = v\left(N u^{\nu}\right)_{;\nu}}$$

"Geometrized law of particle conservation"

The fact that the number # of particles in a volume v is constant is restated alternatively by

$$v = \frac{\#}{N}.$$

Consequently,

$$\frac{1}{N} = \frac{\{\text{mean volume}\}}{\{\text{particle}\}}.$$

This is the mean volume for a single particle of a fluid. Thus the 1st law of thermodynamics, Eq. (17.6) on page 17.3 is

$$p = -\frac{d(v\rho)}{dv} \quad \left(= -\frac{d(\text{energy/particle})}{d(\text{volume/particle})} \right)$$

$$= -\frac{d\left(\frac{\rho}{N}\right)}{d\left(\frac{1}{N}\right)} = N\frac{d\rho}{dN} - \rho$$

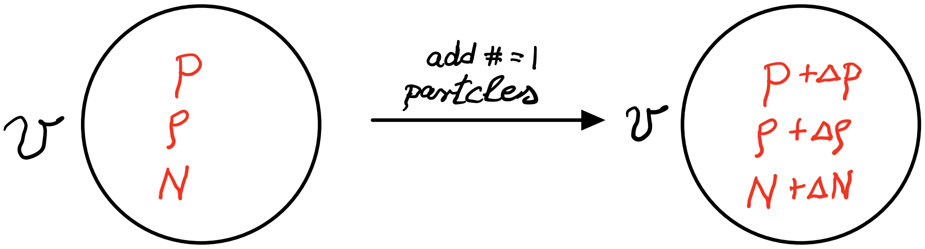
Thus,

$$\boxed{\frac{\rho}{N} + \frac{p}{N} = \frac{d\rho}{dN}}.$$

This is the "chemical potential" of a single particle, better known as the injection energy of a single particle.

To arrive at this concept from physical considerations, consider a comoving volume element with inside pressure p, energy density ρ, and particle density N.

Question: How much energy is needed to inject $\# = 1$ particles into this volume?



Answer:

$$\frac{\Delta\rho}{\Delta N} = \frac{\Delta \frac{(\text{energy})}{(\text{volume})}\, v}{\Delta \frac{(\#\text{ of paricles})}{(\text{volume})}\, v}$$

$$= \frac{(\text{energy})}{(\text{particle})} \equiv \left(\begin{matrix}\text{"injection}\\ \text{energy"}\end{matrix}\right) = \left(\begin{matrix}\text{"chemical}\\ \text{potential"}\end{matrix}\right)$$

Let v be such that $\Delta N v = 1$. In that case $\frac{\Delta\rho}{\Delta N}$ = amount of energy necessary to inject a single particle into the fluid. This is because this energy,

$$\frac{d\rho}{dN} = \frac{p}{N} + \frac{\rho}{N},$$

consists of two parts:

$p \times \frac{1}{N}$ = work necessary to create the volume $\frac{1}{N}$ to accommodate one particle

$\rho \times \frac{1}{N} =$ energy that this particle must have so that once it occupies the created volume, it will be in equilibrium with the surrounding fluid.

Lecture 18

History of Gravitation:

Newton → Einstein → Cartan/Wheeler

In MTW: Read Chapter 15

In "A Journey Into Gravity and Spactime" by J.A. Wheeler (available in the form of a PDF on the internet by "googling" a journey into gravity and spacetime pdf): Read Chapters 6&7.

Einstein's geometrodynamics is summarized by the following two statements:

1. Geometry controls the motion of matter, which is mathematized by

$$T^{\mu\nu}{}_{;\nu}=0.$$

2. Matter controls the geometry of spacetime, which is mathematized by

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^2}T_{\mu\nu}.$$

In his line of reasoning Einstein developed a constellation of fundamental concepts that paved his way towards his geometrical formulation of gravitation

$\boxed{1907}$:

a) Accelerated frame = 1-parameter family of instantaneous inertial frames

= 1-parameter family of tangent spaces

b) Equivalence principle according to which a locally homogeneous, static gravitational field is indistinguishable from a uniformly accelerated frame. (Hence gravitational redshift is a Doppler shift between inertial frames in relative motion).

(See Lecture 8 from Math 5756)

1913

a) Motion of bodies geometrized in terms of geodesics:

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0 \qquad \text{(Math 5757, Lecture 4)}$$

b) Capitalize on the equivalence principle to conclude that Newton's gravitational potential ϕ is part of a spacetime tensor field:

$$g_{00} = -1 - \frac{2}{c^2}\phi \qquad \text{(Math 5757, Lecture 5)}$$

c) Use momenergy conservation

$$T^{\mu\nu}{}_{;\nu} = 0$$

to extend geometrized single particle motion to the motion of the most general aggregate of matter under the influence of gravitational metric potentials

d) In his 1913 paper (Lecture 5-Appendix) Einstein recognized that geometry $g_{\mu\nu}$ including its manifestation via Newton's gravitational potential ϕ, controls the motion of matter.

But in this paper he directed attention to "the main problem of the theory of gravitation". How does one determine the geometry $g_{\mu\nu}$ if one is given the distribution and motion of matter as given by the momenergy tensor $\mathbb{T}$? How does one generalize Newton's field equations

$$\nabla^2 \phi = 4\pi G \rho$$

($G = \frac{1}{15\,000}$ [c.g.s. units], $\frac{1}{15\,000\,000}$ [m k s unit] is Newton's grav'l constant.)

under the stipulation that

$$\phi \simeq -\frac{1}{2} g_{00} \to \frac{1}{2} g_{\mu\nu}$$

$$\rho \to T^{\mu\nu},$$

where $T^{\mu\nu}{}_{;\nu}=0$?

$\boxed{1916}$

a) (i) By means of a tour-de-force Einstein, after three years, arrived at his equations

$$\boxed{\underbrace{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R}_{G_{\mu\nu}\equiv\text{"Einstein tensor"}} = \frac{8\pi G}{c^2}T_{\mu\nu}},$$ "Einstein"

where

$$\boxed{G_{\mu}{}^{\nu}{}_{;\nu}=0}$$

is always satisfied (it is an identity) regardless of the metric-induced curvature components that comprise $G_{\mu\nu}$.

As a consequence these Einstein field

equations always imply

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0}$$ "momenergy conservation"

(ii) Furthermore, relative to torsionless parallel transport, that metric-induced curvature obeys

("2nd Bianchi identity") $$R^{\sigma}{}_{\alpha\beta\gamma;\delta} + R^{\sigma}{}_{\alpha\gamma\delta;\beta} + R^{\sigma}{}_{\alpha\delta\beta;\gamma} \equiv \frac{1}{2!} R^{\sigma}{}_{\alpha[\beta\gamma;\delta]} = 0$$

and

("1st Bianchi identity") $$R^{\sigma}{}_{\alpha\beta\gamma} + R^{\sigma}{}_{\beta\gamma\alpha} + R^{\sigma}{}_{\gamma\alpha\beta} \equiv R^{\sigma}{}_{[\alpha\beta\gamma]} = 0.$$

b) (i) This system of equations is analogous to the Maxwell field equations

$$\boxed{F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu}}$$

"Maxwell"

where

$$\boxed{(F^{\mu\nu}{}_{;\nu})_{;\mu} = 0}$$

is always satisfied regardless of the details of the components that comprise the antisymmetric $F^{\mu\nu}$. As a consequence these Maxwell field equations always imply

$$\boxed{J^{\mu}{}_{;\mu} = 0}.$$

(ii) Furthermore, in light of Faraday's law and the non-existence of magnetic monopoles that electromagnetic tensor obeys

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} \equiv \frac{1}{2!} F_{[\alpha\beta;\gamma]} = 0$$

and

$$F_{\alpha\beta} + F_{\beta\alpha} = 0$$

c) It is evident that momenergy conservation and charge conservation are intrinsic features of ("hardwired" into) Einstein's geometrodynamics and Maxwell's electrodynamics.

In the face of any momenergy or charge creation/annihilation the corresponding Einstein or Maxwell'sfield systems would be silent: under such circumstances they do not apply.

$\boxed{\text{1928 Cartan}}$

The problem with Einstein's

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu} \tag{1}$$

is that its left hand side is ill-defined when Einstein wrote it down in 1916. He constructed this tensorial equation from the components of the curvature tensor merely to (i) satisfy the momenergy conservation law

$$T^{\mu\nu}{}_{;\mu} = 0$$

and (ii) recover the Newtonian gravitational field equation

$$\nabla^2 \phi = 4\pi G \rho \tag{2}$$

in the limit of static weak gravitational fields. Such a construction is necessary but not enough.

In physics and mathematics both sides of an equation (e.g. the stress-strain relation of an elastic madium, F=ma, $\cdots$) must have a well-defined identity. The geometrical meaning of Einstein's l.h.s. and the line of reasoning leading to it need to be specified. Cartan's formulation in 1928, as well Wheeler's in 1964 and in 1990 and Misner's and Wheeler's in 1972, constitute a non-trivial step forward in that direction.

A clue as to the sought-after geometrical meaning of the l.h.s. of Eq.(1) comes from the l.h.s. of Eq.(2). It expresses the following geometrical

Proposition:

For a small sphere of

$$\text{volume} = \frac{4\pi r^3}{3}$$

consider the difference between the mean value of ϕ on the surface of this sphere and its value at the center. Then

$$\nabla^2 \phi = \text{moment of} \left\{ \left(\begin{array}{c} \text{mean value} \\ \text{on the boundary} \\ \text{of the sphere} \end{array} \right) - \left(\begin{array}{c} \text{value at} \\ \text{the center} \\ \text{of the sphere} \end{array} \right) \right\} \frac{8\pi}{\text{volume}}. \tag{3}$$

(Comment: This property of $\nabla^2\phi$ was pointed out by Maxwell already in 1881.) The validity of Eq.(3) is based on the following mathematical reasoning: Consider the mean value (M.V.) of the difference of the potential ϕ on the surface and the center of a small sphere of surface area $4\pi r^2$,

$$\text{M.V.} = \frac{1}{4\pi r^2} \int_0^{\pi} \int_0^{2\pi} [\phi(x,y,z) - \phi(0,0,0)] r^2 \sin\theta \, d\theta \, d\varphi \tag{4}$$

Expand $[\phi(x,y,z)-\phi(0,0,0)]$ to second order and obtain

$$\begin{aligned}[\cdots] =& x\,\phi_{,x}+y\,\phi_{,y}+z\,\phi_{,z} && (5)\\ &+\frac{1}{2}x^2\phi_{,xx}+\frac{1}{2}y^2\phi_{,yy}+\frac{1}{2}z^2\phi_{,zz} && (6)\\ &+xy\,\phi_{,xy}+yz\,\phi_{,yz}+zx\,\phi_{,zx}, && (7)\end{aligned}$$

where all partial derivatives are evaluated at the center $(0,0,0)$ of the sphere. Introduce the spherical coordinates for (x,y,z), do the integration and finf that all linear and off-diagonal quadratic terms integrate to zero. The diagonal terms involving x^2, y^2, and z^2 yield identical results. One obtains

$$(\text{M.V.}) = \frac{1}{4\pi r^2}\frac{1}{2}\frac{4\pi r^4}{3}\,(\phi_{,xx}+\phi_{,yy}+\phi_{,zz})|_{(0,0,0)}\,. \tag{8}$$

Consequently, the moment of this mean deviation is

$$r\times(\text{M.V.}) = \frac{1}{8\pi}(\text{volume})\,\nabla^2\phi\Big|_{(0,0,0)} \tag{9}$$

or

$$\nabla^2\phi = r\times(\text{M.V.})\frac{8\pi}{(\text{volume})} \tag{10}$$

Compare this moment of mean value expression with Newton's gravitational field equation Eq.(2). One obtains for a small sphere of radius r

$$r\times(\text{M.V.} = \frac{G}{2}\rho\,(\text{volume}) = \frac{G}{2}(\text{mass}) \tag{11}$$

$$r\times\{16\pi(\text{M.V.}\} = \frac{8\pi G}{c^2}(\text{mass}\cdot c^2) \tag{12}$$

Thus, Newton's gravitational field equation integrated over a sphere of volume $\frac{4\pi r^3}{3}$ is

$$\text{moment of}\left\{16\pi\left[\begin{array}{c}\text{deviation of}\\ \text{the surface}\\ \text{mean value of}\\ \text{the gravitational}\\ \text{potential on the}\\ \text{boundary of a}\\ \text{3-volume away}\\ \text{from it6s value}\\ \text{at the center}\\ \text{of that 3-volume}\end{array}\right]\right\} = \frac{8\pi G}{c^2}\left(\begin{array}{c}\text{amount of mass}\cdot\text{c}^2\\ \text{inside that 3-volume}\end{array}\right) \tag{13}$$

This is the integral formulation of the Newtoniuan differential field Eq.(2). This equation relates what is in the interior of a volume to the moment of something that is measurable on its boundary.

Cartan, Misner, and Wheeler generalize this moment-based feature to 3-cubes[1] in spacetime. With them the integral formulation of the Einstein field equations get geometrized into the form

$$\text{sum of moments of}\left\{\begin{array}{c}\text{rotation for}\\\text{the 6 faces}\\\text{of a small}\\\text{3-cube}\end{array}\right\}=\frac{8\pi G}{c^2}\left(\begin{array}{c}\text{amount of}\\\text{momenergy}\\\text{inside this}\\\text{3-cube}\end{array}\right)\tag{14}$$

(a) Each 3-cube has associated with it (i) a geometrical object, its total moment of rotation and (ii) a certain vectorial amount of momenergy, which it occupies.

(b) The field Eqs.(1) state that the momenergy in the 3-cube is the source of the moment of rotation. Both are collinjear 4-vectors with Newton's relativized gravitational constant $\frac{8\pi G}{c^2}$ as the constant of proportionality.

(c) Each vector has 4 momenergy components, and there are four 3-cube components ($\Delta x\Delta y\Delta z$, $\Delta t\Delta y\Delta z$, $\Delta t\Delta z\Delta x$, $\Delta t\Delta x\Delta y$) for each volume 4-vector. Consequently, there are $4\times 4=16$ equations as compared to only one equation for the Newtonian gravitation.

(d) To summarize: these equations say that, except for the universal factor $8\pi G/c^2$, the quantity of moment of rotation equals the amount of momenergy in each of these 3-cubes.

---

[1] more precisely, to the components $(\Delta x\Delta y\Delta z),(\Delta t\Delta y\Delta z),(\Delta t\Delta z\Delta x),(\Delta t\Delta x\Delta y)$ of a typical "volume vector"

Lecture 19

The Gravitational Field Equations: Einstein versus Cartan

I. Einstein's tensorial line of reasoning

II. Cartan's, Misner, and Wheeler's geometrization of the E.F. Eq'ns

III. "Rotation" as a tensor

IV. Curvature as rotation

I. Einstein's line of reasoning that led to his gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu},$$

or equivalently

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right),$$

was a multi step tour de force:

(i) Geometrize Newton's 1st Law relative to non-inertial reference frame

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

(ii) Special Relativity:

Uniformly accelerated frame as a sequence of inertial frames.

(iii) Mathematize the dynamical laws governing particles and fields into coordinate frame independent form.

(iv) Recognize and incorporate the Equivalence Principle as the metaphysical corner-

(Here "metaphysical" means: that which pertains to reality, to the nature of things, to existence.)

stone in conceptualizing gravitation:

a) "uniformly acc'd frame ≃ static, homogeneous gravitational field"

or

b) "inertial force = gravi'l force"

(v) Apply the Equivalence Principle (E.P.) to the motion of bodies:

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \rightarrow \frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}$$

$$= \frac{1}{2}g_{00,i} = \left(\text{"inertial force"}\right)$$

$$\underset{\uparrow\,\boxed{\text{E.P.}}}{=} -\Phi_{,i} = \left(\text{"gravitational force"}\right)\Big)$$

(vi) Mathematize the momenergy properties and the dynamics of matter, particles, and fields in geometrical form based on the momenergy tensor $\{T^{\mu\nu}\}$:

$$T^{\mu\nu}{}_{;\nu} = 0 .$$

(vii) Generalize the Newtonian gravitational field equation

$$\nabla^2 \phi = 4\pi G \rho$$

by taking advantage of

a) the special relativistic mass-energy relation and

b) the fact that the Riemann curvature tensor

$$\{R^\alpha{}_{\beta\gamma\delta}\} = \{\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\sigma_{\beta\delta} - \Gamma^\alpha_{\sigma\delta}\Gamma^\sigma_{\beta\gamma}\}$$

is the only tensor containing

2nd derivatives of $g_{\mu\nu}$, including

$$g_{00}{}^{;i}{}_{;i} = \left(-1-\frac{2\Phi}{c^2}\right)^{;i}{}_{;i} = -2\nabla^2\frac{\Phi}{c^2},$$ which

imply that the tensorial generalization of the Newtonian gravitational field equation is

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} = \text{expression in } T_{\mu\nu} \text{ and } g_{\mu\nu} \cdot T^{\alpha}{}_{\alpha}$$

(viii) By demanding that momenergy conservation

$$T^{\mu\nu}{}_{;\nu} = 0$$

be contained in a tensorial way of the tensorially generalized Newtonian equations

$$-\nabla^2(\underbrace{g_{00}}_{\left(-1-2\frac{\Phi}{c^2}\right)}) = \frac{8\pi G}{c^2}\rho,$$

i.e. $$\nabla^2\frac{\Phi}{c^2} = 4\pi G\frac{\rho}{c^2},$$

Einstein arrived at

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha}\right)$$

which is equivalent to

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\equiv G_{\mu\nu}} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

This equation incorporates momenergy conservation

$$G_{\mu}{}^{\nu}{}_{;\nu} = 0$$

identically, and has the Newtonian grav'l equations as an asymptotic limit.

COMMENT:

Such a construction and line of reasoning is necessary, but not enough.

In physics and mathematics both sides, the l.h.s. and the r.h.s. of an equation (e.g. a stress-strain relation, $\vec{F}=m\vec{a}$, etc) must have a well-defined identity.

The r.h.s. of Einstein's equation, $T_{\mu\nu}$, is well-defined geometrically and physically.

However, this is not the case for the l.h.s.

II. In 1928 Cartan, and in 1963, 1972, 1990 Misner and Wheeler filled that cognitive gap by restating Einstein's field equation in geometrical form, both for the l.h.s. and the r.h.s.

$$\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R} = \frac{8\pi G}{c^4} \underbrace{T^{\mu\nu}}$$

$$\text{l.h.s} = \overset{\text{sum of}}{\underline{\text{moments}}}\text{ of} \left\{\begin{matrix}\text{curvature}\\ \text{induced}\\ \underline{\text{rotation}}\\ \text{for the 6}\\ \text{faces of}\\ \text{a small}\\ \text{3-cube}\end{matrix}\right\} = \text{r.h.s} = \frac{8\pi G}{c^4}\left\{\begin{matrix}\text{amount of}\\ \text{momenergy}\\ \text{inside this}\\ \text{3-cube}\end{matrix}\right\}$$

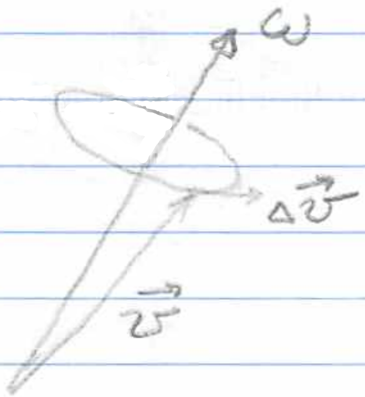
A prerequisite for understanding and using the Einstein field equations is that one grasp the meaning and the geometrical formulation of the concepts

(i) "rotation" and (ii) "moment".

III.) ROTATION AS A TENSOR

The Physical Origin of Rotation.

In three dimensions consider a vector $\vec{v}$ rotating with a given angular velocity around a given axis. The vectorial change $\Delta\vec{v}$ in this vector during time Δt is (recall Figure 4.2 of Lecture 4)



$$\Delta\vec{v} = \Delta t\, \vec{\omega}\times\vec{v}$$

$$= \Delta t \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega^1 & \omega^2 & \omega^3 \\ v^1 & v^2 & v^3 \end{vmatrix}$$

Such a vectorial determinant can be generalized to higher dimensions. But, as far as I know, it will not represent a rotation in that case.

This is because the essential (= most consequential) property of the rotation process is a plane in which the rotation takes place, not around a unique normal. This plane is spanned by a bivector which arises as follows:

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} = -\, e_\ell \wedge e_m \, R^{\ell m} \cdot e_k r^k = -\left(e_\ell \times \underset{\ell}{e_m} - e_m \otimes e_\ell\right) R^{\ell m} \cdot e_k r^k$$

$$= -\, e_\ell R^{\ell m} g_{mk} r^k + e_m R^{\ell m} g_{\ell k} r^k = -\, e_\ell R^{\ell m} r_m + e_m R^{\ell m} r_\ell$$

$$= -\, e_1 \ \omega^2 r_3 - e_2 \omega^3 r_1 - e_3 \omega^1 r_2 + r_1 \ \omega^2 e_3 + r_2 \omega^3 e_1 + r_3 \omega^1 e_2 = e_1(r_2\omega^3 - r_3\omega^2) + e_2(r_3\omega^1 - r_1\omega^3) + e_3 \ (r_1\omega^2 - r_2\omega^1)$$

$$\Delta\vec{v} = \Delta t \, \vec{\omega}\times\vec{v}$$

$$= \Delta t \left[e_1(\omega^2 v^3 - \omega^3 v^2) + e_2(\omega^3 v^1 - \omega^1 v^3) + e_3(\omega^1 v^2 - \omega^2 v^1) \right]$$

$$= -\Delta t \left[\omega^1(e_2\otimes e_3 - e_3\otimes e_2) + \omega^2(e_3\otimes e_1 - e_1\otimes e_3) + \omega^3(e_1\otimes e_2 - e_2\otimes e_1) \right]\cdot\vec{v}$$

$$= -\Delta t \left[\omega^1 e_2\wedge e_3 + \omega^2 e_3\wedge e_1 + \omega^3 e_3\wedge e_1 \right]\cdot\vec{v} \qquad (19.1)$$

This change mathematizes an infinitesimal rotation. It is the sum of three rotations in each of the planes spanned by the three pairs of basis vectors in the ambient Euclidean inner product space.

The bivectors $\{e_1\wedge e_2, e_2\wedge e_3, e_3\wedge e_1\}$ form a basis for a linear space. The ω's are the expansion coefficients for the linear combination, Eq. (19.1). Its coordinate (i.e. observer) independence becomes obvious when expressed as the trace of the product of the two antisymmetric matrices

$$\left[\mathcal{R}^{\ell m}{}_{(\Delta t)}\right] = \begin{bmatrix} 0 & -\omega^3\Delta t & \omega^2\Delta t \\ \omega^3\Delta t & 0 & -\omega^1\Delta t \\ -\omega^2\Delta t & \omega^1\Delta t & 0 \end{bmatrix} \text{ and } \left[E_{mk}\right] = \begin{bmatrix} 0 & -e_1\wedge e_2 & e_3\wedge e_1 \\ e_1\wedge e_2 & 0 & -e_2\wedge e_3 \\ -e_3\wedge e_1 & e_2\wedge e_3 & 0 \end{bmatrix}$$

The three diagonal elements of their product are

$$\mathcal{R}^{1m}E_{m1} = \mathcal{R}^{1m} e_1\wedge e_m = -\omega^3\Delta t e_1\wedge e_2 - \omega^2\Delta t e_3\wedge e_1\,,$$

$$\mathcal{R}^{2m}E_{m2} = \mathcal{R}^{2m} e_2\wedge e_m = -\omega^3\Delta t\, e_1\wedge e_2 - \omega^1\Delta t\, e_2\wedge e_3\,, \text{ and}$$

$$\mathcal{R}^{3m}E_{m3} = \mathcal{R}^{3m} e_3\wedge e_m = -\omega^2\Delta t\, e_3\wedge e_1 - \omega^1\Delta t\, e_2\wedge e_3\,.$$

The trace of the product of $\mathbf{R}$ and E is

$$\operatorname{tr} \mathbf{R}E = \mathbf{R}^{\ell m} e_\ell\wedge e_m = -2\Delta t\left[\omega^1 e_2\wedge e_3 + \omega^2 e_3\wedge e_1 + \omega^3 e_1\wedge e_2\right]$$

The linear combination

$$\mathcal{R}^{\ell m} e_\ell\wedge e_m/2! = \mathbf{R}^{|\ell m|} e_\ell\wedge e_m \qquad (19.2)$$

is called a *rotation*. It is a superposition of

$$\left.\begin{array}{l} \text{a rotation by an angle } \omega^1\Delta t \text{ in the } e_2\text{-}e_3 \text{ plane,} \\ \text{a rotation by an angle } \omega^2\Delta t \text{ in the } e_3\text{-}e_1 \text{ plane,} \\ \text{and a rotation by an angle } \omega^3\Delta t \text{ in the } e_1\text{-}e_2 \text{ plane.} \end{array}\right\} (19.3)$$

It is a tensor of rank $\binom{2}{0}$.

It lends itself to being generalized to dimen- four and greater.

One has therefore

$$\Delta\vec{v} = \frac{1}{2!} R^{\ell m}\, \vec{e}_\ell \wedge \vec{e}_m \cdot \vec{v}$$ (Einstein Summation Convention)

$$\Delta\vec{v} = \quad R^{|\ell m|}\, \vec{e}_\ell \wedge \vec{e}_m \cdot \vec{v}$$ (Summation restricted to $\ell < m$)

By omitting reference to any particular vector $\vec{v}$ one arrives at the concept of rotation as a tensor of rank $\binom{2}{0}$.

Thus one has the following

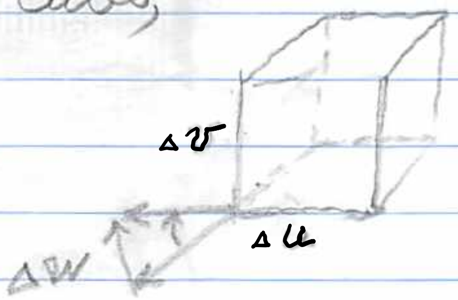
Definition ("rotation")

a) A rotation is a second rank antisymmetric tensor

$$\frac{1}{2!} R^{\ell m}(\Delta t)\, e_\ell \wedge e_m = e_\ell \wedge e_m R^{|\ell m|}(\Delta t) = \text{"rotation"}$$

IV) Curvature as Rotation

The concept of <u>rotation</u> defined this way generalizes to four (and higher) dimensions of spaces with an inner product (i.e, metric structure). Indeed, applying it to the curvature-induced rotational change associated with the Δu-Δv spanned face of a cube,



one has

$$\Delta W = e_\ell \, w^k \, R^\ell{}_{k|\alpha\beta|} \, dx^\alpha \wedge dx^\beta (\Delta u, \Delta v)$$

$$= e_\ell \, w^k \, g_{km} \, R^{\ell m}{}_{|\alpha\beta|} \, dx^\alpha \wedge dx^\beta (\Delta u, \Delta v)$$

$$= e_\ell \, \underbrace{w^k \, e_k} \cdot e_m \, \underset{\sim}{R}^{\ell m} (\Delta u, \Delta v)$$

$$= e_\ell \otimes e_m \cdot W \, \underset{\sim}{R}^{\ell m} (\Delta u, \Delta v)$$

Taking advantage of the curvature's metric-induced antisymmetry,

$R^{\ell m}{}_{\alpha\beta} = -R^{m\ell}{}_{\alpha\beta}$, on has

$$\Delta \mathbf{W} = \frac{1}{2}(e_\ell \otimes e_m - e_m \otimes e_\ell) \cdot \mathbf{W} \underset{\sim}{R}^{\ell m}(\Delta u, \Delta v)$$

$$= e_\ell \wedge e_m \cdot \mathbf{W} \underset{\sim}{R}^{|\ell m|}(\Delta u, \Delta v)$$

Comparing this with the rotation defined on page 19-11, one arrives at

$$\boxed{e_\ell \wedge e_m \underset{\sim}{R}^{|\ell m|}(\Delta u, \Delta v) = \text{"rotation"}}$$

which is induced by the curvature in the area subtended by the vectors u and v. This "rotation" is a $\binom{2}{0}$ tensor. For infinitesimal vectors Δu and Δv its components $\underset{\sim}{R}^{\ell m}(\Delta u, \Delta v)$ are the angles by which vector such as $\mathbf{w}$ get

... rotated in the plane spanned by e_l and e_m.

Nota bene: In the context of spacetime, the rotation can refer to Euclidean rotation, Lorentzian rotation or any of their combinations.

Lecture 20

Curvature-induced Rotation on the Faces of aThree-dimensional Cube

I. Rotation

II. Curvature as a rotation applied to a 3-cube

In MTW Peruse §15.2, Fig 15.1

I. Anglar velocity - induced rotation

Rotation is the directional change enjoyed by a vectorial entity when it is moved. A rotation is quantified in terms of an angle subtended in a plane spanned by the direction of the entity before and after its change.

In three dimensions this mathematization is achieved in terms of the bivector-valued 1-form

$$\overset{\leftrightarrow}{\underset{\sim}{R}} = \frac{e_i \wedge e_j \, \epsilon^{ij}{}_k \, dx^k}{2!} . \qquad (20.1)$$

It is a tensor of rank $\binom{2}{1}$, and, as such, it is a geometrical object, whose master virtue is that it is an invariant independent of one's chosen basis.

It has three unique bivectorial components

$$\frac{e_i \wedge e_j \; \epsilon^{ij}{}_1}{2!}, \quad \frac{e_i \wedge e_j \; \epsilon^{ij}{}_2}{2!}, \text{ and } \frac{e_i \wedge e_j \; \epsilon^{ij}{}_3}{2!} .$$

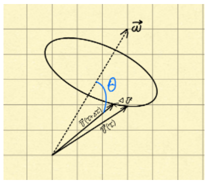
Each one, such as $e_1 \wedge e_2$, is invariant under a rotational transformation $(e_1, e_2) \rightsquigarrow (e_1', e_2')$ in its spanning plane:

$$e_1 \wedge e_2 = e_1' \wedge e_2'$$

Rotation $\underset{\sim}{\overleftrightarrow{R}}$, Eq.(20.1), has its grounding in reality by the fact that it is induced by the angular velocity $\vec{\omega}$. This means that

$$\underset{\sim}{\overleftrightarrow{R}}(\Delta t\vec{\omega}) = \Delta t \, \frac{e_i \wedge e_j}{2!} \, \epsilon^{ij}{}_k \, \omega^k$$

produces the rotational change $\Delta\vec{v}$ when it is applied to the vector $\vec{v}$



$$\Delta\vec{v} = \underset{\sim}{\overleftrightarrow{R}}(\Delta t\vec{\omega})\cdot\vec{v}$$

$$= \Delta t \, \frac{e_i \wedge e_j \cdot \vec{v}}{2!} \, \epsilon^{ij}{}_k \, \omega^k$$

A systematic calculation shows that the squared magnitude of $\Delta\vec{v}$ is

$$\Delta\vec{v}\cdot\Delta\vec{v} = \left(|\vec{v}|^2|\vec{\omega}|^2 - |\vec{v}\cdot\vec{\omega}|^2\right)(\Delta t)^2.$$

Consequently,

$$|\Delta\vec{v}| = |\vec{v}||\vec{\omega}| \sin\theta \, \Delta t.$$

This is the area of the parallelogram spanned by the vectors $\vec{v}$ and $\Delta t\vec{\omega}$.

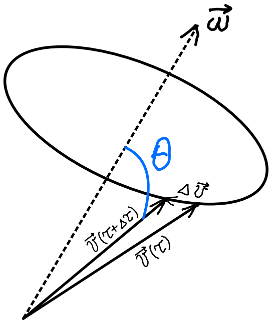


Figure 20.1 The length of $\Delta\vec{v}$ equals the area of the parallelogram spanned by $\vec{v}$ and $\Delta t\vec{\omega}$.

The effect of the rotation $\underset{\sim}{\overleftrightarrow{R}}(\Delta t\vec{\omega})$ on the vector $\vec{v} = v^j e_j$ is* $\boxed{20.3}$

$$\overleftrightarrow{R}\cdot\vec{v} = e_i \wedge e_j \, \epsilon^{ij}{}_k \, \Delta t \omega^k/2! \cdot \vec{v}$$

$$= (e_i v_j - e_j v_i)\epsilon^{ij}{}_k \, \Delta t \omega^k/2!$$

$$= (e_1 v_2 - e_2 v_1)\Delta t \omega^3 + (e_2 v_3 - e_3 v_2)\Delta t \omega^1 + (e_3 v_1 - e_1 v_3)\Delta t \omega^2$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ \Delta t\omega_1 & \Delta t\omega_2 & \Delta t\omega_3 \end{vmatrix}$$

$$= \vec{v}\times\vec{\omega}\Delta t = \Delta\vec{v}$$

*\footnote{ Q: What is $\underset{\sim}{\overleftrightarrow{R}}(\Delta t\vec{\omega})\cdot\vec{v}$ relative to a general non-orthonormal basis $\{e_\ell\}$?

A: With $\vec{v} = e_\ell v^\ell$

$$e_j \cdot e_\ell = g_{j\ell}$$

$\epsilon^{ij}{}_k \omega^k = \epsilon^{ijk}\omega_k$ one has

$$\underset{\sim}{\overleftrightarrow{R}}(\Delta t\vec{\omega})\cdot\vec{v} = \Delta t \frac{e_i \wedge e_j \cdot \vec{v}}{2!} \epsilon^{ij}{}_k \omega^k$$

$$= \Delta t \frac{(e_i g_{j\ell} - e_j g_{i\ell})v^\ell}{2!} \epsilon^{ijk}\omega_k$$

$$= \Delta t \; e_i v_j \omega_k \frac{[ijk]}{\sqrt{g}}$$

$$= \frac{\Delta t}{\sqrt{g}} \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

Let $\{e^*_i\}$ be the basis reciprocal to $\{e_j\}$, i.e.

$$e^*_i \cdot e_j = \delta_{ij} \; ; \quad e^*_i = g^{ik} e_k \, ; \; e_\ell = \sum_{i=1}^{3} g_{\ell i} e^*_i$$

Then

$$\vec{v} = e_\ell v^\ell$$

$$= \sum_i e^*_i g_{i\ell} v^\ell$$

$$= \sum_i e^*_i v_i$$

Thus, $\{v_i\}$ are the component of $\vec{v}$ relative to the reciprol basis $\{e^*_i\}$. These are the components v_i and w_k that go into the entries of the cross product of $\vec{v}$ and $\vec{w}$ relative to a generic non-orthogonal basis

$$\vec{v} \times \vec{w} = \underset{\sim}{\overleftrightarrow{R}}(\vec{w}) \cdot \vec{v} = \frac{1}{\sqrt{g}} \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$= e_i v_j \omega_k \epsilon^{ijk}$$

$$= e_i v_j \omega_k g^{ia} g^{jb} g^{kc} \epsilon_{abc}$$

$$= e_i g^{ia} v^b v^c \epsilon_{abc}$$

$$= \sqrt{g} \begin{vmatrix} e_1^* & e_2^* & e_3^* \\ v^1 & v^2 & v^3 \\ \omega^1 & \omega^2 & \omega^3 \end{vmatrix}$$

\end of footnote }

Page 5 does not exist

II) CURVATURE-INDUCED ROTATION.

The concept of rotation defined by

$$\overleftrightarrow{R} = \frac{\vec{e}_i \wedge \vec{e}_j \, R^{ij}}{2!}$$

generalizes to four (and higher) dimension of spaces with an inner product (i.e. metric) structure. Indeed, recall the curvature-induced rotational change associated with a u-v spanned face of a cube:

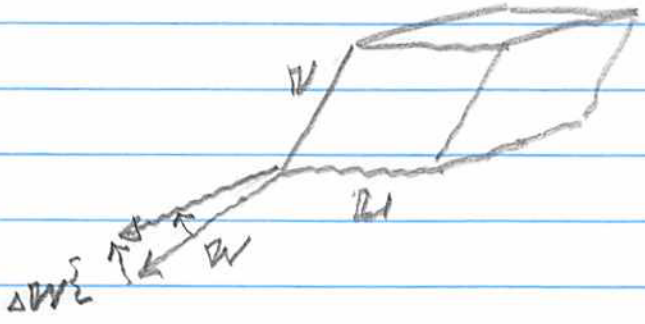


Figure 20.2

One has

$$\overleftrightarrow{\underset{\sim}{R}}(u,v):\ V \longrightarrow V$$

$$W = e_k w^k \rightsquigarrow \Delta W = e_\ell\, R^\ell{}_{k\alpha\beta}\, \frac{dx^\alpha \wedge dx^\beta (u,v)}{2!}\, w^k =$$

$$= e_\ell\, R^{\ell m}{}_{\alpha\beta}\, g_{mk}\, \frac{dx^\alpha \wedge dx^\beta (u,v)}{2!}\, w^k$$

$$= e_\ell \otimes e_m \cdot e_k\, w^k\, R^{\ell m}{}_{\alpha\beta}\, dx^\alpha \wedge dx^\beta (u,v)$$

Taking advantage of metric-induced antisymmetry $R^{\ell m}{}_{\alpha\beta} = -R^{m\ell}{}_{\alpha\beta}$, one one has

$$W \rightsquigarrow \frac{e_\ell \wedge e_m}{2!}\, \underbrace{R^{\ell m}{}_{\alpha\beta}\, \frac{dx^\alpha \wedge dx^\beta (u,v)}{2!}}\cdot W$$

$$\equiv \frac{e_\ell \wedge e_m}{2!}\, \overbrace{\underset{\sim}{R}^{\ell m}(u,v)}\cdot W \equiv \overleftrightarrow{\underset{\sim}{R}}(u,v)\cdot W$$

Compare this with the bivector defined rotation, Eq. (20.1) on page 20.1, one arrives at $\overleftrightarrow{R}(u,v) = \frac{e_\ell \wedge e_m}{2!} R^{\ell m}{}_{\alpha\beta} \frac{u^\alpha \wedge v^\beta}{2!}$ = "rotation"

(20.2)

It is induced by the curvature in the area subtended by the vectors u and v. For infinitesimal vectors u and v, $\underset{\sim}{\overleftrightarrow{R}}(u,v)$'s components $\underset{\sim}{R}^{\ell m}(u,v)$ are the angles by which a vector such as w gets rotated in the (e_ℓ, e_m)-plane.

<u>Nota bene:</u> In the context of spacetime, rotation refers to Euclidean rotation, Lorentz rotation or any of their combinations.

For a 3-d cube permeated by curvature, each of its faces has a the attribute of a rotation proportional to

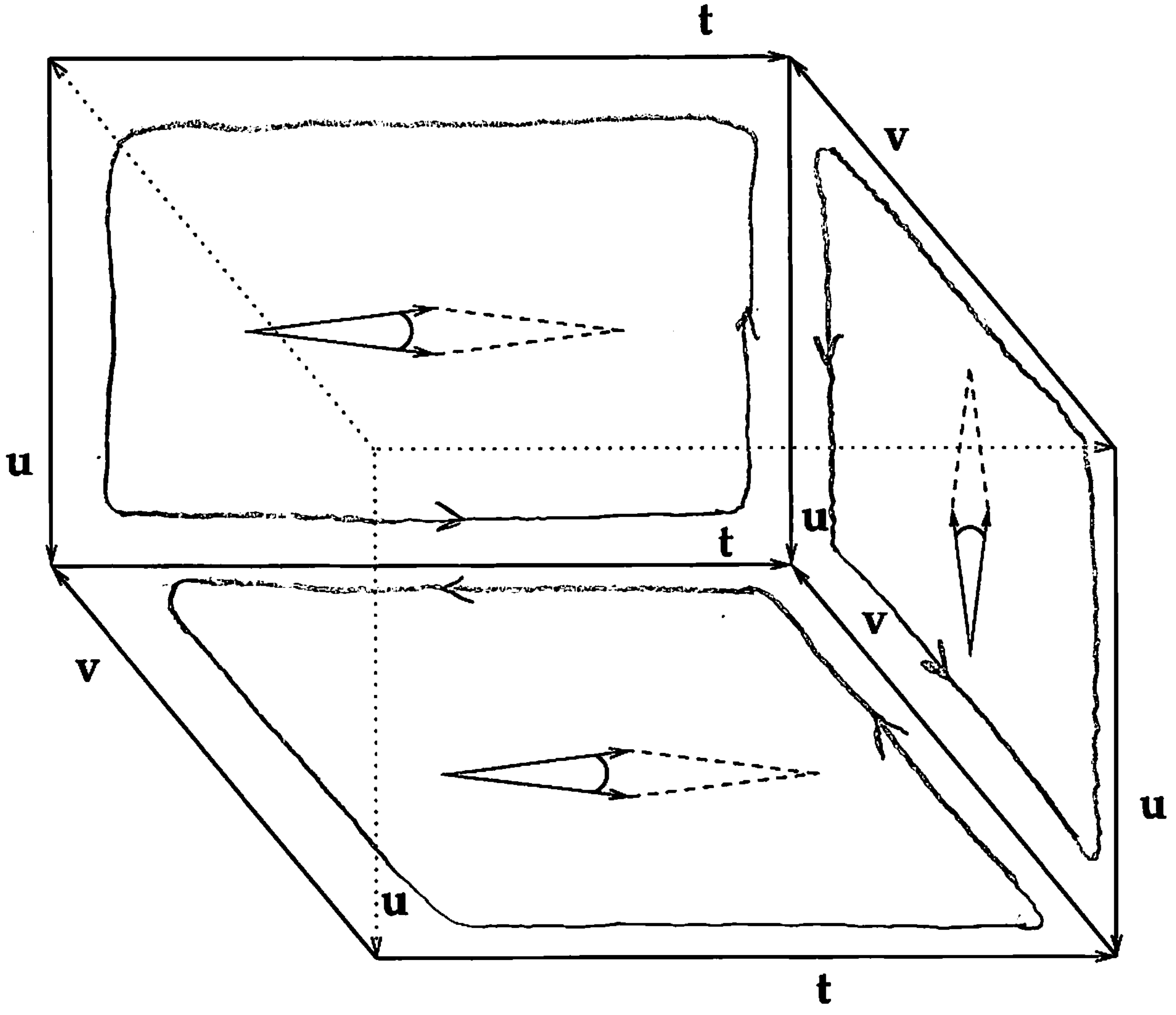


Figure 20.3 3-cube spanned by u, v, and t. Each of the cube's six faces carries a curvature-induced rotation such as Eq. (20.2) on p 20-8.

the area of the respective face.

I CURVATURE-INDUCED ROTATION APPLIED TO THE VECTOR FIELD w ON THE 6 FACES OF A 3-CUBE

The cause of gravitation is the existence of matter in any given 3-volume.

One of the conceptually most efficient ways of geometrizing gravitation is to mathematize the curvature-induced rotation on the face of a 3-d cube in 4-d space-time.

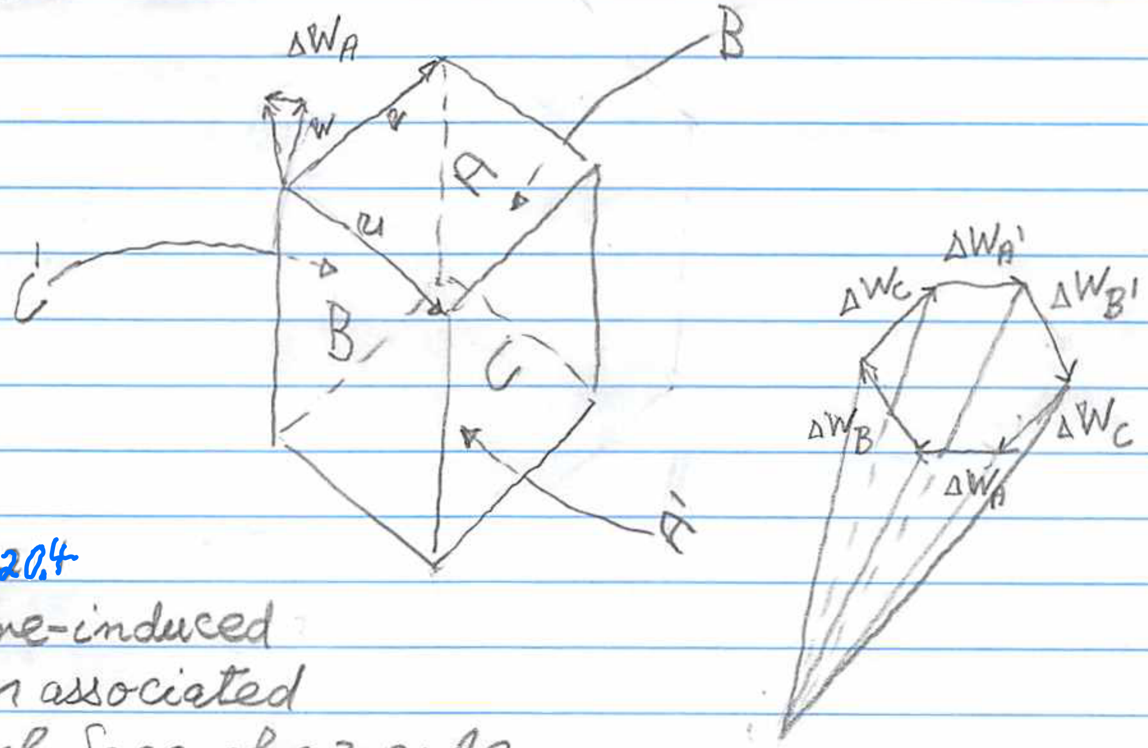


Figure 20.4
Curvature-induced rotation associated with each face of a 3-cube.

Transport the vector $\mathbf{w}$ parallel to itself around the closed square-shaped loop which bounds the face A. The result is the rotated vector $\mathbf{w}+\Delta\mathbf{w}_A$. The amount of this curvature-induced rotation is

$$\Delta \mathbf{w}_A = \frac{\mathbf{e}_l \wedge \mathbf{e}_m \cdot \mathbf{w}}{2!} \underset{\sim}{R}{}^{lm}(\mathbf{u}, \mathbf{v}).$$

The sum total contribution from all six faces vanishes:

$$\Delta \mathbf{w}_A + \Delta \mathbf{w}_B + \Delta \mathbf{w}_C + \Delta \mathbf{w}_{A'} + \Delta \mathbf{w}_{B'} + \Delta \mathbf{w}_{C'} = \mathbf{0}.$$

This is because in getting parallel transported around each of the faces $\mathbf{w}$ gets moved along each edge of two abutting edges twice, but in opposite directions. The result, as depicted in Figure 20.4, is that the sum total is zero.

LECTURE 21

The Bianchi Identities

In MTW peruse §15.1 ("Bianchi Identities in Brief")

I. CURVATURE-INDUCED ROTATION FOR THE SURFACE OF A 3-CUBE

To acquire an understanding of the Einstein field equations (EFE) it is not enough to have a knowledge of curvature. One also needs an understanding of it. Rotation, in particular curvature-induced rotation, is a step into this direction. **In four dimensional spacetime** consider a small 3-cube permeated by curvature.

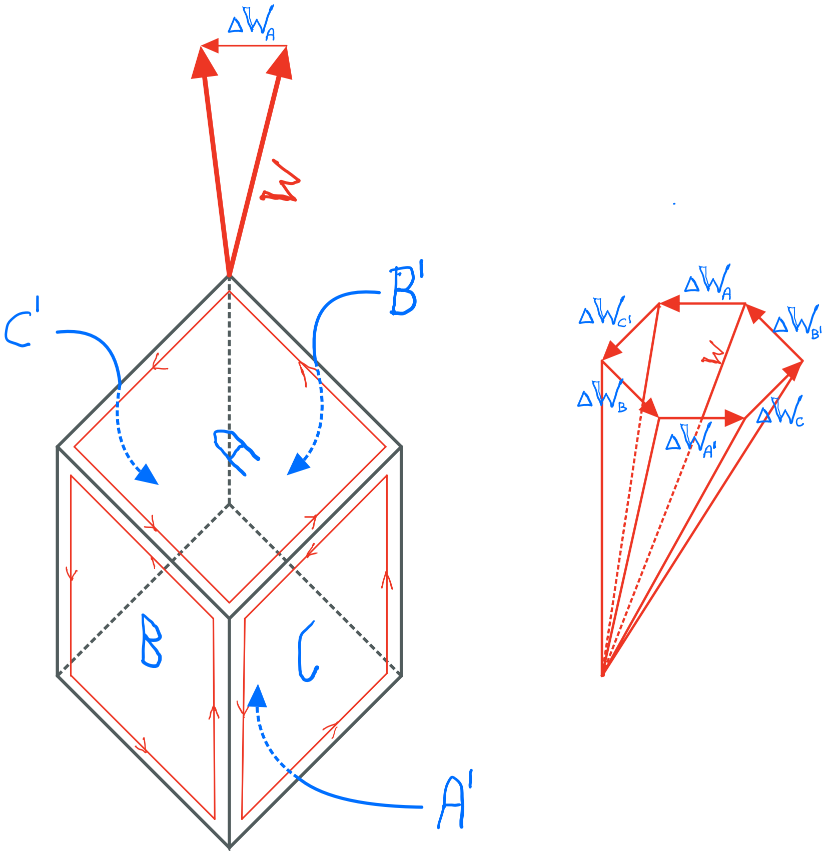


Figure 21.1 Curvature-induced rotation on each of the six faces of 3-cube.

Transport the vector $\mathbf{w}$ parallel to itself around the closed square-shaped loop which bounds the face A. The result is the rotated vector $\mathbf{w}+\Delta\mathbf{w}_A$. The amount of this curvature-induced rotation is

$$\Delta \mathbf{w}_A = \frac{\mathbf{e}_\ell \wedge \mathbf{e}_m \cdot \mathbf{w}}{2!} \underset{\sim}{R}^{\ell m}(\mathbf{u}, \mathbf{v}).$$

The sum total contribution from all six faces vanishes:

$$\Delta\mathbf{w}_A + \Delta\mathbf{w}_B + \Delta\mathbf{w}_C + \Delta\mathbf{w}_{A'} + \Delta\mathbf{w}_{B'} + \Delta\mathbf{w}_{C'} = \mathbf{0}.$$

This is because in getting parallel transported around each of the faces $\mathbf{w}$ gets moved along each edge of two abutting edges twice, but in opposite directions. The result, as shown in Figure 21.1 on page 21.1, is that sum total is zero:*

$$\Delta W_A + \Delta W_{B'} + \Delta W_{C'} + \Delta W_{A'} + \Delta W_B + \Delta W_C = 0 \qquad (21.1)$$

---

*\footnote{Note that the cause of the vanishing of this six term sum is not that terms with opposite sign such as ΔW_A and $\Delta W_{A'}$ cancel. The cause is the fact that the parallel transport of W occurs twice along each edge, but in opposite directions. That the sum is zero is due to the cancellation at each edge of abutting faces.}

To mathematize this geometrical fact consider a vector field W whose domain is on the surface of a 3-cube as well as its interior

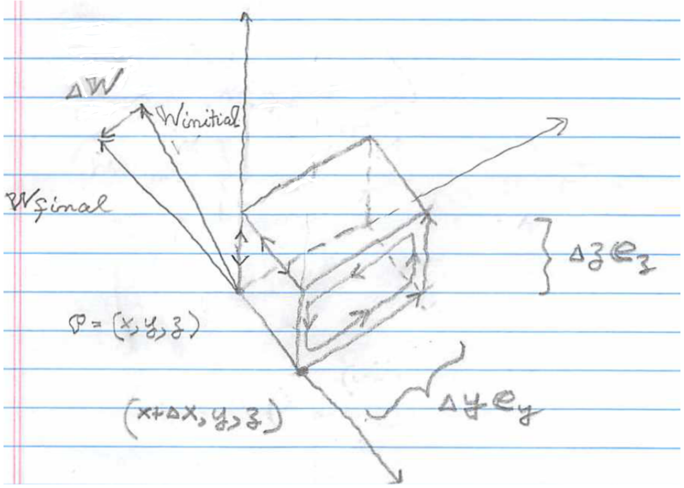


Figure 21.2 Comparing the rotations around the perimeters of two opposing faces of a 3-cube. Only one of them is depicted in the figure.

Parallel transport from a corner point along an edge to one of the 6 faces, around its boundary, and then back again to point P.

The contribution to the vectorial change from the face at x+Δx is

$$\Delta W = e_\ell w^m R^\ell{}_{myz}(x+\Delta x, y, z)\,\Delta y\,\Delta z$$

The opposite face gives a similar contribution. The combined contribution from the pair of faces is

$e_\ell w^m \frac{\partial}{\partial x}\left(R^\ell{}_{myz}\right) \Delta x \Delta y \Delta z$ ("front-back")

Here we have used a vector basis which is parallel ($\Gamma^\mu{}_{\alpha\beta}=0$, but $\partial_\gamma \Gamma^\mu{}_{\alpha\beta} \neq 0$ at the chosen corner P). Such a basis is induced by a "Riemann normal coordinate" system (See Section 11.6 and Exercise 11.9 in MTW).

Relative to such a coordinate system centered at the given point, all covariant derivatives become partial derivatives at this point. This is because the basis vectors are parallel (to 2nd order accuracy) in its neighborhood.

II.) BIANCHI IDENTITIES

Other pairs of faces of that cube give similar contributions. However, the contributions from common edges of abutting faces cancel. Consequently, one has

$$0 = e_\ell w^m \left(\underset{\text{front-back}}{R^\ell{}_{myz;x}} + \underset{\text{right-left}}{R^\ell{}_{mzx;y}} + \underset{\text{top-bottom}}{R^\ell{}_{mxy;z}} \right)$$

More generally (because of the basis independent mathematical frame work) one has

$$\boxed{0 = R^\ell{}_{mij;k} + R^\ell{}_{mjk;i} + R^\ell{}_{mki;j}} \qquad (21.2)$$

These are the "Bianchi identities".

Consider two nearby points on a curve passing through a medium permeated by a scalar f.

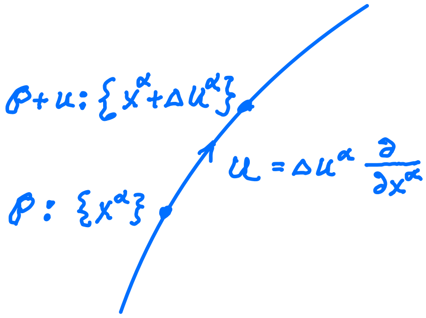


Figure 21.3 Two close-by points on a given curve.

Consider a scalar field f surrounding an infinitesimal curve segment $\overline{\mathcal{P}\,\mathcal{P}+\vec{u}} \equiv u$. The difference between $f(\mathcal{P}+u)$ and $f(\mathcal{P})$ is the line integral

$$\int_{\mathcal{P}}^{\mathcal{P}+u} df = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} \frac{\partial f}{\partial x^\beta}\, dx^\beta = \frac{\partial f}{\partial x^\alpha}\Big|_{\mathcal{P}} \Delta u^\alpha = \langle \frac{\partial f}{\partial x^\alpha}\, dx^\alpha \,|\, u \rangle\Big|_{\mathcal{P}} = \nabla_u f\Big|_{\mathcal{P}}$$

Using the covariant derivative, extend this line of reasoning to vector field $\mathbf{W}$

$$\int_{\mathcal{P}}^{\mathcal{P}+u} d\mathbf{W} = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} d\mathbf{W} = \langle d(\mathbf{e}_\beta W^\beta) \,|\, u \rangle \equiv \nabla_u \mathbf{W}$$

Thus, the line integral of a vectorial 1-form over an infinitesimal curve segment is the covariant derivative of that vectorial one form.

Lecture 22

Curvature as rotation permeating a contour-enclosed area via Stokes and Cartan

In MTW read BOX 4.4: "from definite integrals to integrands"

I VECTOR-VALUED 1-FORM

Consider two nearby points on a curve passing through a medium permeated by a scalar f.

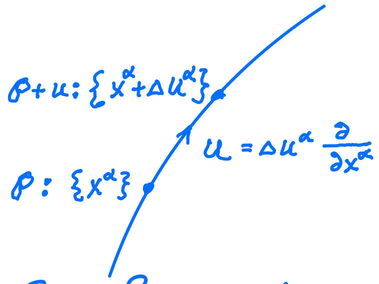


Figure 22.1 Two close-by points on a given curve.

Consider a scalar field f surrounding an infinitesimal curve segment $\overrightarrow{\mathcal{P}(\mathcal{P}+u)} \equiv u$. The difference between $f(\mathcal{P}+u)$ and $f(\mathcal{P})$ is the scalar-valued line integral

$$\int_{\mathcal{P}}^{\mathcal{P}+u} df = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} \frac{\partial f}{\partial x^\beta}\, dx^\beta = \frac{\partial f}{\partial x^\alpha}\bigg|_{\mathcal{P}} \Delta u^\alpha = \langle \frac{\partial f}{\partial x^\alpha}\, dx^\alpha \,|\, u \rangle\Big|_{\mathcal{P}} = \nabla_u f\Big|_{\mathcal{P}}$$

Using the covariant derivative, extend this line of reasoning to vector field $\mathbf{w}$

$$\int_{\mathcal{P}}^{\mathcal{P}+u} d\mathbf{w} = \int_{x^\alpha}^{x^\alpha+\Delta u^\alpha} d\mathbf{w} = \langle d(\mathbf{e}_\beta w^\beta) | u \rangle \equiv \nabla_u \mathbf{w}$$

Thus, the vector-valued line integral of a vectorial 1-form over an infinitesimal curve segment is the covariant derivative of that vectorial one form.

Consider such a line integral of the vectorial 1-form around the closed loop spanned by $u = \Delta u^{\alpha} \frac{\partial}{\partial x^{\alpha}}$ and $v = \Delta v^{\beta} \frac{\partial}{\partial x^{\beta}}$. The evaluation of this loop integral is a 5-step process.

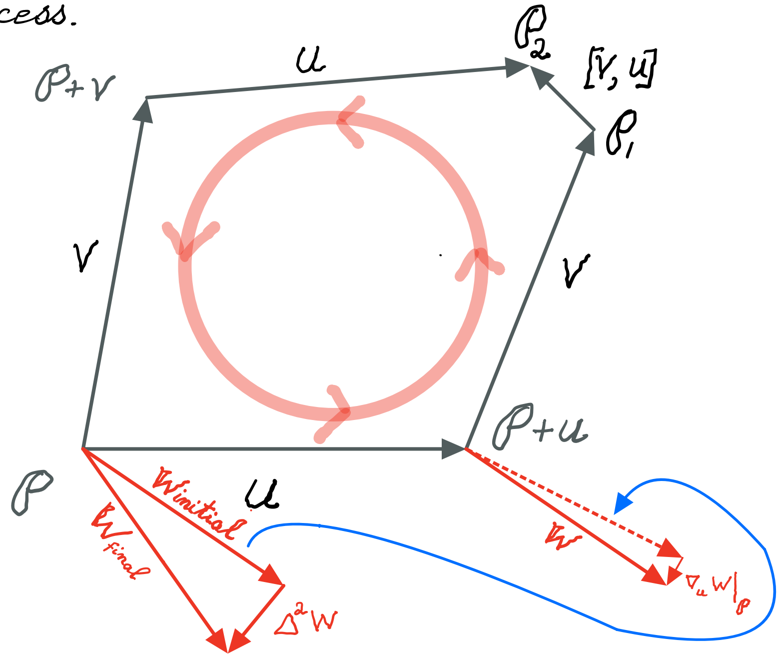


Figure 22.2 The vector-valued loop integral $\oint dW$ is a sum of line integrals

① $$\int_{\mathcal{P}}^{\mathcal{P}+u} dW + \int_{\mathcal{P}+u}^{\mathcal{P}_1} dW + \int_{\mathcal{P}_1}^{\mathcal{P}_2} dW + \int_{\mathcal{P}_2}^{\mathcal{P}+V} dW + \int_{\mathcal{P}+V}^{\mathcal{P}} dW \equiv \oint dW = \Delta^2 W$$

$$\nabla_u W\Big|_{\mathcal{P}} + \nabla_V W\Big|_{\mathcal{P}+u} + \nabla_{[V,u]} W - \nabla_u W\Big|_{\mathcal{P}+V} - \nabla_V W\Big|_{\mathcal{P}} \equiv \oint dW = \Delta^2 W$$

② Combining terms from opposite sides yields

$$\nabla_u \nabla_V W - \nabla_V \nabla_u W - \nabla_{[u,V]} W = \Delta^2 W \qquad (22.1)$$

The l.h.s. is a vector-valued line integral around the boundary of the closed polygon.

③ For notational economy let

$$\underset{\sim}{\vec{\Omega}} \equiv dW = d(e_\alpha w^\alpha)$$

Evaluating this vector-valued covector on the vector u, one obtains

$$\underset{\sim}{\vec{\Omega}}(v) = \langle dW | v \rangle = \nabla_v W. \qquad (22.2)$$

This is merely the extension of the directional derivative of the scalar f, $\langle df | v \rangle = \nabla_v f$, to that of a vector.

④ Introduce Eq. (22.2) into (22.1) and obtain

$$\nabla_u \underset{\sim}{\vec{\Omega}}(v) - \nabla_v \underset{\sim}{\vec{\Omega}}(u) - \underset{\sim}{\vec{\Omega}}([u,v]) = \overset{2}{\Delta} W$$

Apply the 1-2 version of Stokes' theorem,*

$$\boxed{\nabla_u \underset{\sim}{\vec{\Omega}}(v) - \nabla_v \underset{\sim}{\vec{\Omega}}(u) - \underset{\sim}{\vec{\Omega}}([u,v]) = d\underset{\sim}{\vec{\Omega}}(u,v)} \qquad (22.3)$$

,

and obtain $\overset{2}{\Delta} W = d\vec{\Omega}(u,v)$

* \footnote{ Eq. (22.3) is a statement of the vectorial 1-2 Stokes' theorem in its infinitesimal frame invariant form. The scalar 1-2 Stokes' theorem has the same form: Let $\underset{\sim}{\Omega} = g\, df$ be a general scalar valued 1-form. Then the scalar-valued Stokes' theorem is

$$\nabla_u \underset{\sim}{\Omega}(v) - \nabla_v \underset{\sim}{\Omega}(u) - \underset{\sim}{\Omega}([u,v]) = d\underset{\sim}{\Omega}(u,v). \qquad (22.4)$$ }

Problem 22.1 Let $\underset{\sim}{\Omega} = g\,df$ be a scalar-valued 1-form. Show that Eq. (22.4) is valid.

Problem 22.1 Let $\underset{\sim}{\vec{\Omega}} = \vec{\xi}\, df$ be a vector-valued 1-form. Show that Eq.(22.3) is valid. $\boxed{22.4}$

---

⑤ Calculate the exterior derivative $d\underset{\sim}{\vec{\Omega}}$:

$$
\begin{aligned}
d\underset{\sim}{\vec{\Omega}} &= d\,dW = d\,d(e_\alpha w^\alpha) \\
&= d[w^\alpha de_\alpha] + d[e_\alpha dw^\alpha] \\
&= dw^\alpha \wedge de_\alpha + w^\alpha d\,de_\alpha + de_\alpha \wedge dw^\alpha + e_\alpha \underbrace{d\,dw^\alpha}_{\text{zero}} \quad (22.5)
\end{aligned}
$$

$\longleftarrow$ cancel $\longrightarrow$

$$
\begin{aligned}
&= w^\alpha d\,(e_\beta \omega^\beta{}_\alpha) \\
&= w^\alpha [e_\beta\, d\omega^\beta{}_\alpha + e_\gamma \omega^\gamma{}_\beta \wedge \omega^\beta{}_\alpha] \\
d\underset{\sim}{\vec{\Omega}} &= e_\beta (d\omega^\beta{}_\alpha + \omega^\beta{}_\gamma \wedge \omega^\gamma{}_\alpha)\, w^\alpha
\end{aligned}
$$

This is Cartan's 2nd structure equation for his curvature 2-forms

$$\underset{\sim}{\Omega}{}^\beta{}_\alpha = d\omega^\beta{}_\alpha + \omega^\beta{}_\gamma \wedge \omega^\gamma{}_\alpha$$

In terms of the Riemann curvature components his equation is

$$d\underset{\sim}{\vec{\Omega}} = e_\beta (R^\beta{}_{\alpha|\mu\nu|}\, dx^\mu \wedge dx^\nu)\, w^\alpha$$

Conclusion:

$$\boxed{\Delta^2 W \equiv \oint dW = d\underset{\sim}{\vec{\Omega}}(u,v) = e_\beta R^\beta{}_{\alpha\mu\nu} \Delta u^\mu \Delta v^\nu w^\alpha} \qquad (22.6)$$

Comments:

1.) Inspite of the onslaught of derivatives on w in mathematizing $\Delta^2 W$, Eq. (22.1), the vector came through unscathed. On the contrary, the vector enjoys simply a rotation. There was no suffering from first and second order derivatives. They all cancelled out with Eq. (22.5) in the 5th step.

2.) Moreover, in the presence of a metric the motion is attributed to the vector by means of a bivector, the rotation. Indeed, the change, Eq. (22.6), enjoyed by w is

$$\Delta^2 W = e_\beta R^\beta{}_{\alpha\mu\nu}\, \Delta u^\mu \Delta v^\nu\, w^\alpha$$

$$= e_\beta\, g_{\gamma\alpha} R^{\beta\gamma}{}_{|\mu\nu|}\, dx^\mu \wedge dx^\nu (u,v)\, w^\alpha.$$

Because of the metric-induced antisymmetry

$R^{\beta\gamma}{}_{\mu\nu} = -R^{\gamma\beta}{}_{\mu\nu}$, $\Delta^2 W$ separates into two factors:

$$\Delta^2 W = \underbrace{e_\beta \wedge e_\gamma R^{|\beta\gamma|}{}_{|\mu\nu|}\, dx^\mu \wedge dx^\nu (u,v)}_{\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u,v)} \cdot \underbrace{e_\alpha w^\alpha}_{W}.$$

This separation highlights the bivectorial rotation $\overset{\leftrightarrow}{\underset{\sim}{R}}(u,v)$, which acts on the vector W:

$$\boxed{\Delta^2 W \equiv \oint dW = d\underset{\sim}{\vec{\Omega}}(u,v) = \overset{\leftrightarrow}{\underset{\sim}{R}}(u,v)\cdot W} \qquad (22.7)$$

3.) The bivector

$$\overset{\leftrightarrow}{\underset{\sim}{R}}(u,v) = \frac{e_\beta \wedge e_\alpha}{2!} R^{\beta\alpha}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(u,v) \qquad (22.8)$$

is the <u>rotation</u>. It is understood to be applied to a vector W by taking the inner product:

$$\overset{\leftrightarrow}{\underset{\sim}{R}}(u,v)\cdot W = \Delta^2 W.$$

This changes W by the amount $\Delta^2 W$. By leaving the vector W unspecified, this change becomes the concept

$$\overset{\leftrightarrow}{\underset{\sim}{R}}(u,v) = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \Delta u^\mu \Delta v^\nu,$$

which is Eq.(22.8), the rotation(al change) of an as-yet-unspecified vector.

4.) The rotation is a superposition of simple rotations, each taking place in the plane spanned by e_α and e_β. The angle of rotation in this plane is

$$R^{\alpha\beta}{}_{\mu\nu}\,\Delta u^\mu \Delta v^\nu = R^{\alpha\beta}{}_{\mu\nu}\, dx^\mu\wedge dx^\nu/2!\,(u,v).$$

The size of this angle is proportional to (i) the area spanned by the vectors u and v (ii) the magnitude of the curvature $R^{\alpha\beta}{}_{\mu\nu}$ permeating this area.

5.) By omitting explicit reference to these spanning vectors one arrives at the rank $\binom{2}{2}$ tensor

$$\underset{\sim}{\overleftrightarrow{R}} = \frac{e_\alpha\wedge e_\beta}{2!}\, R^{\alpha\beta}{}_{\mu\nu}\,\frac{dx^\mu\wedge dx^\nu}{2!},$$

the curvature-induced rotation due to an as-yet-unspecified area and of an as-yet-unspeciefied -unspecified vector.

Lecture 23-24

Rotation On A 3-cube
Immersed in A Curvature Field

Suggested reading: Chap. 2.8
In "Gravitation and Inertia"
by
I. Ciufolini & J.A. Wheeler

I. Rotational Change Induced by Curvature

23,1

Consider a 3-dimensional domain $\mathcal{D}$ in 4-dimensional spacetime:

$x^\alpha(u,v,t)$; $\alpha = 0,1,2,3$; $a \le u \le b$; $c \le v \le d$; $e \le t \le f$.

At each point-event $\mathcal{P} = \{x^\alpha(u,v,t)\} \in \mathcal{D}$ there are three vectors tangent to $\mathcal{D}$:

$$\begin{aligned} \mathbf{u} &= \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} = u^\alpha e_\alpha \\ \mathbf{v} &= \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta} = v^\beta e_\beta \\ \mathbf{t} &= \frac{\partial x^\gamma}{\partial t}\frac{\partial}{\partial x^\gamma} = t^\gamma e_\gamma \end{aligned} \qquad 23.1$$

Also at each such point consider a 3-d infinitesimal element of volume, a 3-cube spanned by

$$\left.\begin{aligned} \Delta u\, \mathbf{u} &= \Delta u \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} = \Delta u\, u^\alpha e_\alpha \\ \Delta v\, \mathbf{v} &= \Delta v \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta} = \Delta v\, v^\beta e_\beta \\ \Delta t\, \mathbf{t} &= \Delta t \frac{\partial x^\gamma}{\partial t}\frac{\partial}{\partial x^\gamma} = \Delta t\, t^\gamma e_\gamma \end{aligned}\right\} \qquad 23,2$$

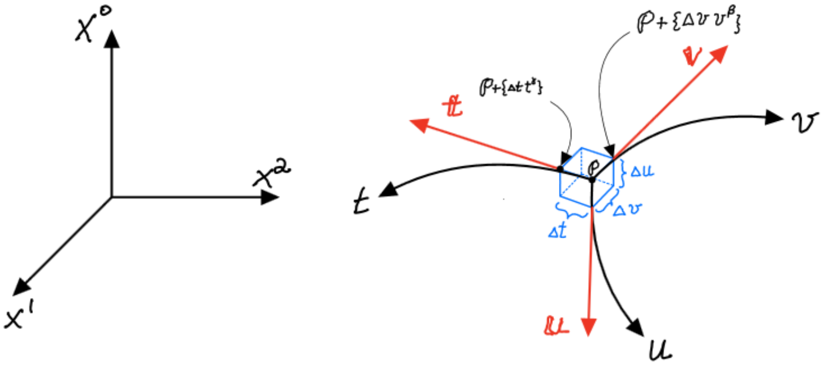
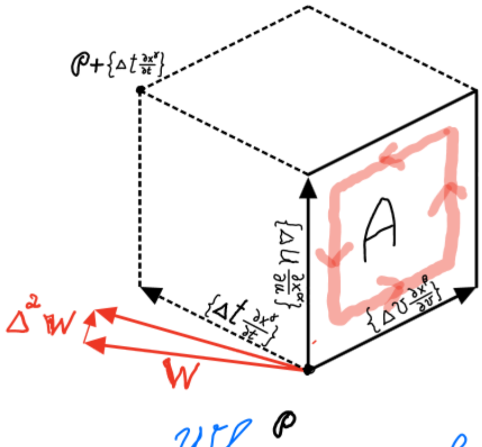


Figure 23.1 3-d cube in spacetime at point-event $\mathcal{P} = \{x^\alpha(u,v,t)\}$ where it is spanned by coordinate increments $\Delta u, \Delta v$, and Δt.

When such a 3-cube is permeated by a curvature field, each of its six faces features a rotation.

A vector W, upon being parallel transported around the boundary of each one of these faces, will be subjected to a rotational change. Each change is proportional to the respective area of each face, to the strength of curvature field, and to the size and direction of that vector.



$$\nabla_u(-)\nabla_v W + \nabla_v \nabla_u W + \nabla_{[u,v]} W$$

$$= -\{\nabla_u \nabla_v W - \nabla_v \nabla_u W - \nabla_{[u,v]} W\}$$

$$= -\underset{\sim}{\overset{\leftrightarrow}{R}}(u,v)$$

$$\oint dW = \nabla_u \Omega(v) - \nabla_v \Omega(u) - \nabla_{[u,v]} \Omega = d\Omega(u,v) \Rightarrow e_\alpha R^\alpha{}_{\sigma\mu\nu} u^\mu v^\nu W^\sigma$$

$$\nabla_u \nabla_v W - \nabla_v \nabla_u W$$

$$= e_\alpha g_{\tau\sigma} W^\sigma R^{\alpha\tau}{}_{\mu\nu} u^\mu v^\nu$$

$$= e_\alpha \otimes e_\tau \cdot e_\sigma W^\sigma R^{\alpha\tau}{}_{\mu\nu} dx^\mu \otimes dx^\nu (u,v)$$

$$= \frac{e_\alpha \wedge e_\tau}{2!} \cdot W R^{\alpha\tau}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} (u,v) = \underset{\sim}{\overset{\leftrightarrow}{R}}(u,v)$$

$$-\oint_A dW$$

$$= \oint_{\partial A} dW$$

Figure 23.2 When a 3-cube is permeated by a curvature field, each of its faces features a rotation. A typical face in the figure is spanned by the two vectors,

$$\Delta u\, \mathbf{u} = \Delta u \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha} \text{ and } \Delta v\, \mathbf{v} = \Delta v \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta},$$

both belonging to the vector space at P.

Rotation, which occurs at every face, is mathematized by the bivector-valued 2-form

$$\overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(\ ,\ ) = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}(\ ,\ ).$$

The specific rotation $\overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v})$, when applied to the vector $\mathbf{w}$, subjects it to the rotational change

$$\Delta^2 \mathbf{w} = \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v}) \cdot \mathbf{w}$$

$$= e_\alpha R^\alpha{}_{\sigma\mu\nu} u^\mu v^\nu w^\sigma \Delta u \Delta v$$

Being a feature of every face, rotation is mathematized by the bivector-valued 2-form,

$$\overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}} = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}. \qquad (23.3)$$

This is a tensor of rank $\binom{2}{2}$, a multilinear map. It resulted, we recall, from evaluating the line integral

$$\oint_{\partial A} d\mathbf{w} \equiv \oint_{\partial A} \vec{\Omega} \ ; \ \mathbf{w} = e_\sigma w^\sigma, \ \vec{\Omega} = d\mathbf{w}$$

around the boundary ∂A of a 2-d domain A spanned by the vectors $\Delta u\, \mathbf{u} = \Delta u \frac{\partial x^\alpha}{\partial u} \frac{\partial}{\partial x^\alpha}$ and $\Delta v\, \mathbf{v} = \Delta v \frac{\partial x^\beta}{\partial v} \frac{\partial}{\partial x^\beta}$.

Use the infinitesimal 1-2 version of Stokes' theorem*

$$\nabla_{\mathbf{u}} \vec{\Omega}(\mathbf{v}) - \nabla_{\mathbf{v}} \vec{\Omega}(\mathbf{u}) - \vec{\Omega}([\mathbf{u},\mathbf{v}]) = d\vec{\Omega}(\mathbf{u},\mathbf{v}), \qquad (23.4)$$

* \footnote{ (23.4)

The infinitesimal version of the vectorial 1-2 Stokes' theorem holds for any combination of vectorial 1-forms. Without loss of generality let $\vec{\Omega} = \vec{f}\, dq$. This leads to the integral formulation of the 1-2 Stokes' theorem,

$$\oint_{\partial A} \vec{\Omega} = \iint_{A} d\vec{\Omega},$$

which is Eq. (22.4) in Lecture 22.}

Problem: Given the vectorial 1-form $\vec{\Omega} = d\mathbf{W}$ and its vectorial value $\vec{\Omega}(\mathbf{v}) = \langle d\mathbf{W}, \mathbf{v}\rangle = \nabla_{\mathbf{v}} \mathbf{W}$ upon operating on $\mathbf{v}$, verify that Eq. (23.4) is a valid equation.

Referring to Figure 22.2 (Lecture 22) one has

$$\oint_{\partial A} d\mathbf{W} = \left[\nabla_{\mathbf{u}} \vec{\Omega}(\mathbf{v}) - \nabla_{\mathbf{v}} \vec{\Omega}(\mathbf{u}) - \vec{\Omega}([\mathbf{u},\mathbf{v}])\right] \Delta u \Delta v$$

$$= d\,d\mathbf{W}(\mathbf{u},\mathbf{v})\, \Delta u\, \Delta v$$

The r.h.s is the Riemann sum approximation consisting of only a single term for the double integral over A:

$$\oint_{\partial A} d\mathbf{W} = \iint_{A} d\,d\mathbf{W}(\mathbf{u},\mathbf{v})$$

Use Cartan's 2nd structure equation to calculate $d\vec{\Omega} \equiv d d\mathbf{W}$:

$$\oint_{\partial A} d\mathbf{W} = \iint_{A} \mathbf{e}_\alpha R^{\alpha}{}_{\sigma\mu\nu}\, u^\mu v^\nu w^\sigma \; du\, dv$$

$$= \iint_A \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu}\, u^\mu v^\nu \cdot W \; du\, dv \qquad \boxed{23.5}$$

$$\equiv \iint_A \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u, v) \cdot W \qquad (23.7)$$

This integral is over the infinitesimal 2-d domain A spanned by $(\Delta u\, u, \Delta v\, v)$.

II. Curvature-induced Rotational Change Due to the 6 Faces of a Cube

A 3-cube, which is spanned by three coordinate induced basis vectors as depicted in Figure 23.3, has 3 pairs of faces.

Q: What is the rotational change from all 3 pairs of opposing faces?

Ans.: In 4 Steps:

Step 1: From Eq. (22.6) (Lecture 22) recall that for faces A' and A the rotational change is

$$\Delta^2 W_{A'} + \Delta^2 W_A =$$

$$\oint_{\partial A'} dW + \oint_{\partial A} dW = \left\{ \iint_{A'} \underset{\sim}{\overleftrightarrow{\mathcal{R}}}\Big|_{\mathcal{P}+\Delta t\,\mathbf{t}} - \iint_A \underset{\sim}{\overleftrightarrow{\mathcal{R}}}\Big|_{\mathcal{P}} \right\} \cdot W \qquad (23.8)$$

$$= \left\{ \iiint \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u, v)\Big|_{\mathcal{P}+\Delta t\,\mathbf{t}} du\, dv - \iint \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(u, v)\Big|_{\mathcal{P}} du\, dv \right\} \cdot W$$

The integrals on the r.h.s. form the difference between two opposing area integrals over the pair of faces A′ and A which are separated by $\Delta t\, t$. This difference is the integral of $\nabla_t \left(\overleftrightarrow{\underset{\sim}{\mathcal{R}}}(u,v)\right)\Big|_{\mathcal{P}}$ over the volume between these two faces:

$$\left\{ \nabla_t \left(\overleftrightarrow{\underset{\sim}{\mathcal{R}}}(u,v)\right)\Big|_{\mathcal{P}} \Delta t\, \Delta u\, \Delta v \right\}\cdot W = \left\{ \int_0^{\Delta t}\int_0^{\Delta u}\int_0^{\Delta v} \nabla_t \overleftrightarrow{\underset{\sim}{\mathcal{R}}}\Big|_{\mathcal{P}} dt\, du\, dv \right\}\cdot W$$

The integrand is the mean value of the directional covariant derivative of the tensor field $\overleftrightarrow{\mathcal{R}}$ at $\mathcal{P}$ in this volume.

Step 2: Apply this mathematization to the other pairs of faces, B & B′ and C′ and C depicted in Figure 23.3 below:

$$\oint_{\partial A'} dW + \oint_{\partial A} dW + \oint_{\partial B} dW + \oint_{\partial B'} dW + \oint_{\partial C'} dW + \oint_{\partial C} dW =$$

$$= \left[\underbrace{\iint_{A'} - \iint_{A}}_{} + \underbrace{\iint_{B} - \iint_{B'}}_{} + \underbrace{\iint_{C'} - \iint_{C}}_{} \right] \overleftrightarrow{\underset{\sim}{\mathcal{R}}}\cdot W =$$

$$= \left[\nabla_t \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(u,v) + \nabla_u \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(v,t) + \nabla_v \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(t,u) \right]_{\mathcal{P}} \Delta t\, \Delta u\, \Delta v \cdot W \equiv \text{r.h.s.} \qquad (23,9)$$

Step 3: Taking into account that the basis vectors, Eq. (23.1), have vanishing commutators, apply the tensorial 2-3 version of Stokes' theorem, namely

$$\nabla_t \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(u,v) + \nabla_u \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(v,t) + \nabla_v \overleftrightarrow{\underset{\sim}{\mathcal{R}}}(t,u)$$
$$-\overleftrightarrow{\underset{\sim}{\mathcal{R}}}([u,v],t) - \overleftrightarrow{\underset{\sim}{\mathcal{R}}}([v,t],u) - \overleftrightarrow{\underset{\sim}{\mathcal{R}}}([t,u],v) = d\overleftrightarrow{\underset{\sim}{\mathcal{R}}}(t,u,v) \qquad (23.10),$$

to the r.h.s. of Eq. (23.9). 23.7

Problem: Given the tensorial 2-form $\overset{\leftrightarrow}{R} = A\, df \wedge dg$, verify that Eq.(23.10) is a valid equation.

As a result of this application the chain, Eqs. (23.9), of line and surface integrals terminates in a volume integral of the 3-cube which is spanned by the triad of vectors $(\Delta u\, \mathbf{u}, \Delta v\, \mathbf{v}, \Delta t\, \mathbf{t})$:

$$\oint_{\partial A'} dw + \oint_{\partial A} dw + \oint_{\partial B} dw + \oint_{\partial B'} dw + \oint_{\partial C'} dw + \oint_{\partial C} dw =$$

$$= \underbrace{\left[\underbrace{\iint_{A'} - \iint_{A}} + \underbrace{\iint_{B} - \iint_{B'}} + \underbrace{\iint_{C'} - \iint_{C}} \right]}_{\partial \mathcal{D}} \overset{\leftrightarrow}{\underset{\sim}{R}} \cdot w = d\overset{\leftrightarrow}{\underset{\sim}{R}}(\mathbf{t},\mathbf{u},\mathbf{v})\, \Delta t \Delta u \Delta v \cdot w$$

$$= \iiint_{\mathcal{D}} d\overset{\leftrightarrow}{\underset{\sim}{R}} \cdot w \qquad (23.11)$$

where $\mathcal{D}$ is the interior of the oriented 3-cube.

Step 4 Mathematize the two-dimensional boundary of a 3-cube and the one-dimensional boundary of that boundary and apply it Eq. (23.11).

a) Let $\mathcal{D} = \{ x^\alpha(\Delta u, \Delta v, \Delta t) \} \equiv \{ x^\alpha = C^\alpha(\lambda_1, \lambda_2, \lambda_3) : 0 \le \lambda_1, \lambda_2, \lambda_3 \le 1 \}$, or more economiclly,

let $\mathcal{D} = C(\lambda_1, \lambda_2, \lambda_3)$ be the interior of the 3-cube

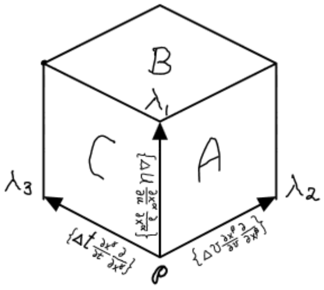


Figure 23.3 A 3-cube spanned by its oriented triad of tangent vectors ($\Delta u\,\mathbb{U} \equiv \Delta u \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha}$, $\Delta v\,\mathbb{V} \equiv \Delta v \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta}$, $\Delta t\,\mathbb{T} \equiv \Delta t \frac{\partial x^\gamma}{\partial t}\frac{\partial}{\partial x^\gamma}$).

b) Let $\partial\mathcal{D}$ be the oriented 2-d boundary of $\mathcal{D}$.

$\partial\mathcal{D}$ consists of the six oriented faces of $\mathcal{D}$, three of which are overtly depicted in Figure 23.3.

$$\left.\begin{array}{ll}
B = C^{1+} = C(1, \lambda_2, \lambda_3) & \text{upper} \\
B' = C^{1-} = C(0, \lambda_2, \lambda_3) & \text{lower} \\
C' = C^{2+} = C(\lambda_1, 1, \lambda_3) & \text{right (back)} \\
C = C^{2-} = C(\lambda_1, 0, \lambda_3) & \text{left (front)} \\
A' = C^{3+} = C(\lambda_1, \lambda_2, 1) & \text{left (back)} \\
A = C^{3-} = C(\lambda_1, \lambda_2, 0) & \text{right (front)}
\end{array}\right\} \qquad (23.12)$$

The oriented 2-d boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is

$$\partial\mathcal{D} = \sum_{j=1}^{3} (-1)^{j-1}\left(C^{j+} - C^{j-}\right) \qquad (23.13)$$

$$= \left(C^{1+} - C^{1-}\right) - \left(C^{2+} - C^{2-}\right) + \left(C^{3+} - C^{3-}\right)$$

$$= \left(B - B'\right) - \left(C' - C\right) + \left(A' - A\right)$$

$$\partial\mathcal{D} = \sum_{\ell=1}^{6} \left[\ell^{th}\,\text{Face}\right] = \left(A' - A\right) - \left(B' - B\right) - \left(C' - C\right)$$

c) The oriented 1-d boundary of $\partial\mathcal{D}$ is

$$\partial(\partial\mathcal{D}) = \partial\Big(\overbrace{C^{1+}(1,\lambda_2,\lambda_3)}^{B} - \overbrace{C^{1-}(0,\lambda_2,\lambda_3)}^{B'} - \overbrace{C^{2+}(\lambda_1,1,\lambda_3)}^{C'} + \underbrace{C^{2-}(\lambda_1,0,\lambda_3)}_{C} + \underbrace{C^{3+}(\lambda_1,\lambda_2,1)}_{A'} - \underbrace{C^{3-}(\lambda_1,\lambda_2,0)}_{A} \Big)$$

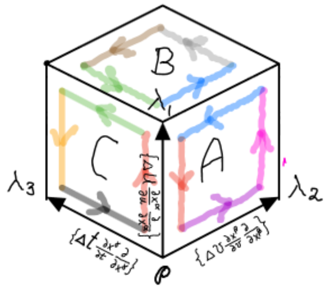


Figure 23.4 The 3-d interior $\mathcal{D}$ of the cube is mathematized by the functions $x^\alpha = C^\alpha(\lambda_1,\lambda_2,\lambda_3)$, $0\le\lambda_i\le 1$, or more economically by $x = C(\lambda_1,\lambda_2,\lambda_3)$. Its oriented boundary $\partial\mathcal{D}$ consists of six opposing faces and is mathematized by the formal sum, Eq.(23.13). Its unoriented components are exhibited in Eq.(23.12).

The boundary of $\partial\mathcal{D}$, namely, $\partial\partial\mathcal{D}$, has 24 components, but only half of them are depicted in the figure. They are directed line segments which have been put into algebraic form by the formal sum $\partial\partial\mathcal{D}$. This equals zero, $\partial\partial\mathcal{D}=0$, because for every line segment there is one which opposes it, and they cancel each other at their common edge.

$$= \overbrace{C^{1+2+}(1,1,\lambda_3) - C^{1+2-}(1,0,\lambda_3) - \left[C^{1+3+}(1,\lambda_2,1) - C^{1+3-}(1,\lambda_2,0)\right]}^{\partial B}$$

$$- \overbrace{\left\{C^{1-2+}(0,1,\lambda_3) - C^{1-2-}(0,0,\lambda_3) - \left[C^{1-3+}(0,\lambda_2,1) - C^{1-3-}(0,\lambda_2 0)\right]\right\}}^{\partial B'}$$

$$- \overbrace{\left\{C^{2+1+}(1,1,\lambda_3) - C^{2+1-}(0,1,\lambda_3) - \left[C^{2+3+}(\lambda_1,1,1) - C^{2+3-}(\lambda_1,1,0)\right]\right\}}^{\partial C'}$$

$$+ \overbrace{C^{2-1+}(1,0,\lambda_3) - C^{2-1-}(0,0,\lambda_3) - \left[C^{2-3+}(\lambda_1,0,1) - C^{2-3-}(\lambda_1,0,0)\right]}^{\partial C}$$

$$+ \overbrace{C^{3+1+}(1,\lambda_2 1) - C^{3+1-}(0,\lambda_2,1) - \left[C^{3+2+}(\lambda_1,1,1) - C^{3+2-}(\lambda_1,0,1)\right]}^{\partial A'}$$

$$- \overbrace{\left\{C^{3-1+}(1,\lambda_2,0) - C^{3-1-}(0,\lambda_2 0) - \left[C^{3-2+}(\lambda_1,1,0) - C^{3-2-}(\lambda_1,0,0)\right]\right\}}^{\partial A}$$

$= 0$ because each term refers to an edge parametrized by $0\le\lambda_i\le 1$.

Hence the conclusion:

$$\boxed{\text{The boundary of the boundary is zero: } \partial\partial\mathcal{D} = 0.}$$

The integration domains of the three definite integrals in Eqs (23.11) have respective components whose respective sums are $\partial\partial\mathcal{D}$, $\partial\mathcal{D}$, and $\mathcal{D}$. Consequently, in its essentials, the chain of Eqs. (23.11) reduces to

$$\int_{\partial\partial\mathcal{D}} d\mathbf{w} = \iint_{\partial\mathcal{D}} \underset{\sim}{\overleftrightarrow{\mathcal{R}}}\cdot\mathbf{w} = \iiint_{\mathcal{D}} d\,\underset{\sim}{\overleftrightarrow{\mathcal{R}}}\cdot\mathbf{w}$$

The fact that $\partial\partial\mathcal{D}=0$ implies

$$0 = \iiint_{\mathcal{D}} d\,\underset{\sim}{\overleftrightarrow{\mathcal{R}}}\cdot\mathbf{w} \equiv \iiint_{\mathcal{D}} d\left(\frac{e_\alpha\wedge e_\beta}{2!}\,R^{\alpha\beta}{}_{\mu\nu}\,\frac{dx^\mu\wedge dx^\nu}{2!}\right)\cdot\mathbf{W}$$

The fact that this equality holds for all 3-cubes $\mathcal{D}$ and all vectors $\mathbf{w}$, implies that

$$\begin{aligned} 0 &= d\,\underset{\sim}{\overleftrightarrow{\mathcal{R}}} \\ &= d\left(e_\alpha\wedge e_\beta\,R^{\alpha\beta}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu\right) \end{aligned}$$

According to this result the existence of a 3-d domain permeated by a curvature-induced rotation field puts two fundamental but related restrictions on it:

(1) The differential of the mathematized rotation field vanishes.

(2) Its components always satisfy the Bianchi identities.

The mathematical validation of these restrictions are consigned to the Appendix 23A.

23.10

Consider (i) the anti-symmetric tensor field

$$\underset{\sim}{F} = F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}/2!$$

permeating spacetime and (ii) a two-dimensional surface A parametrized by u and v, $x^{\alpha}(u,v)$.

The vectors tangent to A are

$$\mathbf{u} = \frac{\partial x^{\alpha}}{\partial u}\frac{\partial}{\partial x^{\alpha}} \equiv u^{\alpha} e_{\alpha}\ , \quad \mathbf{v} = \frac{\partial x^{\beta}}{\partial v}\frac{\partial}{\partial x^{\beta}} = v^{\beta} e_{\beta}$$

The surface integral of $\underset{\sim}{F}$ over A is mathematized by

$$\iint_A \underset{\sim}{F} \equiv \iint_A F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}/2! \qquad (23.3)$$

$$\equiv \iint_A \underset{\sim}{F}(\mathbf{u},\mathbf{v})\ du\, dv$$

$$\equiv \iint_A F_{\mu\nu}(x^{\sigma}(u,v))\, dx^{\mu} \wedge dx^{\nu}/2!\ (\mathbf{u},\mathbf{v})\ du\, dv$$

$$= \iint_A F_{\mu\nu}(x^{\sigma}(u,v))\ u^{\mu} v^{\nu}\, du\, dv$$

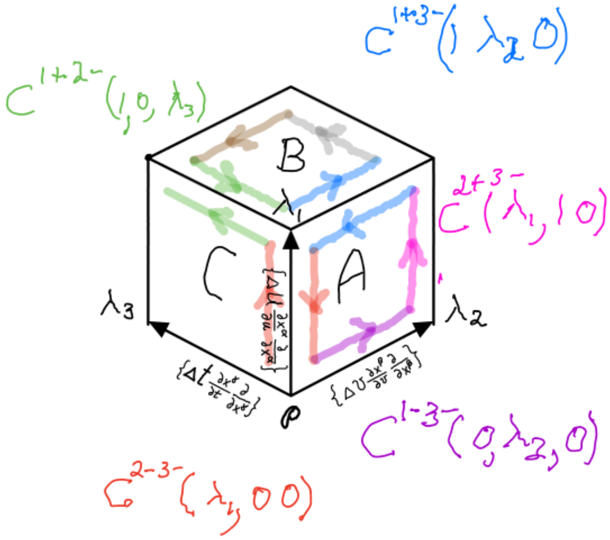
When the integration domain surrounding a point $\mathcal{P} \in A$ is so small that $F_{\mu\nu}$ is constant to lowest order, the value of the integral in the neighborhood of $\mathcal{P}$ is

$$\iint_A \underset{\sim}{F} = \iint_A F_{\mu\nu}(x^{\sigma}(u,v))\ u^{\mu} v^{\nu}\, \Delta u\, \Delta v$$

$$\boxed{\iint_A \underset{\sim}{F} \equiv \iint_A F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}/2! = F_{\mu\nu}\, \Delta u^{\mu}\, \Delta v^{\mu}\Big|_{\mathcal{P} = \{x^{\sigma}(u,v)\}}} \qquad (23.4)$$

Appendix 23

Two equivalent restrictions on the gravitation-induced rotation field



I. The Tensorial Rotation 2-form $\underset{\sim}{\overleftrightarrow{\mathcal{R}}}$ is "Closed".

Consider the rotation field

$$\underset{\sim}{\overleftrightarrow{\mathcal{R}}} = \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} = e_\alpha \otimes e_\beta R^{\alpha\beta}{}_{|\mu\nu|} dx^\mu \wedge dx^\nu$$

where

$$R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} = \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma g^{\beta\gamma} \equiv (d\omega^\alpha{}_\gamma + \omega^\alpha{}_\sigma \wedge \omega^\sigma{}_\gamma) g^{\beta\gamma} \quad .$$

is the component of rotational change* within the α-β plane.

* \ footnote{

Why is that change "rotational? The answer is this: By evaluating $\underset{\sim}{\overleftrightarrow{\mathcal{R}}}$ one obtains

$$\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\Delta u\, \mathbf{U}, \Delta v\, \mathbf{V}) \equiv \frac{e_\alpha \wedge e_\beta}{2!} \underset{\sim}{\mathcal{R}}^{\alpha\beta}(\Delta u\, \mathbf{U}, \Delta v\, \mathbf{V}) \qquad (23.1)$$

$$= \frac{e_\alpha \wedge e_\beta}{2!} \underbrace{R^{\alpha\beta}{}_{\mu\nu}\, u^\mu v^\nu \Delta u \Delta v}_{\theta^{\alpha\beta}} .$$

This is the sum of rotations in each of the $e_\alpha - e_\beta$ planes. The angular amount of this rotation, which is (23.3) induced by the curvature, is

$$\theta^{\alpha\beta} = R^{\alpha\beta}{}_{\mu\nu}\, u^\mu v^\nu \Delta u \Delta v .$$

This is mathematically the same as described in Lecture 19 on page 19.10, but with a physical difference: there Eqs. (19.2) and (19.3) refer instead to rotations which are induced by the angular velocity $\vec{\omega} = e_k \omega^k$ with

$$\theta^{ij} = \epsilon^{ij}{}_k\, \omega^k \Delta t$$

as the angle of rotation in the $e_i - e_j$ plane.

}

Then

$$d \underset{\approx}{\overset{\leftrightarrow}{\mathcal{R}}} = d\left(e_\alpha \otimes e_\beta \underset{\sim}{\mathcal{R}}^{\alpha\beta}\right) = d\left(e_\alpha \otimes e_\beta\, g^{\beta\gamma} \mathcal{R}^\alpha{}_\gamma\right)$$

$$= d\left(e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma\right) \otimes e_\beta\, g^{\beta\gamma} + e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma \otimes\wedge\, d\left(e_\beta\, g^{\beta\gamma}\right) = \underbrace{d\left(e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma\right)}_{①} e_\beta\, g^{\beta\gamma} + e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma \wedge \underbrace{d\left(e_\beta\, g^{\beta\gamma}\right)}_{②} \qquad (23, A, 1)$$

Notice that (i) the vectors e_α and $e_\beta\, g^{\beta\gamma}$ are merely (vector-valued) coefficients for the 2-form $\underset{\sim}{\mathcal{R}}^\alpha{}_\gamma$ and that (ii) the binary operations $\otimes$ and $\wedge$

commute: $\otimes\wedge = \wedge\otimes$. Because of this, e_α and $e_\beta g^{\beta\gamma}$ are mere multipliers of $\overset{\textcircled{3}}{\mathcal{R}^\alpha{}_\gamma}$ and explicit reference to $\otimes$ can be dropped provided one does not interchange e_α and e_β.

The calculation of the exterior derivative ① yields the vector-valued 3-form

$$d(e_\alpha \mathcal{R}^\alpha{}_\gamma) = e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\delta \wedge \omega^\delta{}_\gamma . \qquad (23.A.2)$$

The line of reasoning leading to this depends on using the differential $de_\alpha = e_\sigma \omega^\sigma{}_\alpha$, Cartan's 2nd structure equation $\underset{\sim}{\mathcal{R}}^\alpha{}_\delta = d\omega^\alpha{}_\delta + \omega^\alpha{}_\rho \wedge \omega^\rho{}_\delta$, its exterior derivative, doing a cancellation, and recombining terms to obtain Eq. (23.A.1).

The calculation of the exterior derivative ② yields the vector-valued 1-form

$$d(e_\beta g^{\beta\gamma}) = -\,\omega^\gamma{}_\tau\, g^{\tau\beta} e_\beta \qquad (23.A.3)$$

which is based on taking advantage of the metric compatibility condition

$$d\, g_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$$

and of a cancellation.

Introducing Eq. (23.A.2) and (23.A.3) into (23.A.1) yields

$$d\overleftrightarrow{\underset{\sim}{\mathcal{R}}} \equiv d\left(e_\alpha e_\beta \underset{\sim}{\mathcal{R}}^{\alpha\beta}\right) = e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\delta \wedge \omega^\delta{}_\gamma\, g^{\gamma\beta} e_\beta \;-\; e_\alpha \underset{\sim}{\mathcal{R}}^\alpha{}_\gamma \wedge \omega^\gamma{}_\tau\, g^{\tau\beta} e_\beta$$

$$\boxed{d\overleftrightarrow{\underset{\sim}{\mathcal{R}}} = 0}$$

Thus $\overleftrightarrow{\underset{\sim}{\mathcal{R}}}$ is a "closed" tensorial 2-form.

Question: Does there exist a tensorial 1-form $\overleftrightarrow{\underset{\sim}{A}}$ such that

$$d\overleftrightarrow{\underset{\sim}{A}} = \overleftrightarrow{\underset{\sim}{\mathcal{R}}} \quad \left(= \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}\right) \;?$$

II. $d\underset{\sim}{\overleftrightarrow{\mathcal{R}}}=0 \Longleftrightarrow$ Bianchi Identities

$$\Rightarrow: \quad 0=d\underset{\sim}{\overleftrightarrow{\mathcal{R}}}=d\left[e_\alpha\otimes e_\beta R^{\alpha\beta}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu\right]$$

$$=d\left[e_\alpha\left(R^{\alpha\beta}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu\right)e_\beta\right]$$

STEP 1:

Recall that $de_\alpha=e_\gamma\Gamma^\gamma_{\alpha\sigma}dx^\sigma$. Thus,

$$0=d\underset{\sim}{\overleftrightarrow{\mathcal{R}}}=e_\gamma\Gamma^\gamma{}_{\alpha\sigma}\,dx^\sigma\wedge(R^{\alpha\beta}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu)e_\beta+e_\alpha(R^{\alpha\beta}{}_{\mu\nu,\sigma}\,dx^\sigma\wedge dx^\mu\wedge dx^\nu)e_\beta+e_\alpha(R^{\alpha\beta}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu)\wedge e_\gamma\Gamma^\gamma{}_{\beta\sigma}\,dx^\sigma$$

$$=e_\alpha\left(R^{\alpha\beta}{}_{\mu\nu,\sigma}+R^{\gamma\beta}{}_{\mu\nu}\Gamma^\alpha{}_{\gamma\sigma}+R^{\alpha\gamma}{}_{\mu\nu}\Gamma^\beta{}_{\gamma\sigma}\right)e_\beta\;dx^\sigma\wedge dx^\mu\wedge dx^\nu$$

STER 2:

Augmenting the right hand side by

$$-e_\alpha\left(R^{\alpha\beta}{}_{\delta\nu}\,\Gamma^\delta_{\mu\sigma}+R^{\alpha\beta}{}_{\mu\delta}\Gamma^\delta_{\nu\sigma}\right)e_\beta\;dx^\sigma\wedge dx^\mu\wedge dx^\nu\equiv 0$$

does not alter $d\underset{\sim}{\overleftrightarrow{\mathcal{R}}}$. This is because this augmentation is identically zero, a fact due to (i) the symmetry of the Christoffel symbol under its lower index interchange and (ii) the anti-symmetry of $R^{\alpha\beta}{}_{\mu\nu}$ under the interchange of its lower indeces.

STEP 3:

It follows that

$$0=d\overleftrightarrow{\mathcal{R}}=e_\alpha\left(R^{\alpha\beta}{}_{\mu\nu;\sigma}\right)e_\beta\;dx^\sigma\wedge dx^\mu\wedge dx^\nu$$

Consequently,

$$\boxed{R^{\alpha\beta}{}_{\mu\nu;\sigma}+R^{\alpha\beta}{}_{\nu\sigma;\mu}+R^{\alpha\beta}{}_{\sigma\mu;\nu}=0},$$

which are the Bianchi identies.

$\Leftarrow$: Each of the above three steps is reversible. Thus,

$d\underset{\sim}{\overleftrightarrow{\mathcal{R}}}=0 \Longleftrightarrow$ Bianchi identities indeed.

Lecture 24

(Purpose: Attain mastery of using modern multivariable calculus methods for mathematizing a constellation of key concepts from electrostatics to be exteded to gravitation physics)

I.) EINSTEIN'S EQUATIONS: WHAT THEY MATHEMATIZE

The Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

mathematizes two facts:

(i) geometry controls the motion of matter (via ${T_\mu}^\nu{}_{;\nu} = 0$), and

(ii) matter controls the geometry of spacetime.

Einstein's line of reasoning for arriving at his tensorial equation was guided primarily by the physical and geometrical properties of the right hand side. The l.h.s. came out as a deductive consequence of his inductive line of reasoning applied to the right hand side. Although the l.h.s was a tensorial consequence, Einstein never identified its physical or its geometrical meaning and origin. This gap was filled later by Cartan and Wheeler. They filled it with the geometrical concept of "moment of rotation". Among other things, this resulted in the conservation of momenergy principle to be mathematized by them in terms of the topological principle that "the boundary of a boundary is zero".

The concept "moment of rotation" is an extension of the one familiar from mechanics in 3-d Euclidean space: torque, the moment of force. Both force and torque cause motion of bodies, translation and rotation. But in order to bring out its relevance to the Einstein field equations, both force and the moment of force need to be geometrized in the form surface and volume densities.

II.) DIELECTRIC IN A STATIC FORCE FIELD)

To this end consider a rigid parallelopiped (which for shorthand we will call a "cube", a "3-cube", or a "3-d cube") composed of an array of uniformly distributed and rigidly aligned permanent molecular dipole moments,

$$\vec{p} = q\,\vec{d} = q\,\vec{e}_m d^m$$

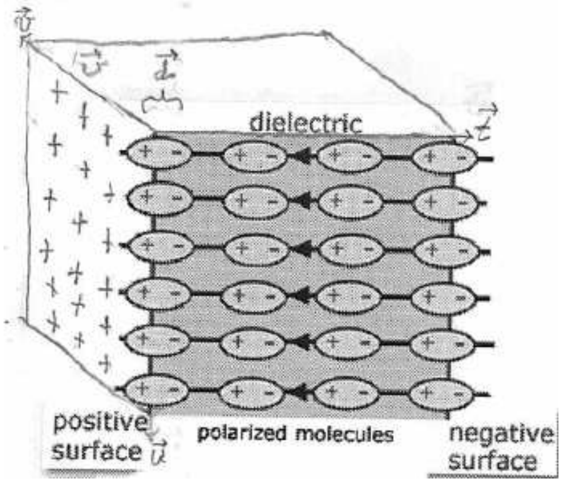


Figure 1: A dielectric parallelopiped of volume $\Delta x^1 \Delta x^2 \delta x^3$ subjected to an external electrostatic field.

The volume of this cube is spanned by the triad of vectors

$$\vec{u} = \Delta x^1 \frac{\partial}{\partial x^1} \equiv \Delta x^1\, \vec{e}_1, \tag{2}$$

$$\vec{v} = \Delta x^2 \frac{\partial}{\partial x^1} \equiv \Delta x^2\, \vec{e}_3, \tag{3}$$

$$\vec{t} = \Delta x^3 \frac{\partial}{\partial x^1} \equiv \Delta x^2\, \vec{e}_3. \tag{4}$$

The electrostatic polarization in this 3-d cube is

$$\overrightarrow{P} = N\,\vec{p} = qN\vec{d}\ \left(= \frac{\textbf{(dipole moment)}}{\textbf{(volume)}} \right) \tag{5}$$

$$\equiv \vec{e}_m d^m\, qN \tag{6}$$

Here

$$N = \frac{\textbf{(\# of molecules)}}{\textbf{(volume)}} \tag{7}$$

is the density of molecules in this cube. The total polarization is

$$\overrightarrow{P} \times \textbf{(volume)} = \begin{pmatrix} \textbf{total} \\ \textbf{dipole} \\ \textbf{moment} \end{pmatrix} \tag{8}$$

The molecular dipoles in their uniform alignment yield surface charge densities on each of the six oriented faces of the cube, namely

$$\begin{pmatrix} \textbf{surface density} \\ \textbf{of molecules} \end{pmatrix} \equiv \sigma^3$$

$$q\ \ \overbrace{Nd^3}\ \ \epsilon_{312}\,\Delta x^1 \Delta x^2|_{x^3+\Delta x^3} = qNd^m \epsilon_{m|ij|} dx^i \wedge dx^j\,(\vec{u},\vec{v})|_{x^3+\Delta x^3},$$

$$q\ Nd^3\ \epsilon_{312}\,\Delta x^1 \Delta x^2|_{x^3} = qNd^m \epsilon_{m|ij|} dx^i \wedge dx^j\,(\vec{u},\vec{v})|_{x^3},$$

and similarly for the other two pairs of faces.

III.) THE FORCE FIELD

Upon subjecting the cube to an electrostatic field

$$\overrightarrow{E} = \vec{e}_k E^k, \tag{9}$$

the force field acting on the charged surfaces is mathematized by

$$\overrightarrow{\underline{F}} = \vec{e}_k E^k\ q\ \ \overbrace{\sigma^m \epsilon_{m|ij|} dx^i \wedge dx^j}^{\begin{pmatrix} \textbf{surface density} \\ \textbf{of molecules on an} \\ \textbf{as-yet-unspecified} \\ \textbf{area} \end{pmatrix}} \tag{10}$$

$$= \vec{e}_k E^k\ \underbrace{q\ \ Nd^m \epsilon_{m|ij|} dx^i \wedge dx^j}_{\begin{pmatrix} \textbf{surface charge} \\ \textbf{density on an} \\ \textbf{as-yet-unspecified} \\ \textbf{area} \end{pmatrix}} \tag{11}$$

or more economically by

$$\boxed{\overrightarrow{\underline{F}} \equiv \overrightarrow{F}_{ij}\; dx^i \wedge dx^j,} \tag{12}$$

the vectorial force field (surface density) acting on the faces of a 3-cube. This is the momentum flux 2-form, a vectorial flux across an as-yet-unspecified element of area[1]. It is a vector-valued 2-form, a tensor of rank $\binom{1}{2}$.

The forces on the opposing oriented faces spanned by $\{\vec{u},\vec{v}\}$ are

$$\begin{aligned} \overrightarrow{\underline{F}}\;(\vec{u},\vec{v})|_{x^3+\Delta x^3} &= \overrightarrow{F}_{12}\;|_{x^3+\Delta x^3}\Delta x^1 \Delta x^2 \\ &= \textbf{force on } (\vec{u},\vec{v})\textbf{-area at } x^3+\Delta x^3 \end{aligned} \tag{16}$$

and

$$\begin{aligned} \overrightarrow{\underline{F}}\;(\vec{v},\vec{u})|_{x^3} &= (-)\;\overrightarrow{F}_{12}\;|_{x^3}\Delta x^1 \Delta x^2 \\ &= \textbf{force on } (\vec{v},\vec{u})\textbf{-area at } x^3, \end{aligned} \tag{17}$$

with similar expressions for the other faces spanned by $\{\vec{v},\vec{t}\}$ and $\{\vec{t},\vec{u}\}$.

IV.) SURFACE FORCES VS. VOLUME FORCE

The total force on these opposing faces, all six of them, is

$$\begin{aligned} \overrightarrow{F}_{total} =& \overrightarrow{\underline{F}}\;(\vec{u},\vec{v})|_{x^3+\Delta x^3} + \overrightarrow{\underline{F}}\;(\vec{v},\vec{u})|_{x^3} \\ &+ \overrightarrow{\underline{F}}\;(\vec{v},\vec{t})|_{x^1+\Delta x^1} + \overrightarrow{\underline{F}}\;(\vec{t},\vec{v})|_{x^1} \\ &+ \overrightarrow{\underline{F}}\;(\vec{t},\vec{u})|_{x^2+\Delta x^2} + \overrightarrow{\underline{F}}\;(\vec{u},\vec{t})|_{x^2} \end{aligned} \tag{18}$$

which in light of Eq.(12) and Eqs.(2)-(4) becomes

$$\overrightarrow{F}_{total} = \nabla_{\vec{t}}\,(\overrightarrow{F}_{12}\;\Delta x^1 \Delta x^2) \tag{19}$$

$$+\,\nabla_{\vec{u}}\,(\overrightarrow{F}_{23}\;\Delta x^2 \Delta x^3) \tag{20}$$

$$+\,\nabla_{\vec{v}}\,(\overrightarrow{F}_{31}\;\Delta x^3 \Delta x^1). \tag{21}$$

---

[1] The force field exerts stresses on the faces of the electretized cube in Figure 1. These stresses are mathematized in terms of the stress tensor familiar from continuum mechanics. Let

$$d^2\Sigma_m = \epsilon_{m|ij|} dx^i \wedge dx^j \quad m = 1,2,3 \tag{13}$$

be the 2-form of an element of area spanned by an as-yet-unspecified pair of vectors. The stress tensor is a vector (force) valued surface-density 2-form

$$\vec{e}_k T^{km} d^2\Sigma_m \equiv \overrightarrow{F}_{|ij|}\; dx^i \wedge dx^j. \tag{14}$$

Its components, in light of Eq.(11) are

$$T^{km} = E^k Nqd^m = E^k \times (\text{dipole moment})^m \tag{15}$$

This stress-tensor is not symmetric: $T^{km} \neq T^{mk}$. This happens when the dipole vector density is not collinear with the applied electrostatic field. Consequently, the stresses acting on the cube exert a non-zero torque on it. This, as we know, imparts angular momentum to the cube.

By introducing the vector valued three-form

$$d\underline{\overrightarrow{F}} = d\,(\overrightarrow{F}_{|ij|}\,dx^i \wedge dx^j) \tag{22}$$
$$= d\,(\overrightarrow{F}_{12}\,dx^1 \wedge dx^2) \tag{23}$$
$$+\,d\,(\overrightarrow{F}_{23}\,dx^2 \wedge dx^3) \tag{24}$$
$$+\,d\,(\overrightarrow{F}_{31}\,dx^3 \wedge dx^1), \tag{25}$$

one recognizes that the total force vector, Eqs.(19)-(21), condenses into the value of that three-form evaluated on the vectors that span the volume of the cube,

$$\boxed{\overrightarrow{F}_{total}= d\,\underline{\overrightarrow{F}}\,(\vec{u},\vec{v},\vec{t}).} \tag{26}$$

Physically this is the total *volume force* acting on and averaged over the cube's interior domain, which is spanned by the three vectors $\vec{u},\vec{v}$ and $\vec{t}$. Mathematically $d\,\underline{\overrightarrow{F}}$ is the familiar *exterior derivative* of $\underline{\overrightarrow{F}}$. Next substitute Eq.(12) into Eq.(26), use the triad of vectors Eqs.(2)-(4) and thus obtain

$$\overrightarrow{F}_{total} = \nabla_{\vec{e}_n}\,\overrightarrow{F}_{|ij|}\,dx^n \wedge dx^i \wedge dx^j(\vec{u},\vec{v},\vec{t}) \tag{27}$$
$$= \left(\nabla_{\vec{e}_1}\,\overrightarrow{F}_{23} + \nabla_{\vec{e}_2}\,\overrightarrow{F}_{31} + \nabla_{\vec{e}_3}\,\overrightarrow{F}_{12}\right) \times \Delta x^1 \Delta x^2 \Delta x^3 \tag{28}$$
$$= \left(\frac{\textbf{(force)}}{\textbf{(volume)}}\right) \times \Delta x^1 \Delta x^2)\Delta x^3, \tag{29}$$

the *volume force* experienced by the cube in its interior. Here

$$\overrightarrow{F}_{ij}= \vec{e}_k E^k\,qNd^m \epsilon_{mij} \tag{30}$$

are the *surface force densities* acting on the ij-labeled faces of the cube.

By equating Eq.(18) to Eq.(28) one obtains

$$\left(\begin{array}{c}\textbf{total force on}\\ \textbf{all 6 faces}\end{array}\right) \equiv \sum_{\ell=1}^{6} \underline{\overrightarrow{F}}\,(\ell^{th}\textbf{face}) \tag{31}$$
$$= \left(\nabla_{\vec{e}_1}\,\overrightarrow{F}_{23} + \nabla_{\vec{e}_2}\,\overrightarrow{F}_{31} + \nabla_{\vec{e}_3}\,\overrightarrow{F}_{12}\right) \times \Delta x^1 \Delta x^2 \Delta x^3 \tag{32}$$

The l.h.s. of this equation refers to the totality of the *surface forces* acting on the (6-faced) boundary of the cube. The r.h.s. of Eq.(32) refers to the *volume force* on the interior of the cube. Thus the 6 conditions on the surface boundary of the cube are sufficient for inferring the mean condition inside:

$$\frac{\sum_{\ell=1}^{6} \underline{\overrightarrow{F}}\,(\ell^{th}\textbf{face})}{\Delta x^1 \Delta x^2)\Delta x^3} = \nabla_{\vec{e}_1}\,\overrightarrow{F}_{23} + \nabla_{\vec{e}_2}\,\overrightarrow{F}_{31} + \nabla_{\vec{e}_3}\,\overrightarrow{F}_{12}\;. \tag{33}$$

When the electrostatic field $\overrightarrow{E}$, Eq.(9), is homogeneous, i.e.

$$\nabla_{\vec{e}_i}\left(\vec{e}_k E^k\right) = 0, \quad (i = 1, 2, 3), \tag{34}$$

Eq.(33) becomes

$$\begin{aligned}\sum_{\ell=1}^{6}\overrightarrow{\underline{F}}\,(\ell^{th}\textbf{face}) &= \vec{e}_k E^k [\,\nabla_{\vec{e}_1}(Nqd^m\epsilon_{m23}) \\ &\qquad +\nabla_{\vec{e}_2}(Nqd^m\epsilon_{m31}) \\ &\qquad +\nabla_{\vec{e}_3}(Nqd^m\epsilon_{m12})]\,\Delta x^1\Delta x^2\Delta x^3 \\ &= \vec{e}_k E^k \quad \sum_{n=1}^{3}\frac{\partial(\sqrt{g}\,Nqd^n)}{\partial x^n}\Delta x^1\Delta x^2\Delta x^3 \\ &= \vec{e}_k E^k\,(N\,q\,d^n)_{;n}\,\underbrace{\sqrt{g}\,\Delta x^1\Delta x^2\Delta x^3}_{\left(\begin{array}{c}\textbf{invariant}\\ \textbf{volume}\end{array}\right)} \end{aligned}\tag{35}$$

Here $\vec{e}_k E^k\,(N\,q\,d^n)_{;n}$ is the force density relative to an physical/orthonormal frame.

V.) ENERGY INJECTED INTO THE CUBE

If the cube undergoes displacement, say $\overrightarrow{w}=\vec{e}_\ell\,w^\ell$, then each of its 6 faces receives mechanical energy from the force field. The amount of that energy is

$$\overrightarrow{w}\cdot\overrightarrow{\underline{F}}\,(\ell^{th}\textbf{face}),\qquad \ell=1,\cdots,6,\tag{36}$$

the work done by each of the respective forces listed in Eq.(18). The union of the 6 oriented faces is the boundary $\partial\mathcal{D}$ of the cube's interior domain $\mathcal{D}$:

$$\bigcup_{\ell=1}^{6}(\ell^{th}face)=\partial\mathcal{D}.$$

The function

$$\overrightarrow{\underline{F}}\,(\cdots) = \cdot\,\overrightarrow{F}_{i,j}\;dx^i\wedge dx^j(\cdots)\tag{37}$$

$$=\overrightarrow{E}\quad N\,q\,d^m\epsilon_{mij}dx^i\wedge dx^j(\cdots)\tag{38}$$

is a linear vector-valued function on its components:

$$\overrightarrow{F}_{total}\left(\bigcup_{\ell=1}^{6}(\ell^{th}face)\right)=\sum_{\ell=1}^{6}\overrightarrow{\underline{F}}\,(\ell^{th}\textbf{face})$$

From this line of reasoning one arrives from Eq.(36) that

$$\boxed{\vec{w}\cdot\,\overrightarrow{\underline{F}}=\overrightarrow{E}\,\cdot\vec{w}\,N\,q\,d^m\epsilon_{mij}dx^i\wedge dx^j}\tag{39}$$

is the translational energy injected into the cube's interior through one of its as-yet-unspecified faces of its boundary.

The total change in mechanical energy of the dielectric cube is therefore

$$\overrightarrow{w}\cdot\overrightarrow{F}_{total}=\overrightarrow{w}\cdot\left[\sum_{\ell=1}^{6}\overrightarrow{\underline{F}}\,(\ell^{th}\textbf{face})\right],\tag{40}$$

In light of Eqs.(32) this total is

$$\overrightarrow{w}\cdot\overrightarrow{F}_{total}=\overrightarrow{w}\cdot\left(\nabla_{\vec{e}_1}\overrightarrow{F}_{23}+\nabla_{\vec{e}_2}\overrightarrow{F}_{31}+\nabla_{\vec{e}_3}\overrightarrow{F}_{12}\right)\times\Delta x^1\Delta x^2\Delta x^3,\tag{41}$$

or, equivalently, because of Eq.(26),

$$\overrightarrow{w}\cdot\overrightarrow{F}_{total} = \overrightarrow{w}\cdot d\,\underline{\overrightarrow{F}}\,(\vec{u},\vec{v},\vec{t}). \tag{42}$$

VI.) TRANSLATIONAL EQUILIBRIUM

Even though the opposing faces carry non-zero charges, the total charge on the 3-d cube is zero. Such a cube, when subjected to an $\overrightarrow{E}$-field, which we take to be homogeneous ($\nabla_{\vec{u}}\overrightarrow{E} = \nabla_{\vec{v}}\overrightarrow{E} = \nabla_{\vec{t}}\overrightarrow{E} = 0$), exerts no net force on the cube. The sum total of the forces on the cube's 6 faces vanishes:

$$\overrightarrow{F}_{total} = \sum_{\ell=1}^{6}\underline{\overrightarrow{F}}\,(\ell^{th}\textbf{face}) = \overrightarrow{0}\ . \tag{43}$$

Thus the cube is in a state of *translational equilibrium.* It gains no translational energy. In light of the fact that Eq.(43) holds for any set of spanning vectors in Eq.(26), one concludes that

$$\boxed{d\,\underline{\overrightarrow{F}} = 0} \tag{44}$$

mathematizes that condition for translational equilibrium. In light of Eq.(33) this is equivalent to

$$\nabla_{\vec{e}_1}\overrightarrow{F}_{23} + \nabla_{\vec{e}_2}\overrightarrow{F}_{31} + \nabla_{\vec{e}_3}\overrightarrow{F}_{12} = 0 \tag{45}$$

or more compactly

$$\boxed{\nabla_i\overrightarrow{F}_{jk} + \nabla_j\overrightarrow{F}_{ki} + \nabla_k\overrightarrow{F}_{ij} = 0} \tag{46}$$

VI.) DIVERGENCELESS VECTOR FIELD

The internal charge structure of the cube consists of dipoles distributed uniformly throughout its interior. If $\vec{e}_m d^m(\overrightarrow{E})q$ is the molecular dipole moment[2], then

$$\overrightarrow{P} = \vec{e}_n d^n q\, N \equiv \vec{e}_n P^n$$

in Eq.(6) is the macroscopic polarization vector field. Compare its components with those in the volume force, Eq.(35), experienced by the cube under the condition of translational equilibrium, Eq.(43). Based on this, the conclusion is that the divergence of the polarization vector field vanishes:

$$0 = (N\,q\,d^n)_{;n} \equiv P^n_{\ ;n} \equiv \nabla\cdot\overrightarrow{P} \equiv \textbf{div (polarization)} \tag{47}$$

---

[2] The molecular charge separation vector $\{d^m(\overrightarrow{E})\}$ is typically a linear, but not necessarily a coliniear, function of the externally applied electrostatic field.

VII.) CONCLUSION

Mathematically the Einstein field equations (EFE), Eq.(1), is a geometrical extension[3] of the interaction between a dielectric body and the electrostatic forces acting on it. The forces in Euclidean space have two types of causal attributes:

1. those that result in translational motion and
2. those that result in rotational motion.

For Einstein's field equations (EFE) both types need to be extended to the 4-d spacetime. Moreover, both of them require the conservation laws stated in the form of the generalized vectorial and tensorial Stokes' theorem, the relation between vectorial, as well as tensorial, physical attributes inside a given 3-d domain to those on its 2-d boundary.

	Electrostatics	Gravitation
Eq.(#)	Electrostatic-induced force field:	Curvature-induced rotation field:
Eq.(12)	$\underline{\vec{F}}=\vec{F}_{i,j}\; dx^i \wedge dx^j$	$\underline{\overset{\leftrightarrow}{\mathcal{R}}}=\overset{\leftrightarrow}{R}_{i,j}\; dx^i \wedge dx^j$
Eq.(11)	$=\vec{E}\quad N\; q\, d^m \epsilon_{m\|ij\|} dx^i \wedge dx^j$	$= \vec{e}_\mu \wedge \vec{e}_\nu R^{\mu\nu}{}_{\alpha\beta} dx^\alpha dx^\beta$
	Volume force:	
Eq.(26)	$\vec{F}_{total}= d\; \underline{\vec{F}}\; (\vec{u},\vec{v},\vec{t})$	$d\; \underline{\overset{\leftrightarrow}{\mathcal{R}}}\; (\mathbf{u},\mathbf{v},\mathbf{t})$
	Change in energy due to displacement shift $\vec{w}$:	Rotational change in movement due to displacement shift $\mathbf{w} = \mathbf{e}_\nu w^\nu$:
Eq.(39)	$\underline{\vec{F}}\; \cdot \vec{w} =\vec{E}\; \cdot \vec{w}\, N q\, d^m \epsilon_{mij} dx^i \wedge dx^j$	$\underline{\overset{\leftrightarrow}{\mathcal{R}}}\; \cdot (\mathbf{e}_\nu w^\nu) = \vec{e}_\mu w^\nu R^\mu{}_{\nu\alpha\beta} dx^\alpha dx^\beta$
	Translational Equilibrium:	Bianchi Identity:
Eq.(44)	$d\; \underline{\vec{F}}= 0$	$d\; \underline{\overset{\leftrightarrow}{\mathcal{R}}}= 0$
Eq.(46)	$\nabla_i\; \vec{F}_{jk} +\nabla_j\; \vec{F}_{ki} +\nabla_k\; \vec{F}_{ij}= 0$	$\nabla_\gamma\; \overset{\leftrightarrow}{R}_{\alpha\beta} +\nabla_\alpha\; \overset{\leftrightarrow}{R}_{\beta\gamma} +\nabla_\beta\; \overset{\leftrightarrow}{R}_{\gamma\alpha}= 0$

Table 1 above highlights the extension of vectorial concepts from electrostatics in a 3-d Euclidean environment to tensorial concepts in 4-d spacetime necessary for the EFE.

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[3] The extension is one from vectors in Euclidean space to tensors in 4-d spacetime.

Lecture 25

The boundary of a boundary=0
implies the Bianchi identities

A 3-cube has 3 pairs of faces.

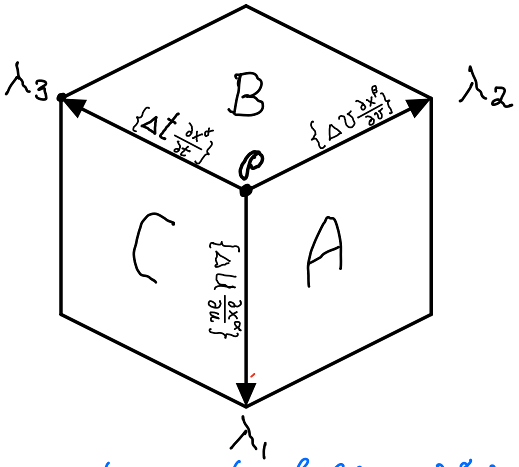


Figure 25.1 Given 3 commuting vector fields, $\mathbf{u} = \frac{\partial x^\alpha}{\partial u}\frac{\partial}{\partial x^\alpha}$, $\mathbf{v} = \frac{\partial x^\beta}{\partial v}\frac{\partial}{\partial x^\beta}$, and $\mathbf{t} = \frac{\partial x^\delta}{\partial t}\frac{\partial}{\partial x^\delta}$, each such triad spans a 3-cube with 6 faces. However, if these vectors do not commute, then, as depicted in the Appendix, form a polyhedron with 10 faces, a "chipped" cube.

Q: What is the rotational change from all 3 pairs of opposing faces?

Ans.: In 3 Steps:

Step 1: From faces A' and A the rotational change is

$$\overset{2}{\Delta} W_{A'} + \overset{2}{\Delta} W_A =$$

$$\oint_{\partial A'} dw + \oint_{\partial A} dW = \Big\{ \iint_{A'} \underset{\sim}{\overleftrightarrow{\mathcal{R}}}\Big|_{P+\Delta t\,\mathbf{t}} - \iint_{A} \underset{\sim}{\overleftrightarrow{\mathcal{R}}}\Big|_{P} \Big\} \cdot w$$

$$= \Big\{ \iint \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\mathbf{u},\mathbf{v})\Big|_{P+\Delta t\,\mathbf{t}} du\,dv - \iint \underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\mathbf{u},\mathbf{v})\Big|_{P} du\,dv \Big\} \cdot W$$

$$= \Big\{ \nabla_{\mathbf{t}} \Big(\underset{\sim}{\overleftrightarrow{\mathcal{R}}}(\mathbf{u},\mathbf{v}) \Big)\Big|_{P} \Delta t\, \Delta u\, \Delta v \Big\} \cdot W$$

Step 2: Apply this mathematization to the other pairs of faces, B' & B and C' and C:

$$\Delta^2 W_{A'} + \Delta^2 W_{A} + \Delta^2 W_{B'} + \Delta^2 W_{B} + \Delta^2 W_{C'} + \Delta^2 W_{C} =$$

$$\oint_{\partial A'} dW + \oint_{\partial A} dW + \oint_{\partial B'} dW + \oint_{\partial B} dW + \oint_{\partial C'} dW + \oint_{\partial C} dW = \qquad (25.1)$$

$$= \Big[\underbrace{\iint_{A'} - \iint_{A}}_{} + \underbrace{\iint_{B'} - \iint_{B}}_{} + \underbrace{\iint_{C'} - \iint_{C}}_{} \Big] \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}} \cdot W =$$

$$= \Big[\nabla_t \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(u,v) + \nabla_u \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(v,t) + \nabla_v \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(t,u) \Big] \Delta t \Delta u \Delta v \} \cdot W$$

(25.2)

Each term is a covariant directional derivative of a sum of bivectors, tensors of rank $\binom{2}{0}$.

Step 3 Take advantage of the tensorial 2-3 version of Stokes' theorem

$$\nabla_t \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(u,v) + \nabla_u \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(v,t) + \nabla_v \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}(t,u)$$

$$- \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}([u,v],t) - \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}([v,t],u) - \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}}([t,u],v) = \underset{\sim}{d} \overset{\leftrightarrow}{\underset{\sim}{\mathcal{R}}} (u,v,t)$$

(25.3)

If the spanning vector fields u, v, and t have non-zero commutators then additional line integrals need to be added to the sum, Eq.(25.1), and hence to Eq.(25.2). These line integrals are over paths that enclosed the areas of a chipped cube spanned by $[\mathbf{u},\mathbf{v}]$ and $\mathbf{t}$, $[\mathbf{v},\mathbf{t}]$ and $\mathbf{u}$, as well as $[\mathbf{t},\mathbf{u}]$ and $\mathbf{v}$. They are depicted in the Appendix

The polyhedron with these areas has more than the six faces that characterize the 3-cube on page 25.1. Thus, for commuting spanning vectors one has

$$\Delta^2 w_{A'} + \Delta^2 w_A + \Delta^2 w_{B'} + \Delta^2 w_B + \Delta^2 w_{C'} + \Delta^2 w_C =$$

$$\oint_{\partial A'} dw + \oint_{\partial A} dw + \oint_{\partial B'} dw + \oint_{\partial B} dw + \oint_{\partial C'} dw + \oint_{\partial C} dw =$$

$$= \left[\iint_{A'} - \iint_{A} + \iint_{B'} - \iint_{B} + \iint_{C'} - \iint_{C} \right] \overleftrightarrow{\mathcal{R}} \cdot w =$$

$$= d \underset{\sim}{\overleftrightarrow{\mathcal{R}}} (\Delta u\,\mathbf{u}, \Delta v\,\mathbf{v}, \Delta t\,\mathbf{t}) \Big|_{\mathcal{P}} \cdot w$$

$$= \iiint_{3-cube} \underset{\sim}{d} \underset{\sim}{\overleftrightarrow{\mathcal{R}}} (\mathbf{u},\mathbf{v},\mathbf{t})\, du\, dv\, dt \cdot W$$

Together the line integrals are over $\partial\partial$(3-cube), the boundary of the boundary of the 3-cube:

$$\int_{\partial\partial(3-cube)} dw = \iiint_{3-cube} \underset{\sim}{d} \underset{\sim}{\overleftrightarrow{\mathcal{R}}} (\mathbf{u},\mathbf{v},\mathbf{t})\, du\, dv\, dt \cdot W$$

However,

$$\partial\partial = 0 .$$

Thus

$$\iiint_{3\text{-cube}} d\overset{\leftrightarrow}{R}(u,v,t)\, du\, dv\, dt \cdot w = 0$$

The vanishing this integral hold for arbitrarily chosen vectors u, v, t, an w. Consequently,

$$0 = d\,\underset{\sim}{\overset{\leftrightarrow}{R}} \equiv d\left\{ \frac{e_\alpha \wedge e_\beta}{2!} R^{\alpha\beta}{}_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!} \right\}$$

The vanishing of this exterior derivative is also validated by means of a direct calculation, namely by calculating

$$d\left\{ \frac{e_\alpha \wedge e_\beta}{2}\, g^{\beta\sigma} \left(d\,\omega^\alpha{}_\sigma + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\sigma \right) \right\} = 0 .$$

It is also a way of proving the Bianchi identities

$$R^{\alpha\beta}{}_{\mu\nu;\rho} + R^{\alpha\beta}{}_{\nu\rho;\mu} + R^{\alpha\beta}{}_{\rho\mu;\nu} = 0.$$

Appendix

Chipped Cube: Jacobi Identity

Chipped cube solution to Exercise 9.12 in MTW

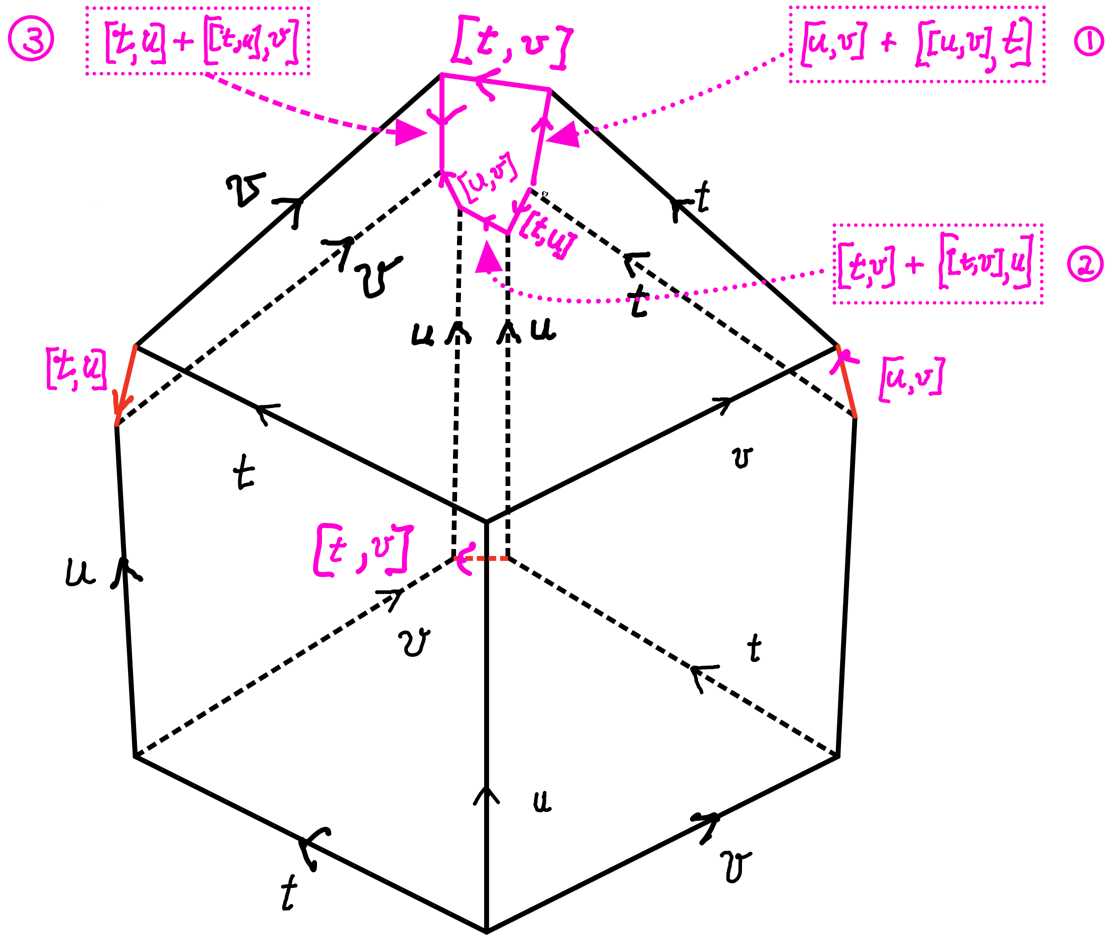


Figure 25.2 a "chipped cube"

is a 10-sided polyhedron whose edges are formed by three non-commuting vector fields u, v, and t. The sum of the commutators $\textcircled{1} + [t, v] + \textcircled{3} - [u,v] - \textcircled{2} - [t, u]$

at the top far end is a vector sum which vanishes:

$$0 = \textcircled{1} + [t, v] + \textcircled{3} - [u,v] - \textcircled{2} - [t, u] = [u,v] + \big[[u,v],t\big] + [t,v] + [t,u] + \big[[t,u],v\big]$$
$$- [u,v] - [t,v] - \big[[t,v],u\big] - [t,u]$$

Consequently,

$$\big[[u,v],t\big] + \big[[t,u],v\big] + \big[[v,t],u\big] = 0,$$

which is "Jacobi's identity."

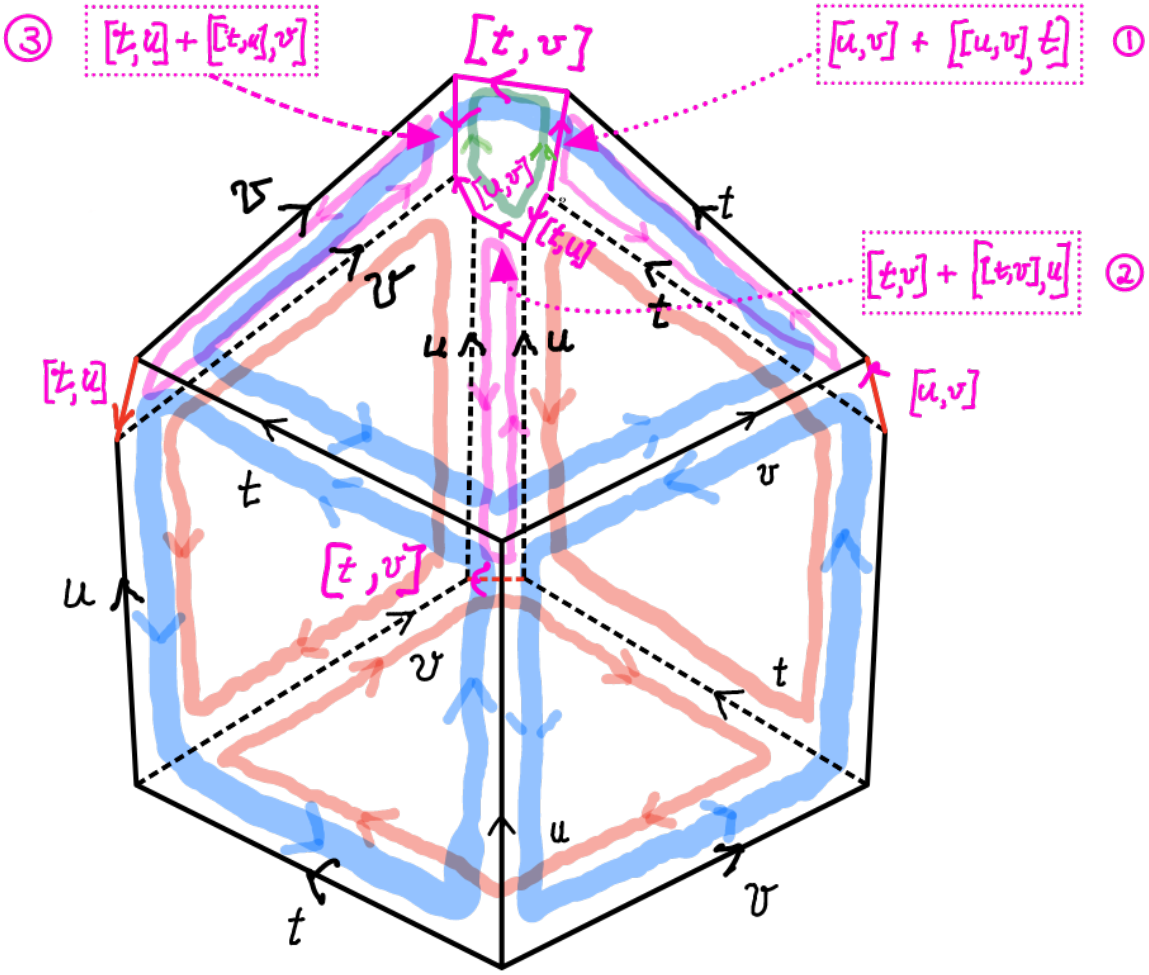
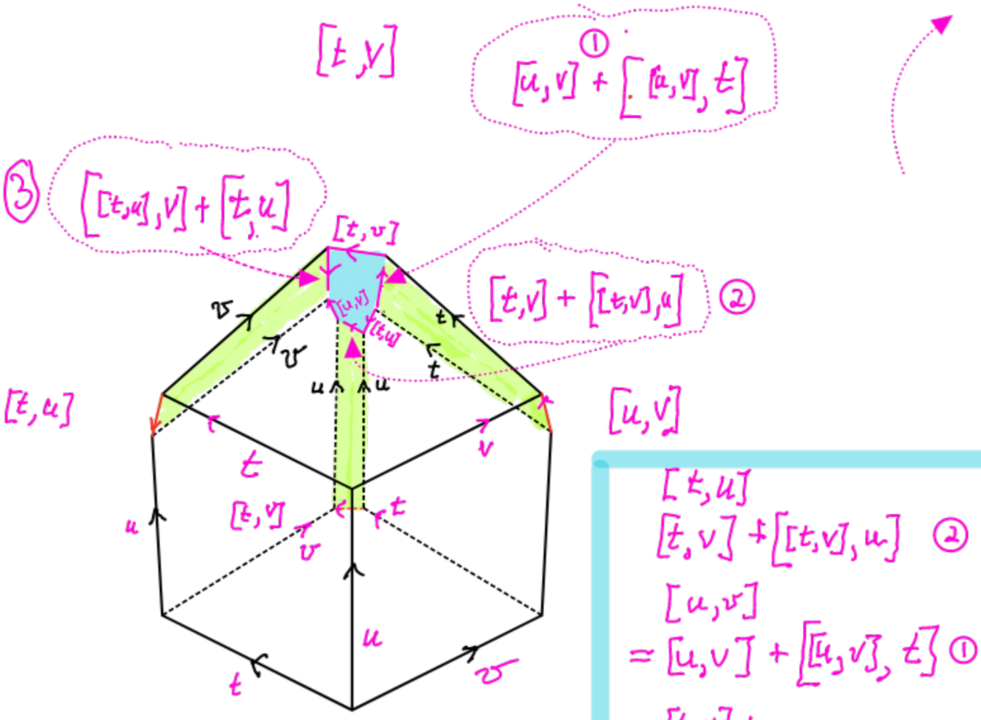


Figure 25.3 Closed line integration paths along the edges of the facets of a chipped cube.



$[t,u]$

$[t,v] + [[t,v],u]$ ②

$[u,v]$

$= [u,v] + [[u,v],t]$ ①

$[t,v] +$

$+ [t,u] + [[t,u],v]$ ③

$$-[[t,v],u] + [[u,v],t] + [[t,u],v] = 0$$

$$\boxed{[[v,t],u] + [[t,u],v] + [[u,v],t]}$$

Lecture 26

Surface force density mathematized. Translational equilibrium.

Reading assignment

1. Typeset "Lecture 24"
2. In MTW § 15.3

I. SURFACE FORCE DENSITY

Q: What is the cause of the force experienced by a cube with its plane faces?

A.: The force experienced by a 3-cube comes from the pressure and the shear stresses on each of the six faces $\Delta\vec{A}_{(\ell)}$, $\ell = 1, 2, \cdots, 6$:

$$\begin{bmatrix} \Delta F^1{}_{(\ell)} \\ \Delta F^2{}_{(\ell)} \\ \Delta F^3{}_{(\ell)} \end{bmatrix} = \begin{bmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{bmatrix} \begin{bmatrix} \Delta A^1{}_{(\ell)} \\ \Delta A^2{}_{(\ell)} \\ \Delta A^3{}_{(\ell)} \end{bmatrix}$$

The total force from all six faces is

$$\Delta\vec{F} = \sum_{\ell=1}^{6} e_i T^i{}_j \Big|_{\ell^{th}\text{face}} \Delta A^j{}_{(\ell)} = \sum_{\ell=1}^{6} e_i T^i{}_j \Big|_{\ell^{th}\text{face}} \epsilon^j{}_{|km|} dx^k \wedge dx^m (\vec{u}_\ell, \vec{v}_\ell)$$

From the mechanics of a rigid body subjected to force fields one knows that they have two causal attributes:

1. Those that result in translational motion and
2. those that result in rotational motion of a

given body.*

*\footnote{The driving force behind mathematizing these two constellations of concepts is that their mathematical extension to 4-d spacetime is what is needed in order to understand the E.F.E. in particular the l.h.s, i.e. the Einstein tensor.}

To mathematize the difference and the relation between the two, concretize the force field by an electrostatic field interacting with a dielectric.

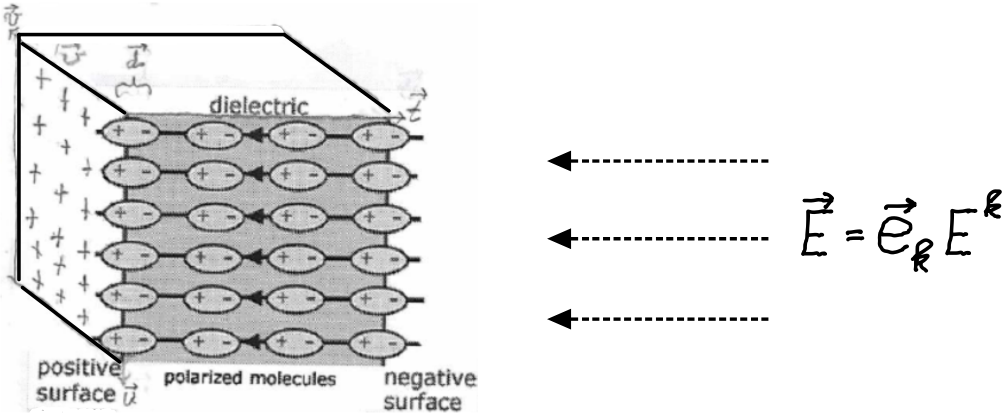


Figure 26.1 Polarized dielectric cube in an electric field

A dielectric consists of an array of polarizable molecules, each having a dipole moment

$$\vec{p}(\vec{E}) = e_m\, r^m(\vec{E})\, q$$

when subjected to a homogeneous electrostatic field

$$\vec{E} = e_k E^k.$$

For a dielectric cube having volume

$$\epsilon_{|ijk|}\, dx^i \wedge dx^j \wedge dx^k(\Delta u\,\vec{u}, \Delta v\,\vec{v}, \Delta t\,\vec{t}) \simeq \begin{vmatrix} u^2 & u & u^3 \\ v^1 & v^2 & v^3 \\ t^1 & t^2 & t^3 \end{vmatrix} \Delta u \Delta v \Delta t$$

spanned by the triad $\vec{u}$, $\vec{v}$, and $\vec{t}$, the total number of polarized molecules is

$$N\, \epsilon_{|ijk|}\, dx^i \wedge dx^j \wedge dx^k(\Delta u\,\vec{u}, \Delta v\,\vec{v}, \Delta t\,\vec{t}),$$

where N is the measured density of molecules.

The force acting on the $(\Delta u\,\vec{u}, \Delta v\,\vec{v})$-spanned face* is

* \ footnote{ whose normal is $\vec{u}\times\vec{v} = e_m\, \epsilon^m{}_{|ij|}\, dx^i \wedge x^j(\vec{u},\vec{v})$}

$$\text{"force"} = \underset{\sim}{\vec{F}}(\Delta u\,\vec{u}, \Delta v\,\vec{v}) = \underbrace{\vec{E}\, q\, \overbrace{\underbrace{N\, r^m(\vec{E})\, \epsilon_{m|ij|}}_{\boxed{\text{surface density of dipoles}}} dx^i \wedge dx^j(\Delta u\,\vec{u}, \Delta v\,\vec{v})}^{\boxed{\#\text{ of charges on a } (\vec{u},\vec{v})\text{-face}}}}_{\vec{F}_{|ij|}\, dx^i \wedge dx^j} \qquad (26.1)$$

The volume of a single-layered slab of molecular dipoles is

$$r^m(\vec{E})\,\epsilon_{m|ij|}\,dx^i\wedge dx^j(\vec{u},\vec{v})\,\Delta u\,\Delta v$$

The surface density of dipoles in this layer is a new concept. It is mathematized by the <u>surface density 2-form</u>

$$N\,r^m(\vec{E})\,\epsilon_{m|ij|}\,dx^i\wedge dx^j, \qquad (\text{"surface density"})$$

and it is understood to be evaluated on a pair of vectors, in which case it yields the number dipoles in the slab.

In the presence of an electric field this dipole slab experiences the force given by Eq. (26.1).

By omitting explicit reference to that pair of vectors under the principle that Eq. (26.1) holds for <u>some</u> pair of vectors but holds for <u>any</u> pair (i.e. <u>the pair exists, but is not specified</u>), one arrives at the concept

$$\boxed{\underset{\sim}{\vec{F}} = \vec{F}_{ij}\,\frac{dx^i\wedge dx^j}{2!} = \frac{(\text{force})}{(\text{area})} = \begin{pmatrix}\text{surface force}\\ \text{density}\end{pmatrix}} \qquad (26.2)$$

Here

$$\vec{F}_{ij} = \vec{E}\,q\,N\,r^m\,\epsilon_{mij}$$

are the coordinate components of the force on an as-yet-unspecified surface area.

This rank $\binom{1}{2}$ tensor mathematizes a stress field. It acts on all faces of the rigid cube. It has two causal attributes which

1. result in the cube's translational motion, and
2. result in the cube's rotational motion.

II. TRANSLATIONAL EQUILIBRIUM

The dielectric cube with zero total charge will experience a zero total force from the homogeneous stress field of a homogeneous electric field acting on the cube

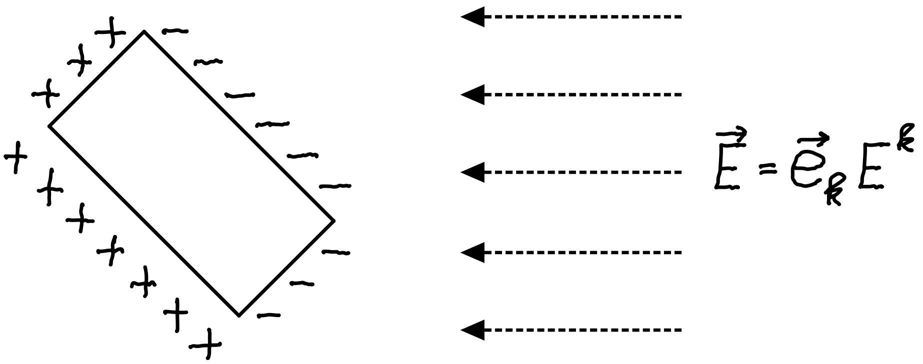


Figure 26.2 The total force due to a homogeneous electric field acting on all faces of the cube is zero.

The sum total of the forces acting on the 6 faces vanishes

$$\boxed{\vec{F}_{total} = \sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\,\text{face}) = 0} \qquad (26.3)$$

The boundary $\partial \mathcal{D}$ of the cube's interior domain consists of the union of its 6 faces,

$$\partial \mathcal{D} = \bigcup_{\ell=1}^{6} (\ell^{th}\,\text{face}),$$

and they come in pairs of opposing faces having opposite orientation. Evaluating $\underset{\sim}{\vec{F}}$ on each pair

$$(\vec{u},\vec{v}),\ (\vec{v},\vec{u});\ (\vec{v},\vec{t}),\ (\vec{t},\vec{v});\ (\vec{t},\vec{u}),\ (\vec{u},\vec{t}),$$

one finds that*

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th}\,\text{face}) = d\underset{\sim}{\vec{F}}(\Delta u\vec{u}, \Delta v\vec{v}, \Delta t\vec{t})$$

* \footnote{ The ensuing line of reasoning parallels the one leading to Eq.(25.2), page 25.2. Evaluate $\underset{\sim}{\vec{F}}$ on each of the six faces. They are located at

$$\mathcal{P}+\delta\mathcal{P}=\{x^\alpha+\Delta t\, t^\alpha\} \quad \mathcal{P}+\Delta\mathcal{P}=\{x^\alpha+\Delta u\, u^\alpha\} \quad \mathcal{P}+d\mathcal{P}=\{X^\alpha+\Delta v\, v^\alpha\}$$

and just $\mathcal{P}=\{x^\alpha\}$ for their opposing faces.

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th} \text{face}) = \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|^{\Delta u \Delta v}_{\mathcal{P}+\{\Delta t\, t^\alpha\}} - \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|^{\Delta u \Delta v}_{\mathcal{P}} + \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Big|^{\Delta v \Delta t}_{\mathcal{P}+\{\Delta u\, u^\alpha\}} - \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Big|^{\Delta v \Delta t}_{\mathcal{P}} + \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Big|^{\Delta t \Delta u}_{\mathcal{P}+\{\Delta v\, v^\alpha\}} - \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Big|^{\Delta t \Delta u}_{\mathcal{P}}$$

$$= \nabla_{\vec{t}}\, \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Delta t \Delta u \Delta v + \nabla_{\vec{u}}\, \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Delta u \Delta v \Delta t + \nabla_{\vec{v}}\, \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Delta v \Delta t \Delta u$$

Use the vectorial version of the 2-3 Stokes' theorem,

$$\nabla_{\vec{t}}\, \underset{\sim}{\vec{\Omega}}(\vec{u},\vec{v}) + \nabla_{\vec{u}}\, \underset{\sim}{\vec{\Omega}}(\vec{v},\vec{t}) + \nabla_{\vec{v}}\, \underset{\sim}{\vec{\Omega}}(\vec{t},\vec{u})$$

$$-\underset{\sim}{\vec{\Omega}}([\vec{u},\vec{v}],\vec{t}) - \underset{\sim}{\vec{\Omega}}([\vec{v},\vec{t}],\vec{u}) - \underset{\sim}{\vec{\Omega}}([\vec{t},\vec{u}],\vec{v}) = d\underset{\sim}{\vec{\Omega}}(\vec{u},\vec{v},\vec{t}),$$

which holds without loss of generality for

$$\underset{\sim}{\vec{\Omega}} = \vec{A}\, df \wedge dg.$$

Consequently,

$$\sum_{\ell=1}^{6} \underset{\sim}{\vec{F}}(\ell^{th} \text{face}) = d\underset{\sim}{\vec{F}}(\vec{u},\vec{v},\vec{t})\Delta u \Delta v \Delta t \quad \}$$

The condition for <u>translational equilibrium</u>, Eq.(26.3), holds for all cubes spanned by triads of vectors such as $\{u, v, t\}$. Consequently, translational equilibrium is mathematized by

$$0 = d\underset{\sim}{\vec{F}} = \left(\vec{F}_{ij,k} + \vec{F}_{jk,i} + \vec{F}_{ki,j}\right) dx^i \wedge dx^j \wedge dx^k \qquad (26.4)$$

Comment 26.1

It is an instructive exercise to show that Eq.(26.4) is equivalent to

$$0 = \vec{F}_{ij;k} + \vec{F}_{jk;i} + \vec{F}_{ki;j}\,. \qquad (26.5)$$

Appendix to Lecture 10 and 26

The vectorial measure of an as-yet-to-be specified area is

$$e_\ell d^2\underset{\sim}{\Sigma}^\ell \equiv d^2\vec{\underset{\sim}{\Sigma}} \equiv d^2\vec{\Sigma} = e_\ell \,\epsilon^\ell{}_{ij}\, \frac{dx^i\wedge dx^j}{2!};$$

We have 1.) $e_\ell\, \epsilon^\ell{}_{ij}\, dx^i\wedge dx^j/2!\, (\vec{u},\vec{v}) \equiv \vec{u}\times\vec{v}$

and 2.) $d\left(e_\ell\, d^2\Sigma^\ell\right) = 0$

PROOF:

$$d\left(e_\ell\, \epsilon^\ell{}_{ij}\, \frac{dx^i\wedge dx^j}{2!}\right) = d\left(e_\ell\, g^{\ell k}\sqrt{g}\,[k\,i\,j]\,\frac{dx^i\wedge dx^j}{2!}\right)$$

$$= \left[de_\ell\; g^{\ell k} + e_\ell\, dg^{\ell k} + e_\ell\, g^{\ell k}\frac{d\sqrt{g}}{\sqrt{g}}\right]\sqrt{g}\,[k\,i\,j]\wedge\frac{dx^i\wedge dx^j}{2!}$$

$$= \Big[\underset{①}{e_n\Gamma^n{}_{\ell r}\, dx^r g^{\ell k}} + \underset{②}{e_\ell(-)g^{\ell r}g^{sk}\, dg_{rs}} + \underset{③}{e_\ell\, g^{\ell k}\frac{d\sqrt{g}}{\sqrt{g}}}\Big]\sqrt{g}\,[k\,i\,j]\,\frac{dx^i\wedge dx^j}{2!}$$

Recall that

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}\left(\sqrt{g}\,u^r\right) = u^r{}_{;r} = u^r{}_{,r} + u^s\,\Gamma^r{}_{sr}$$

$$u^r{}_{,r} + u^r\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^r} = u^r{}_{,r} + u^r\,\Gamma_r{}^s{}_s \Rightarrow \boxed{\Gamma_r{}^s{}_s = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^r} = \frac{1}{2}g^{sm}\left(\cancel{g_{mr,s}} + g_{ms,r} - \cancel{g_{rs,m}}\right)}$$

Thus

$$d\left(e_\ell\, d^2\Sigma^\ell\right) = e_n\Gamma^n{}_{\ell r}\, g^{\ell k}\, dx^r\wedge\sqrt{g}\,[k\,i\,j]\,\frac{dx^i\wedge dx^j}{2!} + ② + ③$$

$$\delta^r{}_k\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$= \underbrace{e_n\Gamma^n{}_{\ell r}\, g^{\ell r}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3}_{①} + ② + ③ = ① + ② + ③$$

$$① = e_n\,\frac{1}{2}g^{nm}\left(g_{m\ell,r} + g_{mr,\ell} - g_{r\ell,m}\right)g^{\ell r}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$= e_n\left(g^{nm}g_{m\ell,r}\,g^{\ell r} - \frac{1}{2}g^{nm}g^{\ell r}g_{r\ell,m}\right)\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$② = e_\ell(-)g^{\ell r}g^{sk}\, dg_{rs}\,\sqrt{g}\,[k\,i\,j]\; dx^i\wedge dx^j/2!$$

$$= -e_\ell\, g^{\ell r}g^{sk}\, g_{rs,p}\sqrt{g}\,[k\,i\,j]\; dx^p\wedge dx^i\wedge dx^j/2!$$

$$= -e_\ell\, g^{\ell r}g^{sk}\, g_{rs,p}\sqrt{g}\;\delta^p{}_k\; dx^1\wedge dx^2\wedge dx^3$$

$$= -e_\ell\, g^{\ell r}g^{sp}\, g_{rs,p}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$= -e_n\, g^{nm}g^{sp}\, g_{ms,p}\sqrt{g}\; dx^1\wedge dx^2\wedge dx^3$$

$$③ = e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}}\, d\sqrt{g} \wedge [k\,ij]\, dx^i \wedge dx^j / 2!$$

$$= e_\ell\, g^{\ell k} \frac{1}{\sqrt{g}}\, \frac{\partial \sqrt{g}}{\partial x^p}\, [k\,ij]\, dx^p \wedge dx^i \wedge dx^j / 2!$$

$$= e_\ell\, g^{\ell k} \frac{1}{2}\, g^{ms}\, g_{ms,p}\, \delta^p_k \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= e_\ell\, g^{\ell k} \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3 ,$$

$$= e_n\, g^{nk} \frac{1}{2}\, g^{ms}\, g_{ms,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$① + ② + ③ = e_n \left(g^{nm}\, g_{m\ell,r}\, g^{\ell r} - \frac{1}{2} g^{nm}\, g^{\ell r}\, g_{r\ell,m} \right) \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$- e_n\, g^{nm}\, g^{sp}\, g_{ms,p} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$+ e_n\, g^{nk} \frac{1}{2}\, g^{\ell r}\, g_{\ell r,k} \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$d\left(e_\ell\, d^2\Sigma^\ell\right) = 0$$

$$d\left(e_\ell\, \epsilon^\ell{}_{ij}\, dx^i \wedge dx^j\right) = 0$$

Lecture 27

Force and Torque on a Dielectric Dipole via Cartan's Calculus

I. Fulcrums, Levers, and Moments of Force

II. Translational Equilibrium and Rotational Non-equilibrium

III. Torque as Moment of Force.

I. Fulcrums, Levers, and Moments of Force:

Force vs. Torque.

Consider a neutrally charged macroscopic body but with a non-zero surface charge, for example, an electret made out of quartz or teflon. Such a body has a dipole moment.

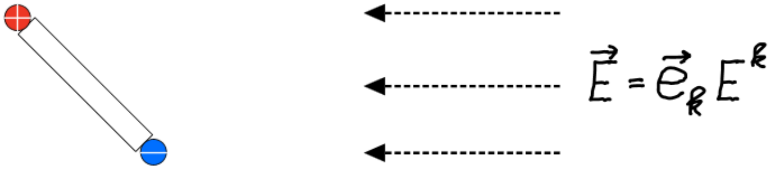


Figure 27.1 Macroscopic dipole in an electri force field.

When subjected to a uniform electric force field, this body will experience no net force, and thus remain in translational equilibrium, but not in rotational equilibrium. This is because the force field subjects the dipole to a "non-zero moment of force", a torque.

Equilibrium or non-equilibrium, the effect of the force field on the body's translation is via the body's surface, while the body's rotation is a volume effect.

II. Cartan's Unit Tensor $d\mathcal{P}$.

Mathematize these effect not only by expressing them in terms of the familiar system of a fulcrum and its levers which extend to the surface areas of the body, but also by doing so in terms of Cartan's unit tensor

$$e_1 dx^1 + e_2 dx^2 + e_3 dx^3 = e_i dx^i = \frac{\partial}{\partial x^i} dx^i \equiv d\mathcal{P} \in \binom{1}{1},$$

which Cartan calls it "the displacement vector."

When it comes to geometrizing the Einstein field equations, it becomes necessary to do so in terms of lever arms and their moments, which in spacetime must be done in terms of Cartan's unit tensor: Its dictionary definition would be

$$\boxed{d\mathcal{P} = \frac{\partial}{\partial x^i} \otimes dx^i = \begin{pmatrix} \text{change of an} \\ \text{as-yet-unspecified} \\ \text{scalar into an} \\ \text{as-yet-unspecified} \\ \text{direction.} \end{pmatrix}}$$

For a specified scalar, say ψ, the change into an as-yet-unspecified direction is

$$d\mathcal{P}(\psi, \ ) = \frac{\partial \psi}{\partial x^i} dx^i = D\psi.$$

For a specified direction, say $\vec{w} = \Delta x^k \frac{\partial}{\partial x^k}$,
the change

$$d\varphi(\psi, \vec{w}) = \langle \frac{\partial \psi}{\partial x^i} dx^i ; \Delta x^k \frac{\partial}{\partial x^k} \rangle = \frac{\partial \psi}{\partial x^i} \delta^i_k \Delta x^k = \Delta x^k \frac{\partial \psi}{\partial x^k} = D_{\vec{w}} \psi \qquad \{27.1\}$$

III. Dielectric Dipole in a Homogeneous Electric Field.

The task of mastering the mathematical method of fulcrums, their lever arms, and their moments in terms of Cartan's unit tensor is achieved most economically with the help of a charged dipole body in a homogeneous electrostatic field.

Consider a cubical dielectric dipole spanned by vectors u, v, and t. Each of the cube's six faces has a uniform surface charge. Being immersed in a uniform electric force field, the charge on each face experiences a force. In spite of the fact that for a dielectric with no net charge the sum total force on all six faces vanishes, there is a non-zero torque.

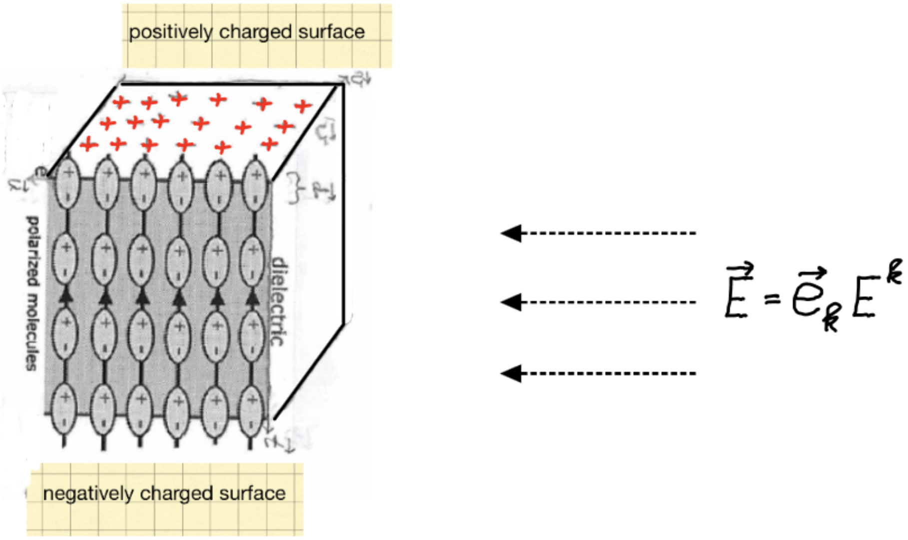


Figure 27.2 Macroscopic cubical dipole subjected to a uniform electric force field. This force force field exerts a force density, uniform but different, on each of the cube's six faces. The resulting torque is proportional to the cube's volume.

IV. Fulcrum and Lever

This torque is mathematized by an arbitrarily placed fulcrum, say $\mathcal{P}'$, and the lever arms emanating from it.

Let $\mathcal{P}_3^\pm$ be two points at the center of two opposing faces of the 3-cube. The fulcrum $\mathcal{P}'$ gives

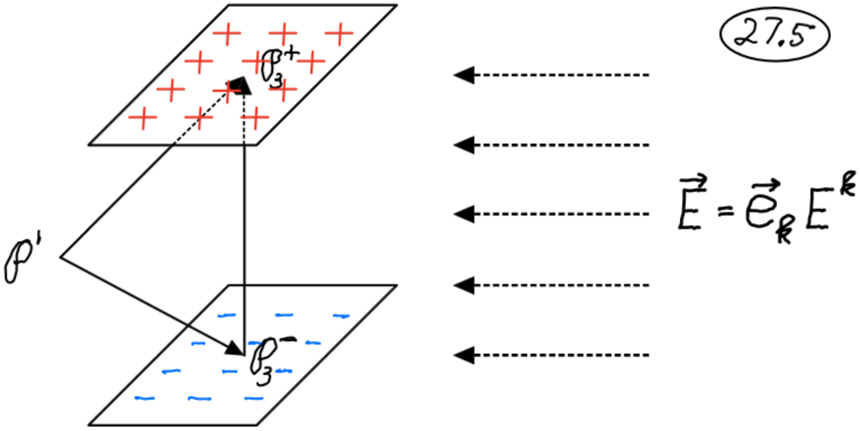


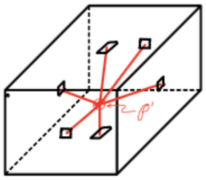
Figure 27.3 The centers between two opposing faces $\mathcal{P}_3^-$ and $\mathcal{P}_3^+$ are separated by the displacement vector

$$\overrightarrow{\mathcal{P}_3^+ - \mathcal{P}_3^-} \equiv \Delta x^3 e_3 .$$

The arbitrarily placed fulcrum point $\mathcal{P}'$ may be inside or outside the 3-cube.

rise to 6 displacement vectors

$$\overrightarrow{\mathcal{P}_i^{\pm} - \mathcal{P}'} ; \quad i = 1,2,3$$



between $\mathcal{P}'$ and the centers of each of the six faces.

V. Moment of Force.

Next consider the two forces acting on each pair of oppositely oriented faces at the end of the two respective levers emanating from the arbitrarily placed fulcrum $\mathcal{P}'$. They determine the moments of force applied to each pair of opposite faces,

$$\left(\overleftrightarrow{\mathcal{T}}\right)_3 = (\mathcal{P}_3^+ - \mathcal{P}')\wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+} + (\mathcal{P}_3^- - \mathcal{P}')\wedge \underset{\sim}{\vec{F}}(\underset{\uparrow}{\vec{v},\vec{u}})\Big|_{\mathcal{P}_3^-}$$

("opposite orientation")

The vector $\underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+} = \vec{F}_{|i<j|}\,dx^i\wedge x^j(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+}$ is the force acting on the face at $\mathcal{P}_3^+$.

The vector $\underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^-} = \vec{F}_{|i<j|}\,dx^i\wedge x^j(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^-}$ is the force acting on the face at $\mathcal{P}_3^-$.

Similar expressions hold for the other four faces. The total moment of force is

$$\overleftrightarrow{\mathcal{T}} = (\mathcal{P}_3^+ - \mathcal{P}_3^-)\wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v}) + (\mathcal{P}_2^+ - \mathcal{P}_2^-)\wedge \underset{\sim}{\vec{F}}(\vec{t},\vec{u}) + (\mathcal{P}_1^+ - \mathcal{P}_1^-)\wedge \underset{\sim}{\vec{F}}(\vec{v},\vec{t}) \qquad (27.1)$$

$$-\mathcal{P}'\underbrace{\left[\underset{\sim}{\vec{F}}(\vec{u},\vec{v})\Big|_{\mathcal{P}_3^+} + \underset{\sim}{\vec{F}}(\vec{v},\vec{u})\Big|_{\mathcal{P}_3^-} + \underset{\sim}{\vec{F}}(\vec{t},\vec{u})\Big|_{\mathcal{P}_2^+} + \underset{\sim}{\vec{F}}(\vec{u},\vec{t})\Big|_{\mathcal{P}_2^-} + \underset{\sim}{\vec{F}}(\vec{v},\vec{t})\Big|_{\mathcal{P}_1^+} + \underset{\sim}{\vec{F}}(\vec{t},\vec{v})\Big|_{\mathcal{P}_1^-}\right]}_{\substack{\| \\ \sum_{\ell=1}^{6}\underset{\sim}{\vec{F}}(\ell^{th}\text{ face}) = \left(\text{total force on 3-cube}\right)}}$$

When the dielectric 3-cube carries a net charge, the total force $\sum_{\ell=1}^{6}\underset{\sim}{\vec{F}}(\ell^{th}\text{face}) \neq 0$. In that case the 3-cube is not in translational equilibrium; it will be pushed away from its initial location, and the moment of force $\overleftrightarrow{\mathcal{T}}$ will depend on the fulcrum $\mathcal{P}'$.

By contrast, if the dielectric carries no net charge,

$$\sum_{\ell=}^{6} \vec{\underset{\sim}{F}}(\ell^{th} face) = 0,$$

the 3-cube will be in translational equilibrium, and the moment of force, Eq.(27.1), will be independent of the location of the fulcrum $\mathcal{P}'$.

VI. Torque as a bivector-valued volume-form.

There are three pairs of opposing faces. Each pair is connected by the respective three pairs of connecting levers. As depicted in Figure 27.3, they are

$$\left.\begin{aligned} \mathcal{P}_3^+ - \mathcal{P}_3^- &= \Delta x^3 \;\; e_3 = \vec{t} \\ \mathcal{P}_2^+ - \mathcal{P}_2^- &= \Delta x^2 \;\; e_2 = \vec{v} \\ \mathcal{P}_1^+ - \mathcal{P}_1^- &= \Delta x^1 \;\; e_1 = \vec{u} \end{aligned}\right\} \qquad (27.2)$$

These, together with Eq.(26.2)* [in Lecture 26], imply that the total moment of force, Eq.(27.1), is

$$\overleftrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}) = \underbrace{\Delta x^3 e_3}_{\vec{t}} \wedge \vec{F}_{|ij|} dx^i \wedge dx^j (\underbrace{\Delta x^1 e_1}_{\vec{u}}, \underbrace{\Delta x^2 e_2}_{\vec{v}}) \qquad e_3 \langle dx^3, t\rangle \wedge \vec{F}_{|ij|} \; dx^i \wedge dx^j (u\, v)$$

$$e_2 \langle dx^2 | \, v \rangle \wedge \vec{F}_{|ij|} \; dx^i \wedge dx^j (t\, u)$$

$$+ \underbrace{\Delta x^2 e_2}_{\vec{v}} \wedge \vec{F}_{|ij|} \, dx^i \wedge dx^j (\underbrace{\Delta x^3 e_3}_{\vec{t}}, \underbrace{\Delta x^1 e_1}_{\vec{u}}) \qquad e_1 \langle dx^1 | \, u \rangle \wedge \vec{F}_{ji} \, dx^j \wedge dx^i (v, t)$$

$$+\underbrace{\Delta x^1 e_1}_{\vec{u}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j (\underbrace{\Delta x^2 e_2}_{\vec{v}}, \underbrace{\Delta x^3 e_3}_{\vec{t}}) \qquad (27.3) \qquad \textcircled{27.8}$$

\footnote{ The vectorial coefficients $\vec{F}_{ij}$ of the 2-form $\underset{\sim}{\vec{F}} = \vec{F}_{ij}\, dx^i \wedge dx^j/2!$ are

$$\vec{F}_{ij} = e_k E^k q N r^m \epsilon_{mij} \quad \}$$

In spite of superficial appearances to the contrary, the expression for the total moment of force, Eq. (27.3), is a coordinate frame invariant. Indeed, using the basis expansions Eq. (27.2), Eq. (27.3) becomes

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}}(\vec{u}, \vec{v}, \vec{t}) = e_m\, dx^m \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j (\vec{u}, \vec{v}, \vec{t}),$$

which in terms of Cartan's unit tensor / "displacement vector"

$$d\mathcal{P} = e_m\, dx^m$$

is

$$\overleftrightarrow{\mathcal{T}}(\vec{u}, \vec{v}, \vec{t}) = d\mathcal{P} \wedge \underset{\sim}{\vec{F}}(\vec{u}, \vec{v}, \vec{t})$$

or

$$\boxed{\begin{aligned} \underset{\sim}{\overleftrightarrow{\mathcal{T}}} &= d\mathcal{P} \wedge \vec{F}_{ij}\, dx^i \wedge dx^j/2! \\ &= e_m \wedge \vec{F}_{ij}\, dx^m \wedge dx^i \wedge dx^j/2! \end{aligned}}$$

This moment of force is a tensor of rank $\binom{2}{3}$. Physically it is the torque exerted by the homogeneous electrostatic field $\vec{E} = e_k E^k$ on a 3-cube spanned by a triad of as-yet-unspecified vectors,

$$\overleftrightarrow{\mathcal{T}} = e_m \wedge e_k E^k q N r^n \epsilon_{n|ij|}\, dx^m \wedge dx^i \wedge dx^j$$

Lecture 28

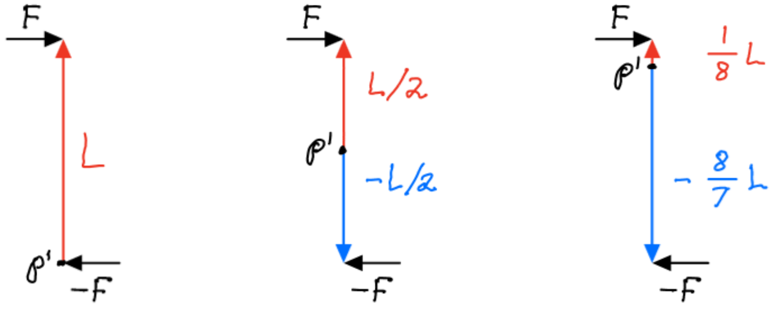
Torque Via Cartan's Moment

I. MOMENT of FORCE as a $\binom{2}{3}$ TENSOR FIELD (a BIVECTOR-valued 3-FORM)

II. MOMENT of FORCE as TORQUE

Read in MTW § 15.3, 15.4, 15.5

As depicted in Figure 28.1, the moment of force on a cube in tranlational equilibrium is independent of the fulcrum location $\mathcal{P}'$.



$$\mathsf{T} = L\wedge F + 0\wedge(-F) = \frac{L}{2}\wedge F + \frac{-L}{2}\wedge(-F) = \frac{L}{8}\wedge F + \left(-\frac{8}{7}L\right)\wedge(-F)$$

Figure 28.1 The moment of force on a 1-dimensional cube in translational equilibrium is independent of the fulcrum location $\mathcal{P}'$.

I. MOMENT of FORCE as a $\binom{2}{3}$ TENSOR FIELD (a BIVECTOR-valued 2-FORM)

A dielectric 3-cube with no net charge but immersed in a homogeneous electrostatic field experiences the moment of force

$$\overleftrightarrow{\mathfrak{T}} = (\mathcal{P}_3^+ - \mathcal{P}_3^-)\wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v}) + (\mathcal{P}_2^+ - \mathcal{P}_2^-)\wedge \underset{\sim}{\vec{F}}(\vec{t},\vec{u}) + (\mathcal{P}_1^+ - \mathcal{P}_1^-)\wedge \underset{\sim}{\vec{F}}(\vec{v},\vec{t}). \tag{28.1}$$

Here

$$\left.\begin{aligned}
\mathcal{P}_3^+ - \mathcal{P}_3^- &\equiv \vec{t} = \Delta x^3 \; e_3 = e_j \langle dx^j, \vec{t}\rangle \\
\mathcal{P}_2^+ - \mathcal{P}_2^- &\equiv \vec{v} = \Delta x^2 \; e_2 = e_i \langle dx^i, \vec{v}\rangle \\
\mathcal{P}_1^+ - \mathcal{P}_1^- &\equiv \vec{u} = \Delta x^1 \; e_1 = e_m \langle dx^m, \vec{u}\rangle
\end{aligned}\right\} \qquad (28.2)$$

are the displacement vectors that separate the opposing faces of the dielectric 3-cube.

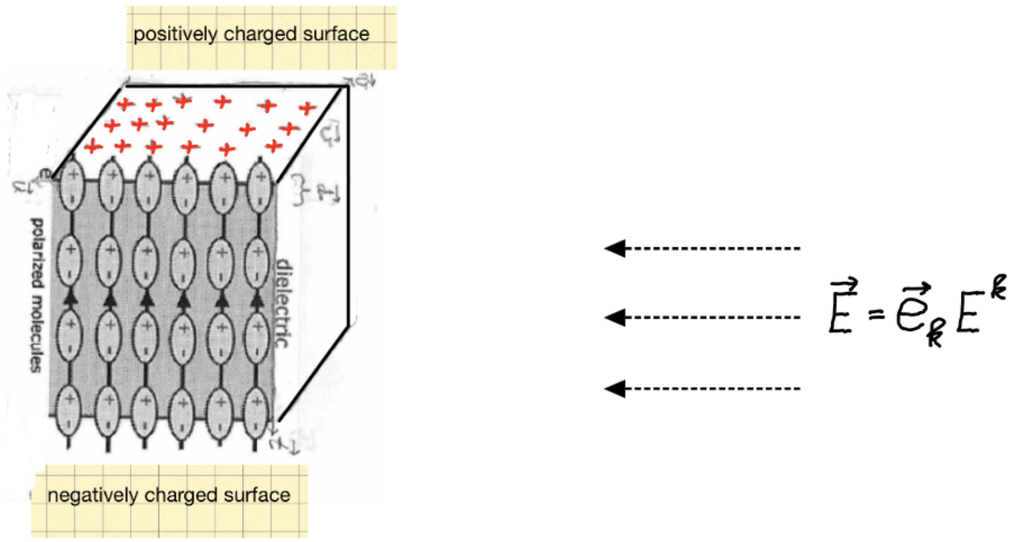


Figure 28.2 Dielectric 3-cube immersed in a homogeneous electrostatic field. The separation between each pair of opposing faces is given by the respective vectors $\vec{u}$, $\vec{v}$, and $\vec{t}$ exhibited by Eq. (28.2).

The introduction of Eq. (28.2) into Eq. (28.1) leads to a non-trivial simplification in the expression for the moment of force, Eq. (28.1).

First of all, recall that all paired forces on the opposing faces of the 3-cube are obtained by evaluating Eq. (26.1'), the surface force density 2-form

$$\underset{\sim}{\vec{F}} = \vec{F}_{ij}\, dx^i \wedge dx^j/2! = e_k E^k q N r^m \epsilon_{mij}\, dx^i \wedge dx^j/2! \qquad (28.3)$$

on the appropriate pair of spanning vectors $(\vec{u},\vec{v})$, $(\vec{v},\vec{t})$, and $(\vec{t},\vec{u})$. Consequently, the expression for the moment of force the 3-cube is subjected to is

$$\overleftrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}) = \underbrace{\Delta x^3 e_3}_{\vec{t}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{u},\vec{v})$$

$$+ \underbrace{\Delta x^2 e_2}_{\vec{v}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{t},\vec{u})$$

$$+ \underbrace{\Delta x^1 e_1}_{\vec{u}} \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{v},\vec{t}) \qquad (28.4)$$

Secondly, in spite of superficial appearances to the contrary, the expression for the total moment of force, Eq. (28.4), is a coordinate frame invariant. Indeed, using the basis expansions

(28.4) .

$$e_3 \Delta x^3 = e_m \langle dx^m, \vec{t} \rangle$$
$$e_2 \Delta x^2 = e_m \langle dx^m, \vec{v} \rangle$$
$$e_1 \Delta x^1 = e_m \langle dx^m, \vec{u} \rangle$$

Eq.(28.4) becomes

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}}(\vec{u},\vec{v},\vec{t}) = \begin{array}{r} e_m \wedge \vec{F}_{|ij|} \\ +\, e_m \wedge \vec{F}_{|ij|} \\ +\, e_m \wedge \vec{F}_{|ij|} \end{array} \boxed{\begin{array}{l} \langle dx^m, t \rangle\, dx^i \wedge dx^j(\vec{u},\vec{v}) \\ \langle dx^m, \vec{v} \rangle\, dx^i \wedge dx^j(\vec{t},\vec{u}) \\ \langle dx^m, \vec{u} \rangle\, dx^i \wedge dx^j(\vec{v},\vec{t}) \end{array}} \rightarrow \boxed{\begin{array}{l} \text{Their sum equals} \\ dx^m \wedge dx^i \wedge dx^j(\vec{u},\vec{v},\vec{t}) \end{array}}$$

$$= e_m\, dx^m \wedge \vec{F}_{|ij|}\, dx^i \wedge dx^j(\vec{u},\vec{v},\vec{t}),$$

which in terms of Cartan's unit tensor / "displacement vector"

$$d\mathcal{P} = e_m\, dx^m$$

is

$$\overleftrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}) = d\mathcal{P} \wedge \underset{\sim}{\vec{F}}(\vec{u},\vec{v},\vec{t})$$

or

$$\boxed{\begin{aligned} \underset{\sim}{\overleftrightarrow{\mathcal{T}}} &= d\mathcal{P} \wedge \vec{F}_{ij}\, dx^i \wedge dx^j/2! \\ &= e_m \wedge \vec{F}_{ij}\, dx^m \wedge dx^i \wedge dx^j/2! \end{aligned}} \qquad (28,6)$$

III. Moment of Force as Torque

(28.5)

The familiar representation of torque is in terms of the vector cross-product

$$\vec{T} = \vec{R} \times \vec{F}.$$

However, the moment of force density

$$\overleftrightarrow{T} = e_m dx^m \wedge \vec{F}_{|ij|} dx^i \wedge dx^j$$

$$= e_m \wedge e_k E^k qN r^n \epsilon_{n|ij|} dx^n \wedge dx^i \wedge dx^j \qquad (28.6)$$

evaluated on the volume of the 3-cube spanned by the triad of vectors $\vec{u}$, $\vec{v}$, and $\vec{t}$ is a bivector.

In spite of their difference, the two representations agree on one key aspect: they are linear spaces with the same dimension,

$$\dim \Lambda^2(E^3) = \dim(E^3).$$

Their bases are

$$\{e_m \wedge e_k : {}^m_k\} = 1,2,3\} \subset \Lambda^2(E^3)$$

and

$$\{e_\ell : \ell = 1,2,3\} \subset E^3.$$

Thus there exists an isomorphism ☆ (a special case of the "Hodge duality" mapping),

$$\star : \Lambda^2(E^3) \longrightarrow E^3$$

$$e_m \wedge e_k \rightsquigarrow \star(e_m \wedge e_k) = e_\ell \epsilon^\ell{}_{mk}$$

$$= e_\ell g^{\ell n} [n m k] \sqrt{g}$$

28.6

Apply this $\star$ tranformation to Eq. (28.6), a bivector-valued 3-form. The result is the vector-valued 3-form

$$\underset{\sim}{\vec{\mathcal{T}}} = \star(\underset{\sim}{\overleftrightarrow{\mathcal{T}}}) = \star(e_m \wedge e_k E^k q N r^n \epsilon_{n|ij|}\, dx^m \wedge dx^i \wedge dx^j)$$

$$= e_\ell \epsilon^{\ell}{}_{mk} E^k q N \; r^n \underbrace{\epsilon_{n|ij|}\,[m\,i\,j]}_{\sqrt{g}\;\delta^m_n} dx^1 \wedge dx^2 \wedge dx^3$$

which reduces to

$$= e_\ell \underbrace{\epsilon^{\ell m k}}_{\frac{1}{\sqrt{g}}[\ell m k]} E_k r_m q N \underbrace{\sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3}_{\text{coordinate invariant}}$$

Evaluate this 3-form on the triad of spanning vectors $(\vec{u}, \vec{v}, \vec{t})$ and obtain

$$= \underbrace{\frac{1}{\sqrt{g}} \begin{vmatrix} e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix}}_{\vec{r} \times \vec{F}} \underbrace{N \begin{pmatrix} \text{volume} \\ \text{spanned} \\ \text{by } \vec{u}, \vec{v}, \vec{t} \end{pmatrix}}_{\#\text{ of dipoles}}$$

This is the moment of force suffered by # dipoles,

$$\# = N \sqrt{g}\, dx^1 \wedge dx^2 \wedge dx^3\, (\vec{u}, \vec{v}, \vec{t}),$$

each subjected to the torque

$$\vec{r} \times \vec{F} = \frac{1}{\sqrt{g}} \begin{vmatrix} e_1 & e_2 & e_3 \\ r_1 & r_2 & r_3 \\ qE_1 & qE_2 & qE_3 \end{vmatrix} .$$

Thus the moment of force applied to the faces of a dielectric 3-cube physically equals (modulo the Hodge isomorphism $\star$) the sum total torque applied to each and everyone of the molecular dipoles occupying the volume of the 3-cube:

$$\utilde{\overleftrightarrow{\mathcal{T}}}(\vec{u},\vec{v},\vec{t}\,) \overset{\star}{\approx} \overrightarrow{\mathcal{T}}(\vec{u},\vec{v},\vec{t}\,) = \star\Big(\big(\vec{r}\times\vec{F}\big)\cdot\#\Big)$$

The face representation $\utilde{\overleftrightarrow{\mathcal{T}}}$ of the stressed dielectric 3-cube is related to its volume representation $\utilde{\overrightarrow{\mathcal{T}}}$ by means of the Hodge isomorphism $\star$.

Lecture 29

Electric Field induced Moment of Force
vs
Curvature induced Moment of Rotation

I. Moment of Force vs. Torque

  A. Moment of force

  B. Moment of rotation

II. Rotational "$\vec{F}=m\vec{a}$"

  vs

  Einstein Field Equation

III Two equivalent momenergy representations

In MTW read all of Ch. 15.

I. Moment of Force vs. Torque

A homogeneous electrostatic field subjects a polarized dielectric 3-cube to a moment of force, which is mathematized by the bivectorial 3-form

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}} = d\mathcal{P} \wedge \underset{\sim}{\vec{F}} = e_\ell \wedge e_k F^k_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j.$$

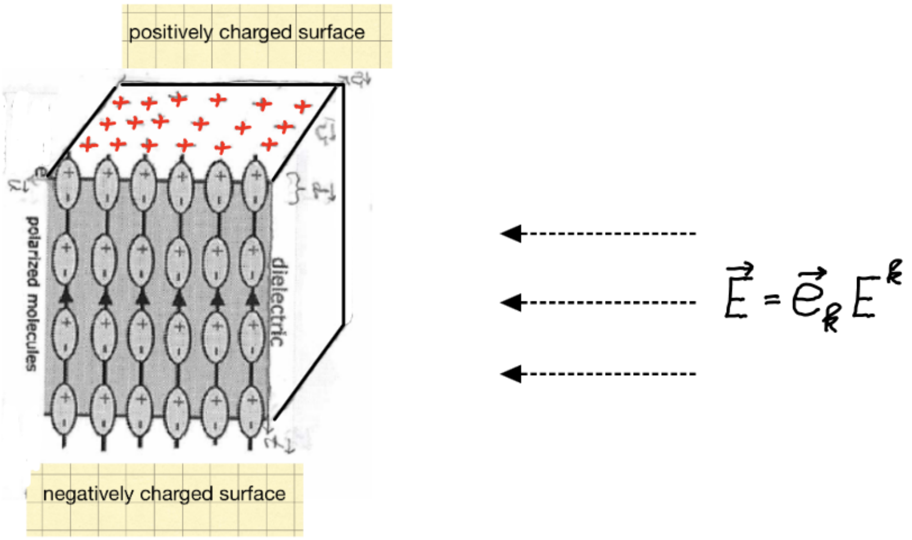


Figure 29.1 The surface charges on the boundary of a polarized dielectric 3-cube interacting with the electrostatic field $\vec{E}$ cause the 3-cube to be subjected to a moment of force.

Although the cause of this force moment is confined to the 3-cube's surface, its magnitude is proportional to the volume, and hence to the number of molecular dipole moments $q\vec{r}$ of the 3-cube. Indeed, the vectorial torque on the 3-cube spanned by the 3-d vectors $\vec{u}$, $\vec{v}$, and $\vec{t}$ is

$$\vec{\underset{\sim}{\mathcal{T}}}(\vec{u},\vec{v},\vec{t}) = \star\left(\overleftrightarrow{\underset{\sim}{\mathcal{T}}}(\vec{u},\vec{v},\vec{t})\right) = \underbrace{\frac{1}{\sqrt{g}}\begin{vmatrix} e_1 & e_2 & e_3 \\ q r_1 & q r_2 & q r_3 \\ E_1 & E_2 & E_3 \end{vmatrix}}\underbrace{N \times \begin{pmatrix}\text{volume}\\ \text{spanned}\\ \text{by } \vec{u},\vec{v},\vec{t}\end{pmatrix}}$$

$$= \left(q\vec{r}\times\vec{E}\right) \times \left(\#\text{ of dipoles}\right)$$

page 29.3 does not exist

29.4

I. B) 4-d SPACETIME

The generalization to 4-d space is now straight and proceeds as follows:

1. 3-cube in E^3	1, 3-cube in 4-d spacetime
2.  Vector-valued $\Sigma\vec{E}$ force field density	2,  Bivector-valued, curvature-induced rotation field density
$\underset{\sim}{\vec{F}} = \vec{e}_k F^k_{\;|ij|}\, dx^i \wedge dx^j$	$\underset{\sim}{\overleftrightarrow{\mathcal{R}}} = e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta$
3, Translational equilibrium	3. Bianchi identity
a) $\sum_{\ell=1}^{6} \vec{F}(\ell^{th}\text{ face}) = 0$	a) $\sum_{\ell=1}^{6} \overleftrightarrow{\mathcal{R}}(\ell^{th}\text{ face}) = 0$
b) $d\underset{\sim}{\vec{F}} = 0$	b) $d\overleftrightarrow{\mathcal{R}} = 0$
c) $F^k{}_{[ij;n]} = 0$	c) $R^{\lambda\mu}{}_{[\alpha\beta;\gamma]} = 0$
4. Moment of Force (volume)	Moment of Rotation / volume
$\underset{\sim}{\overleftrightarrow{\mathcal{T}}} = d\mathcal{P} \wedge \underset{\sim}{\vec{\mathbb{F}}}$	$d\mathcal{P} \wedge \overleftrightarrow{\mathcal{R}} =$
$= e_\ell \wedge e_k F^k_{\;|ij|}\, dx^\ell \wedge dx^i \wedge dx^j$	$= e_\nu \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta$
$\vec{\mathcal{T}} = \bigstar(\overleftrightarrow{\mathcal{T}})$	$\bigstar(d\mathcal{P} \wedge \overleftrightarrow{\mathcal{R}}) = e_\sigma \epsilon_{\nu\lambda\mu}{}^{\sigma} R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu \wedge dx^\alpha \wedge dx^\beta = \frac{8\pi G}{c^2} * \vec{\mathbb{T}}$
$= e_m \epsilon^m{}_{\ell k} F^k_{\;|ij|}\, dx^\ell \wedge dx^i \wedge dx^j$	$= \frac{8\pi G}{c^2} e_\sigma T^{\sigma\tau} \epsilon_{\tau|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$

II. EINSTEIN'S FIELD EQ'N

A.) The geometrical statement of the Einstein field equations is

$$\boxed{\frac{\begin{pmatrix}\text{Moment of}\\ \text{rotation}\end{pmatrix}}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}} = \frac{8\pi G}{c^4}\,\frac{(\text{Momenergy})}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}}}$$

The mathematized version of this statement is

$$\boxed{d\mathcal{P}\wedge \underset{\sim}{\overset{\leftrightarrow}{R}} = \frac{8\pi G}{c^4}\,\star^{-1}({*}\mathbb{T})}$$

or, equivalently

$$\boxed{\star\left(d\mathcal{P}\wedge \underset{\sim}{\overset{\leftrightarrow}{R}}\right) = \frac{8\pi}{c^4}\,{*}\mathbb{T}}$$

The momenergy-valued 3-form is a bulk property.

III. Momenergy/volume has two equivalent representation

1.) As a vector-valued 3-volume density

$$*\mathbb{T} = e_\rho T^{\rho\sigma} \underbrace{\epsilon_{\sigma\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma}_{d^3\Sigma_\sigma = (\text{invt vol})_\sigma}$$

and

2.) as a tri-vector valued 3-volume density

$$\star^{-1}(*\mathbb{T}) = e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu}{}_\rho T^{\rho\sigma} d^3\Sigma_\sigma$$

This equivalence is based on the isomorphism between two 4-dimensional linear spaces

$$\begin{bmatrix} \star \\ \star^{-1} \end{bmatrix} : R^4 \wedge R^4 \wedge R^4 \underset{\star^{-1}}{\overset{\star}{\rightleftarrows}} R^4$$

$$e_\nu \wedge e_\lambda \wedge e_\mu \rightsquigarrow \star(e_\nu \wedge e_\lambda \wedge e_\mu) = \epsilon_{\nu\lambda\mu}{}^\rho e_\rho$$

$$e_\rho \rightsquigarrow \star^{-1}(e_\rho) = \frac{1}{3!} e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda M}{}_\rho$$

$$\star\star^{-1}(e_\rho) = e_\rho$$

B.) Component formulation of the E.F.E.:

$$d\mathcal{P}\wedge \underset{\sim}{\overset{\leftrightarrow}{\mathcal{R}}} = e_\nu\wedge e_\lambda\wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu\wedge dx^\alpha\wedge dx^\beta = \frac{8\pi G}{c^4} \star^{-1}(*\mathbb{T})$$

$$= \frac{8\pi G}{c^4}\, \frac{e_\nu\wedge e_\lambda\wedge e_\mu}{3!}\, \epsilon^{\nu\lambda\mu}{}_\rho T^{\rho\sigma} d^3\Sigma_\sigma$$

OR equivalently

$$\star\left(d\mathcal{P}\wedge \underset{\approx}{\overset{\leftrightarrow}{\mathcal{R}}}\right) = \epsilon_{\nu\lambda\mu}{}^{\rho} e_\rho R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\nu\wedge dx^\alpha\wedge dx^\beta$$

$$= \frac{8\pi G}{c^4} e_\rho T^{\rho\sigma} d^3\Sigma_\sigma$$

Introduce

$R^{\gamma\mu}{}_{\gamma\nu} \equiv R^\mu{}_\nu$ (Ricci)

$R^\delta{}_\delta = R$ (Curvature invt)

and obtain

$$\boxed{R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = \frac{8\pi G}{c^4} T^\mu{}_\nu}$$

APPENDIX TO LECTURE 29

5.) Hodge dual on Euclidean space E^3	Hodge dual on Minkowski spacetime R^4
a) $\star : \Lambda^2(E^3) \rightarrow E^3$	a) $\star : \Lambda^3(R^4) \longrightarrow R^4$
$e_\ell \wedge e_k \rightsquigarrow \star(e_\ell \wedge e_k) =$ $= e_n \epsilon^n{}_{\ell k}$	$e_\nu \wedge e_\lambda \wedge e_\mu \rightsquigarrow \star(e_\nu \wedge e_\lambda \wedge e_\mu) =$ $= \epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma$
$\star(d\mathcal{P} \wedge \underset{\sim}{\vec{F}}) =$ $= \vec{e}_n \epsilon^n{}_{\ell k} F^k{}_{\vert ij \vert} dx^\ell \wedge dx^i \wedge dx^j$	$\star(d\mathcal{P} \wedge \underset{\sim}{\overleftrightarrow{R}}) =$ $= \epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma R^{\lambda\mu}{}_{\vert\alpha\beta\vert} dx^\nu \wedge dx^\alpha \wedge dx^\beta$
b) Inverse Hodge dual	b) Inverse Hodge dual
$\star^{-1} : e_m \rightsquigarrow \star^{-1}(e_m) =$ $= \frac{1}{2!} e_\ell \wedge e_k \epsilon^{\ell k}{}_m$	$\star^{-1} : e_\rho \rightsquigarrow \star^{-1}(e_\rho) =$ $= \frac{-1}{3!} e_\nu \wedge e_\lambda \wedge e_\mu \epsilon^{\nu\lambda\mu}{}_\rho$
c) $\star\star^{-1}(e_m) = \frac{1}{2!} e_n \epsilon^n{}_{\ell k} \epsilon^{\ell k}{}_m$ $= e_n \delta^n{}_m$ (identity!)	$\star\star^{-1}(e_\rho) = \frac{-1}{3!} \epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma \epsilon^{\nu\lambda\mu}{}_\rho$ $= (+)\delta^\sigma_\rho e_\sigma$ (identity!)
d) $\star^{-1}\star(e_\ell \wedge e_k) = \star^{-1}(e_n \epsilon^n{}_{\ell k})$ $= \frac{1}{2!} e_i \wedge e_j \epsilon^{ij}{}_n \epsilon^n{}_{\ell k}$ $= \frac{1}{2!} e_i \wedge e_j \delta^{ij}_{\ell k}$ $= e_\ell \wedge e_k$ (identity!)	$\star^{-1}\star(e_\nu \wedge e_\lambda \wedge e_\mu) = \star^{-1}(\epsilon_{\nu\lambda\mu}{}^\sigma e_\sigma)$ $= \epsilon_{\nu\lambda\mu}{}^\sigma \frac{-1}{3!} e_\alpha \wedge e_\beta \wedge e_\gamma \epsilon^{\alpha\beta\gamma}{}_\sigma$ $= \frac{1}{3!} e_\alpha \wedge e_\beta \wedge e_\gamma \delta^{\alpha\beta\gamma}_{\nu\lambda\mu}$ $= e_\nu \wedge e_\lambda \wedge e_\mu$ (identity!)

Lecture 30

EFEs: Moment of Rotation per 3-volume = Momenergy per 3-volume

I. The Einstein Field Equations.

Einstain started his process of mathematizing gravitation in 1907 when he introduced two fundamental concepts: (1) An accelerated frame as a one-parameter family of instantaneous inertial ("free float") frames, mathematicians nowadays call "tangent spaces," and (2) his equivalence principle, according to which the behavior of things in a uniformly accelerated frame, e.g. the free motion of bodies in a rocket), is indistinguishable from the behavior of things in a local uniform gravitation field; inother words "inertial" forces are indistinguishable of forces due to gravition.

He took his next fundamental step in 1913 when he realized that, to mathematize (a) the source of gravitation and (b) the motion of bodies under its influence, one must do so in geometrical terms, namely using the methods developed by Gauss, Riemann, Ricci, Levi-Civita, and others.

By a subsequent tour de force he arrived in 1915 at his field equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^2}T_{\mu\nu}. \qquad (30.1)$$

His line of reasoning was based on his 1913 recognition that his equations had to be tensorial in nature. His guiding principle was based on the nature and the conservation laws of the r.h.s. of his equation, the momentum and energy ("momenergy") tensor, which is the source of the gravitational field.

Indeed, by applying his 1913 recognition to the Poisson equation for the Newtonian gravitational potential,

$$\nabla^2 \phi_{NEWTON} = 4\pi G \rho$$

he made three inferences:

(i) from his mass energy relation, he inferred that the source of that equation,

$$\nabla^2 \phi_{NEWTON} = \frac{4\pi G}{c^2} \rho c^2$$

is the mass-energy density, which is only part of the momentum-energy ("momenergy") tensor, and

(ii) from the geometrization of Newton's first law of motion relative to non-inertial reference frames plus his equivalence priciple, he inferred that, for weak gravitational fields, the Newtonian potential ϕ_{NEWTON} is related (see Eq.(5.9) on page 5.13) to the g_{00} component of the space time tensor

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

by the equation

$$g_{00} = -1 - \frac{2G}{c^2}\phi_{NEWTON}.$$

(iii) from the fact that the only tensor involving the 2nd derivatives of the metric tensor is the Riemann-Christoffel curvature tensor, he inferred that what he must look for is a tensorial equation equation for the components of the metric, based on the curvature tensor with the momergy tensor as the source.

Its a consequence of these three inferences
the tensorial l.h.s. of his equations
was a mathematically deductive consequence.
As to its geometrical and physical nature, that was
left in a shroud of mystery until E. Cartan in 1925
from a geometrical perspective, and J.A. Wheeler in 1964 (with help
from his student C.W. Misner) from a physics perspective,
identify the l.h.s. of Eq.(30.1) as the moment of curvature-induced
rotation. Because of this, the E.F.Eq'ns state a causal
relationship between gravitation and mater:
For any given volume element in spacetime,

$$\text{Moment of rotation} = \frac{8\pi G}{C^2}\,\text{Momenergy}$$

or

$$\frac{\begin{pmatrix}\text{Moment}\\ \text{of rotation}\end{pmatrix}}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}} = \frac{8\pi G}{C^2}\,\frac{(\text{Momenergy})}{\begin{pmatrix}\text{Spacetime}\\ \text{3-volume}\end{pmatrix}} \tag{30.2}$$

The line of reasoning leading to this statement of the E.F.E.s is to start with well-known concepts from mechanics and electrostatic, mathematize them in term of differential forms, and then extend them from 3-d Euclidean space to the 4-d spacetime.

II. Moment of Electrostatic Force

A dielectric with non-zero polarization when immersed into the force field of a homogeneous electrostatic

$$\vec{E} = e_k E^k$$

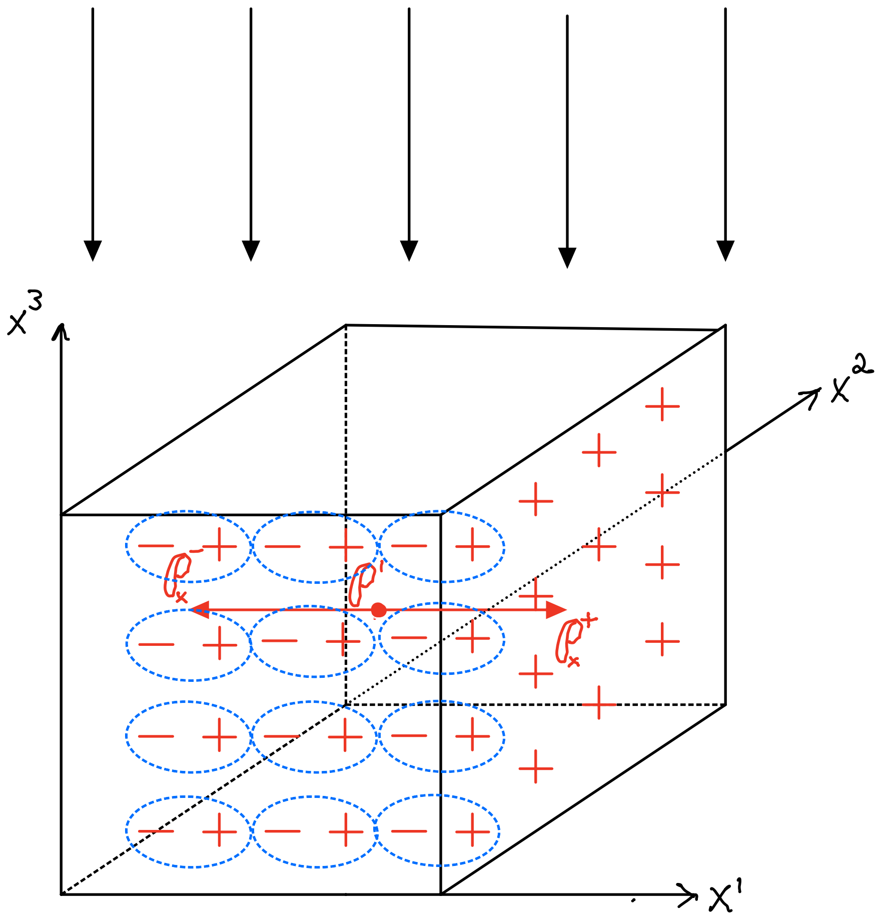


Figure 30.1 As depicted, a force couple acts on a pair of oppositely charged opposing faces of a polarized dielectric cube immersed in the homogeneous electrostatic field $\vec{E} = e_k E^k$.
The vectorial separation $\mathcal{P}^+ - \mathcal{P}^-$ is a lever arm. Together with the two opposite forces of the couple, it furnishes the moment of force, which tends to make the cube rotate.

field, as depicted in Figure 30.1, gets subjected to a shear force couple. It acts on the oppositly charged areas of the opposing pair of faces. These two forces are mathematized by evaluating the vectorial surface force density $\left(\frac{(\text{force})}{(\text{area})}\right)$

$$\underset{\sim}{\vec{F}} = e_k F^k_{|ij|}\, dx^i \wedge dx^j = e_k E^k q N r^\ell \epsilon_{\ell|ij|}\, dx^i \wedge dx^j \qquad (30.3)$$

on each of the two faces.
The displacement vector

$$\mathcal{P}^+_x - \mathcal{P}^-_x = d\mathcal{P}(e_1 \Delta x^1) = e_\ell\, dx^\ell(e_1 \Delta x^1) = e_\ell \langle dx^\ell | e_1 \rangle \Delta x^1 = e_1 \Delta x^1,$$

which separates the two faces, or — for that matter separates any other pair of faces, is mathematized by Cartan's "unit tensor"

$$d\mathcal{P} = e_\ell\, dx^\ell = e_\ell\, \delta^\ell_j \otimes dx^j \qquad (30.4)$$

This vectorial 1-form is a vector which refers to an as-yet-unspecified displacement away from its fulcrum. Together with the force, Eq. (30.3) it forms a new mathematical concept*, namely that of the moment of force,

* \footnote{The formation of the concept "moment of force" is illustrate pictorially in Figure 28.1. For a philosophically precise explanation on how to form a concept see Chapter 2 in "Introduction to Objectivist Epistemlogy" by Ayn Rand. The "Conceptual Common Denominator" in that chapter is all instances of the moment of force possess a fulcrum as a common feature, fulcrum whose particular location exists but can be omitted from explicit reference to the concept "moment of force". This irrelevance of the particular of the particular location of the fulcrum is illustrated very graphically in Figure 27.3 and 28.1 and is also known as the principle of measurement omission in the theory of concept formation.

$$\underset{\sim}{\overleftrightarrow{\mathcal{T}}} = d\mathcal{P}\wedge \underset{\sim}{\vec{F}} \tag{30.5}$$

$$= e_\ell\, dx^\ell \wedge \underset{\sim}{\vec{F}} = e_\ell \wedge e_k\, F^k_{|ij|}\, dx^\ell \wedge dx^i \wedge dx^j \tag{30.6}$$

This is a bivector-valued 3-form. Evaluate it on the three vectors that span the volume of the 3-cube depicted in Figure 30.1. The result is a linear combination of bivectors. They are elements of a linear space which is 3-dimensional. It follows that this space is isomorphic to a 3-dimensional space of vectors. There is a one-to-one correspondence between bivectors and vectors. This correspondence ("isomorphism") is unique. It is a special case of the Hodge duality map. It maps elements of area spanned by a pair of 3-d vectors into a vector perpendicular to that area. It is mathematized by the definition

$$\left.\begin{array}{ll} \star : \Lambda^2(E^3) \longrightarrow E^3 & (\text{"genus"}) \\ e_\ell \wedge e_k \rightsquigarrow \star(e_\ell \wedge e_k) = e_m \epsilon^m{}_{\ell k} & (\text{"differentia"}) \end{array}\right\} \quad (30.7)$$

Apply this isomorphic map to the moment of density ("shear force per unit area") and obtain

$$\star(\overleftrightarrow{\underset{\sim}{T}})=\star(d\mathcal{P}\wedge\vec{\underset{\sim}{F}})$$

30.9

$$=e_m\epsilon^m{}_{\ell k}F^k_{ij}\,dx^\ell\wedge dx^i\wedge dx^j/2!\equiv\vec{\underset{\sim}{T}}\qquad(\text{"}\underline{\text{torque}}\text{ 3-form"})$$

(30.8)

This is the vector ("torque")-valued volume form.

The vectorial coefficient of this 3-form is readily obtained by noting that

$$dx^\ell\wedge dx^i\wedge dx^j=[\ell\, i\, j]\,dx^1\wedge dx^2\wedge dx^3.$$

Consequently, the torque 3-form is

$$\vec{\underset{\sim}{T}}=e_m g^{mn}\sqrt{g}\,[n\,\ell\,k]\,F^k_{ij}\,[\ell\,i\,j]\,dx^1\wedge dx^2\wedge dx^3/2! \qquad (30.9)$$

Sum over the repeated indices of the product of the two permutation symbols. The result is the generalized Kronecker delta:

$$[n\,\ell\,k][\ell\,i\,j]=(-)\delta^{ij}_{nk}=(-)\begin{vmatrix}\delta^i_n & \delta^i_k\\ \delta^j_n & \delta^j_k\end{vmatrix}=-(\delta^i_n\delta^j_k-\delta^i_k\delta^j_n)$$

It follow that the vectorial 3-form is

$$\vec{\underset{\sim}{T}}=e_m\,g^{mn}(-)F^k{}_{kn}\sqrt{g}\,dx^1\wedge dx^2\wedge dx^3$$

or explicitly, using Eq. (30.3)

$$\vec{\underset{\sim}{T}}=e_m\,\epsilon^m{}_{\ell k}\,r^\ell E^k qN\sqrt{g}\,dx^1\wedge dx^2\wedge dx^3$$

$$=\frac{Nq}{g}\begin{vmatrix}e_1 & e_2 & e_3\\ r_1 & r_2 & r_3\\ E_1 & E_2 & E_3\end{vmatrix}\sqrt{g}\,dx^1\wedge dx^2\wedge dx^3 \qquad (30.10)$$

III. Moment of Curvature-induced Rotation.

The above mathematical method of moments, which Cartan introduced in terms of his differential forms, applies also to higher dimensional spaces, including spacetime, which is 4-dimensional. This application is mandatory if one wishes to understand gravitation on a level that approaches that of electromagnetism*

*\footnote{See Box 15.1.H - 15.1.I in MTW}

However, understanding the meaning of Einstein's tensor on the l.h.s. of Eq. (30.1) requires a different type of moment, namely the moment of curvature-induced rotation which is depicted in Figure 30.2 below.

Parallel transport is the means for comparing vectors at different events of spacetime and hence for mathematizing their changes. Being compatible with parallel transport, the Lorentz metric at each event is the means for expressing these changes in terms of inner products and their angles of rotation.

A vector parallel transported around the boundary of a face gets rotated by an angle such as the purple one depicted in the red parallelograms in Figure 30.2. In that figure this curvature-induced rotation for the $\vec{v}$-$\vec{t}$ spanned face is

$$\underset{\sim}{\overset{\leftrightarrow}{\mathcal{R}}}(\vec{v},\vec{t}) = e_\lambda \wedge e_\mu \, R^{|\lambda\mu|}{}_{|\alpha\beta|} \, dx^\alpha \wedge dx^\beta(\vec{v},\vec{t}) \qquad (30.11)$$

This rotation is a linear combination of bivectors, each one in the plane spanned by the pair e_λ and e_μ; the angle of rotation is

$$\underset{\sim}{\theta}^{\lambda\mu}(\vec{v},\vec{t}) = R^{\lambda\mu}{}_{|\alpha\beta|} \, dx^\alpha \wedge dx^\beta(\vec{v},\vec{t}). \qquad (30.12)$$

The vectorial change due to parallel transporting the vector $\vec{w}$ around the area spanned by vectors $\vec{v}$ and $\vec{t}$ is

$$\Delta^2\vec{w} = e_\lambda \wedge e_\mu \, \underset{\sim}{\theta}^{\lambda\mu}(\vec{v},\vec{t}) \cdot \vec{w}$$

This is a rotational change in $\vec{w}$. Because of this, one has the

Definition

Given the metric compatible parallel transport whose curvature operator is $\mathcal{R}(\vec{u},\vec{v})\vec{w} \equiv \left[\nabla_{\vec{u}}\nabla_{\vec{v}} - \nabla_{\vec{v}}\nabla_{\vec{u}} - \nabla_{[\vec{u},\vec{v}]}\right] w = e_\lambda R^{\lambda}{}_{\mu\alpha\beta}\, u^\alpha v^\beta w^\mu$ (30.12)

then

$$\boxed{e_\lambda \wedge e_\mu\, \theta^{\lambda\mu} = e_\lambda \wedge e_\mu\, R^{|\lambda\mu|}{}_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta = \text{"ROTATION"}}$$

is called the <u>rotation</u> due to parallel transport around the boundary of the oriented area spanned by two as-yet-unspecified vectors.

Comment

From its very inception, the concept "curvature" has had its observational roots in curves, curved surfaces, and other entities that exist in a higher dimensional environment. The additional dimension(s) that accommodate the existence of lower dimensional entities are perfectly valid in that they have their basis in the world observed and processed by one's faculties.

However, in trying to apply the concept "curvature" to spacetime, one is confronted with a metaphysical (i.e. pertaining to the nature of reality) problem: there does not exist a higher dimensional environment or additional dimensions that would warrant the conclusion: spacetime is "curved" in the same observational and measurable sense that the trajetory of a particle or the surface of a soap bubble is curved. A higher dimensional environment with a fictitious extra dimension simply does not exist in reality. However, the introduction of a fictitious additional dimension quite often is a valid mathematical method for conceptualing in quantitative form (see Figure 33.2 in Lecture 33-34) the properties of the domain in one's focus. But one must not equate a mathematical method with metaphysics.

To avoid the dangers of such epistemic confusions and be logically precise, one needs to abstain from misleading nomenclature such as "the curvature of spacetime," from implied dimension(s), fictitious and not rooted in the world. By contrast, rotations, Euclian or Lorentzian, are processes that are part and parcel of the physical world. Unlike fictitious dimensions, such rotations are observable in the domain of space and time. Subsuming such rotations, "the holonomy of spacetime" is their condensation in the context of gravitation. The mathematization of this holonomy is consigned to the Appendix to Lecture 30.

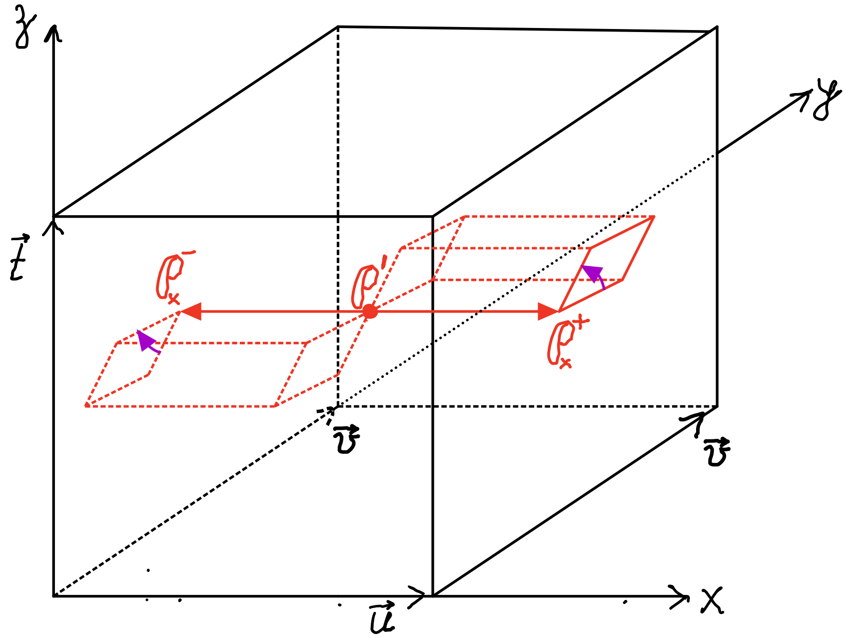


Figure 30.2 Moment of rotation induced by the curvature permeating two opposing faces of a 3-cube.

ρ' = "fulcrum", an arbitrarily located point in or near the 3-cube.

There are two lever arms emanating from the arbitrarily located fulcrum point P';

$$\overrightarrow{P^{+}-P'} \text{ and } (P^{-}-P'),$$

and terminating at P^{+} and P^{-} on two opposing faces of the 3-cube in Figure 30.2. However, their difference,

$$(P^{+}-P')-(P^{-}-P')=P^{+}-P^{-}$$

is independent of the fulcrum P'. Instead, it is a lever arm that connects the two opposing.

IV. Cartan's Unit Tensor as a Fulcrum-Based Lever arm.

From the collection of lever arms ("displacement vectors") in a coordinate neighborhood, focus on those that emanate from a common fulcrum P'.

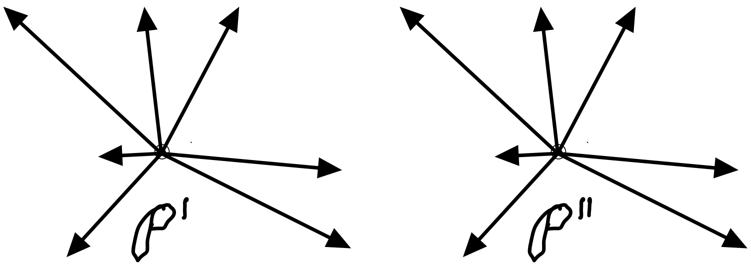


Figure 30.3 The domain of $d\mathcal{P}\big|_{\mathcal{P}'}$ is the vector space $V_{\mathcal{P}'}$, while that of $d\mathcal{P}\big|_{\mathcal{P}''}$ is the vector space $V_{\mathcal{P}''}$ at $\mathcal{P}''$.

These lever arms are mathematized quite trivially in terms of Cartan's unit tensor at fulrum $\mathcal{P}'$,

$$d\mathcal{P}\big|_{\mathcal{P}'}: \quad V_{\mathcal{P}'} \longrightarrow V_{\mathcal{P}'}$$

$$\mathcal{P}-\mathcal{P}' = \Delta x^{\tau}\frac{\partial}{\partial x^{\tau}} \rightsquigarrow d\mathcal{P}(\mathcal{P}-\mathcal{P}') = \overbrace{e_{\sigma}\, \langle dx^{\sigma}}^{d\mathcal{P}}, \overbrace{\Delta x^{\tau}\frac{\partial}{\partial x^{\tau}}}^{\mathcal{P}-\mathcal{P}'}\rangle = e_{\sigma}\Delta x^{\tau} = \mathcal{P}-\mathcal{P}'$$

One says that $d\mathcal{P}\big|_{\mathcal{P}'}$ is a vector at $\mathcal{P}'$ that refers to an as-yet-unspecified displacement away from $\mathcal{P}'$.

$d\mathcal{P}$ at fulcrum $\mathcal{P}'$ is to be contrasted with $d\mathcal{P}$ at a different fulcrum $\mathcal{P}''$.

V. Moment of Rotation as a Vector-valued volume form.

$$\overset{\leftrightarrow}{\underset{\sim}{R}} = e_{\mu}\wedge e_{\nu}\, R^{|\mu\nu}{}_{|\alpha\beta|}\, dx^{\alpha}\wedge dx^{\beta}$$

The flux of rotation flux featured by

The curvature of spacetime

That general relativity puts forth the proposition that gravitation is due to the "curva-

ture of spacetime" hinges on the premise that
implies

If spacetime grips matter, telling it how to move, then it is not surprising to discover that matter grips spacetime, telling it how to curve ... to understand this ... let's imagine what free-float spacetime-driven motion would look like if spacetime were not curved.

Every object in free float would would not enjoy the company of the Sun. Each would float away on its own proud, disregarding course. Conceivable though such a universe is, it is not the universe that we know. Faced with this difficulty, we could give up the idea that spacetime tells mass how to move. But if we want to retain this idea, despite the observed curvature of planetary orbits and the identical curvature of a ball and a bullet through spacetime, we will say with Einstein, that spacetime is curved. Moreover, this curvature is greater at and within the Earth than it is far away from the Earth.

In brief, mass grips spacetime, telling it how to curve.

Parallel transport is the means for comparing vectors at different events of spacetime and hence for mathematizing their changes. Being compatible with parallel transport, the Lorentz metric at each event is the means for expressing these changes in terms of inner products and their angles of rotation.

Lecture 31 & 32

The 2plus2 Decomposition
of
Spacetime

I Spherically Symmetric Tensor Fields

A tensor field is said to be symmetric if it is invariant under the transformation generated by the vector field

$\xi^\mu(x^\alpha)\frac{\partial}{\partial x^\mu}$,

$$\begin{aligned} x^\alpha &\longrightarrow x'^\alpha = x^\alpha + \varepsilon\,\xi^\alpha(x^\gamma) \\ x'^\gamma &\longrightarrow x^\gamma = x'^\gamma - \varepsilon\,\xi^\gamma(x'^\alpha)\ , \quad \varepsilon \ll 1. \end{aligned} \qquad (31.1)$$

Thus, whenever

$$v_\mu(x^\alpha)\, d(x^\mu + \varepsilon\,\xi^\mu(x^\gamma)) = v_\mu(x^\alpha)\, dx^\alpha$$

$$g_{\mu\nu}(x^\alpha + \varepsilon\,\xi^\alpha)\, d(x^\mu + \varepsilon\,\xi^\mu(x^\gamma))\, d(x^\nu + \varepsilon\,\xi^\nu(x^\gamma)) = g_{\mu\nu}(x^\alpha)\, dx^\mu dx^\nu$$

$$v^\mu(x'^\gamma)\frac{\partial}{\partial x'^\mu} \equiv v^\mu(x^\gamma + \varepsilon\,\xi^\gamma)\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial}{\partial x^\gamma} = v^\mu\frac{\partial}{\partial x^\mu}$$

to first order in ε, one says that the covector field $v_\mu\, dx^\mu$, the tensor field $g_{\mu\nu}\,dx^\mu dx^\nu$, and the vector field $v^\mu\frac{\partial}{\partial x^\mu}$ are invariant under the ξ^μ-generated transformation.

From their representations relative to the coordinate system

$$\{x^\mu : \underbrace{x^0,\ x^1,}_{\substack{\text{longitudinal}\\ \text{coordinates}\\ \{x^A:\ A=0,1\}}}\ \underbrace{x^2 = \theta,\ x^3 = \varphi}_{\substack{\text{transverse}\\ \text{coordinates}\\ \{x^a:\ a=\theta,\varphi\}}}\}$$

one infers that the

$$\xi^\mu_\theta \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \theta} \text{ and } \xi^\mu_\varphi \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \varphi}$$

generated transformations are symmetry transformations. This is because they leave each of the following geometrical and physical tensor fields in spacetime invariant.

1. Metric tensor:

$$g_{\mu\nu}dx^\mu dx^\nu = g_{AB}(x^C)\,dx^A dx^B + \overbrace{r^2(x^C)\underbrace{(d\theta^2+\sin^2\theta\, d\varphi^2)}_{\gamma_{ab}\,dx^a dx^b}}^{g_{ab}\,dx^a dx^b} : \left[\begin{array}{c|cc} g_{AB} & 0 & 0 \\ & 0 & 0 \\ \hline 0\ 0 & & \\ 0\ 0 & \multicolumn{2}{c}{r^2\gamma_{ab}} \end{array}\right]$$

$g_{AB}(x^C)\,dx^A dx^B$

$r^2(x^C)$

$\gamma_{ab}\,dx^a dx^b = d\theta^2 + \sin^2\theta\, d\varphi^2$

2. Momenergy tensor:

$$t_{\mu\nu}dx^\mu dx^\nu = t_{AB}(x^C)\,dx^A dx^B + \overbrace{\underbrace{t(x^C)\,r^2(d\theta^2+\sin^2\theta\, d\varphi^2)}_{t\,g_{ab}\,dx^a dx^b}}^{t_{ab}\,dx^a dx^b} : \left[\begin{array}{c|cc} t_{AB} & 0 & 0 \\ & 0 & 0 \\ \hline 0\ 0 & & \\ 0\ 0 & \multicolumn{2}{c}{t\,g_{ab}} \end{array}\right]$$

$t_{AB}(x^C)\,dx^A dx^B$

$t\,(x^C)$

3. General covector:

$$v_\mu\, dx^\mu : \begin{bmatrix} v_A \\ 0 \\ 0 \end{bmatrix}$$

4. The Klein-Gordon field equation

$$\Box\psi - \frac{m^2c^2}{\hbar^2}\psi \equiv \left(g^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\right)_{;\nu} - \lambda_c^{-2}\psi$$

$$= \left(g^{AB}\frac{\partial\psi}{\partial x^A}\right)_{|A} + \frac{1}{r^2}\left(\gamma^{ab}\frac{\partial\psi}{\partial x^a}\right)_{:b} - \frac{1}{\lambda^2}\psi$$

$$= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left[\sqrt{-g}\, g^{AB}\frac{\partial\psi}{\partial x^B}\right] + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\varphi^2}\right\} - \frac{1}{\lambda_c^2}\psi$$

Let $\psi = \psi_{\ell m}(x^a)\, Y_\ell^m(\theta,\varphi)$ be a spherical normal mode solution. It satisfies

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\varphi^2} = -\ell(\ell+1)\psi$$

The amplitude $\psi_{\ell m}(x^a)$ satifies

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left[\sqrt{-g}\, g^{AB}\frac{\partial\psi_{\ell m}}{\partial x^B}\right] - \left(\frac{\ell(\ell+1)}{r^2} + \frac{1}{\lambda_c^2}\right)\psi_{\ell m} = 0$$

4. Einstein tensor

$$G_{\mu\nu}\,dx^\mu dx^\nu \equiv \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) dx^\mu dx^\nu \qquad \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{G_{AB}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{G_{ab}} \\ 0 & 0 & & \end{array}\right]$$

$$= G_{AB}\,dx^A dx^B + \underbrace{G_{ab}}_{\frac{1}{2} G_d{}^d g_{ab}} dx^a dx^b$$

$$G_{AB} = \frac{1}{r^2}\left\{-2\, r_{;A|B} + \left(2\, r\, r_{;c}{}^{|c} + r_{;c}\, r^{;c} - 1\right) g_{AB}\right\}$$

$$G_{ab} \equiv \tfrac{1}{2} G_d{}^d\, g_{ab} = \left(\frac{r_{;c}\, r^{;c}}{r} - \mathcal{R}\right) g_{ab}$$

where $\mathcal{R}$ is the Gaussian curvature defined by

$${}^{(2)}R^{AB}{}_{CD} = \mathcal{R}\left(\delta^A_C\, \delta^B_D - \delta^A_D\, \delta^B_C\right)$$

5. The Einstein field equations

$$G_{AB} = -2\, r\, r_{;A|B} + g_{AB}\left(2\, r\, r_{;c}{}^{|c} + r_{;c}\, r_{;D}\, g^{CD} - 1\right) = \frac{8\pi G}{c^2}\, r^2\, t_{AB}$$

$$\tfrac{1}{2} G_a{}^a = \frac{r_{;C|D}\, g^{CD}}{r} - \mathcal{R} = \frac{8\pi G}{c^2}\, t$$

6. The conservation equation

$$G_\mu{}^\nu{}_{;\nu} \equiv 0 = t_\mu{}^\nu{}_{;\nu}$$

$$\left(r^2\, G_A{}^B\right)_{|B} - r\, r_{,A}\, G_a{}^a = 0$$

$$\left(r^2\, t_A{}^B\right)_{|B} - r\, r_{,A}\, 2t = 0$$

Lecture 32 & 33

Integration of the Einstein Field
Equations using
a conservation law

I. The 2+2 Decomposition of Spacetime with Spherical Symmetry

Given a spherically symmetric system, split its dynamics, (which is governed by the E.F.E.s) into two subsystems:

One coordinatized by the spherical coordinates on the 2-sphere S^2, the "transverse manifold";
the other by a radial and time coordinate, the "longitudinal manifold" M^2. The result of this split is that the spacetime manifold M^4 got factored into two submanifolds:

$$M^4 = M^2 \times S^2.$$

The metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ in the representation which reflects this decomposition is

$$ds^2 = g_{AB}(x^a)\, dx^A dx^B + r^2(x^c)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Thus, all spherically symmetric gravitational systems are mathematized by two degrees of freedom on M^2,

$g_{AB}(x^a)\, dx^A dx^B$ ("metric tensor field on M^2")
$r^2(x^a)$ (scalar field on M^2)

The Einstein field equations $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \underbrace{t_{\mu\nu}}$

t_{00} in units of energy density

have the same type of decomposition:

$$\boxed{r^2 G_{AB} = -2 r \, r_{,A|B} + g_{AB} \left(2 r \, r_{,c}{}^{|C} + r_{,c} \, r_{,D} \, g^{CD} - 1 \right) = \frac{8\pi G}{c^4} r^2 t_{AB}} \qquad (32.1)$$

("tensor field equation on M^2)

$$\boxed{G_{ab} = \left(\frac{r_{,C|D} \, g^{CD}}{r} - \mathcal{R} \right) r^2 \gamma_{ab} = \frac{8\pi G}{c^4} t \, r^2 \gamma_{ab}} \qquad (32.2)$$

where ${}^{(2)}R^{AB}{}_{CD} = \left(\delta^A_C \delta^B_D - \delta^A_D \delta^B_C \right) \mathcal{R}$.

("scalar field equation on M^2)

II. Partial integration of the E.F.E.s for spherically symmetric system.

The gravitational field equations can be integrated in part by combining the Bianchi identity*

$$\left(r^2 G_A{}^B \right)_{|B} - r \, r_{,A} \left(\frac{r_{,C|D} \, g^{CD}}{r} - \mathcal{R} \right) = 0$$

with the E.F.E., Eq. (32.1)

* \footnote{ which is the only vectorial identity on M^2, which resulted from the 2+2 decomposition of the Bianchi identidy $G_\mu{}^\nu{}_{;\nu} = 0$.}

The process consists of 3 steps.

Step 1.

Multiply $r^2 G_A{}^B$ by $-\frac{1}{2} r_{,C}\, \epsilon^{CA}$, obtain the M^2 vector

$$-\tfrac{1}{2} r_{,C}\, \epsilon^{CA}\, r^2\, G_A{}^B \equiv J^B, \qquad (32.3)$$

and find that its divergence vanishes:

$$J^B{}_{|B} = 0. \qquad (32.4)$$

Step 2.

The divergence condition, Eq. (32.4), implies that there exist a scalar ψ on M^2 with a gradient whose components

$$\psi_{,E} = -J^B\, \epsilon_{BE}\,. \quad (\text{"conservative" vector field on } M^2)$$

This conclusion is based on integrating the ordinary first order differential equation implied by Eq. (32.4).

Indeed, recall that the covariant divergence of any vector field can always be expressed in terms of an ordinary divergence

$$0 = J^B{}_{|B} \equiv \frac{1}{\sqrt{-g}}\, \frac{\partial\left(\sqrt{-g}\, J^B\right)}{\partial x^B}$$

$$= \frac{1}{\sqrt{-g}} \left[\underbrace{\frac{\partial\left(\sqrt{-g}\, J^0\right)}{\partial x^0}}_{\equiv\ \partial M/\partial x^0} + \underbrace{\frac{\partial\left(\sqrt{-g}\, J^1\right)}{\partial x^1}}_{\equiv\ \partial (-) N/\partial x^1} \right].$$

Here

$$\text{and}\quad \begin{aligned} M &= \sqrt{-g}\,J^0 \\ N &= -\sqrt{-g}\,J^1 \end{aligned} \qquad (32,5)$$

The vanishing of $J^B{}_{|B} = 0$ guarantees that $M\,dx^1 + N\,dx^0$ is an exact differential.

"Exact differential" means that there exists a scalar ψ such that

$$M\,dx^1 + N\,dx^0 = \frac{\partial \psi}{\partial x^1}dx^1 + \frac{\partial \psi}{\partial x^0}dx^0 \qquad (32,6)$$

This is because $J^B{}_{|B} = 0$ implies that

$$\frac{\partial M}{\partial x^0} = \frac{\partial N}{\partial x^1}\,.$$

It follows that

$$\frac{\partial \psi}{\partial x^1} = M = \sqrt{-g}\,J^0$$

and

$$\frac{\partial \psi}{\partial x^0} = N = -\sqrt{-g}\,J^0$$

or

$$\boxed{\frac{\partial \psi}{\partial x^E} = -J^B \epsilon_{BE}} \qquad (32,7)$$

Step 3.

Find the scalar ψ by applying the boxed Eq. (32.7) to Eq. (32.3).

$$\frac{\partial \psi}{\partial x^E} = \frac{1}{2} r_{,C}\, \epsilon^{CA}\, r^2\, G_A{}^B \epsilon_{BE} \left(= -J^B \epsilon_{BE}\right) \qquad (32,8)$$

Insert Eq. (32.1), the expression for

$$r^2 G_A{}^B = -2r\,r_{,A}{}^{|B} + \delta_A^B\left(2r\,r_{,C}{}^{|C} + r_{,C}\,r_{,D}\,g^{CD} - 1\right),$$

simplify and find

$$\frac{\partial \Psi}{\partial x^E} = \left[\frac{1}{2}\,r\,(1 - r_{,D}\,r^{|D})\right]_{,E} \qquad (32.9)$$

The 3-step mathematical deducton that the $G_A{}^B$ expression on the right hand side of Eq. (32.8) is a conservative vector field is a step forward in integrating the E.F.E.s. The scalar whose gradient is this field is by inspection

$$\Psi = \frac{1}{2}\,r\,(1 - r_{,C}\,r_{,D}\,g^{CD}). \qquad (32.10)$$

The r-r coefficient of the inverse metric is therefore

$$g^{rr} = 1 - \frac{2\Psi}{r}. \qquad (32.11)$$

III. Conservation of Spherical Mass-energy.

The physical meaning of the scalar function $\Psi(x^0, x^1 = r)$ is furnish by the E.F.E.s.

On one hand the r.h.s. of Eq. (32.9) features the gradient of this scalar; on the other hand Eq. (32.8) is, via the E.F.E.s, proportional to the momergy density,

$$\frac{\partial \psi}{\partial x^E} = -\frac{1}{2} r^2 r_{,C}\, \epsilon^{CA}\, \frac{8\pi G}{c^4}\, t_A{}^B \epsilon_{BE}. \qquad (32.12) \qquad \boxed{32.6}$$

The equality of these two mathematizes the conservation of gravitational mass-energy. No matter how violent and complex the spherical process, that mass-energy is conserved.

Indeed, being the gradient of a scalar, the line integral of the r.h.s. of Eq. (32.12),

$$\Psi(x^0;x^1) = \int^{(x^0;x^1)} \frac{\partial \psi}{\partial x^E} dx^E = -\frac{1}{2}\int^{(x^0;x^1)} r_{,C}\, \epsilon^{CA} r^2 \frac{8\pi G}{c^4} t_A{}^B \epsilon_{BE}\, dx^E, \qquad (32.13)$$

is independent of its path between two fixed events in M^2, the 2-d longitudinal spacetime manifold.* The

\footnote{Physically each of its events $(x^0;x^1)$ is associated with a sphere of area $4\pi r^2(x^0;x^1)$.}

integral depends only on its end point-events, and therefore vanishes over a closed path.

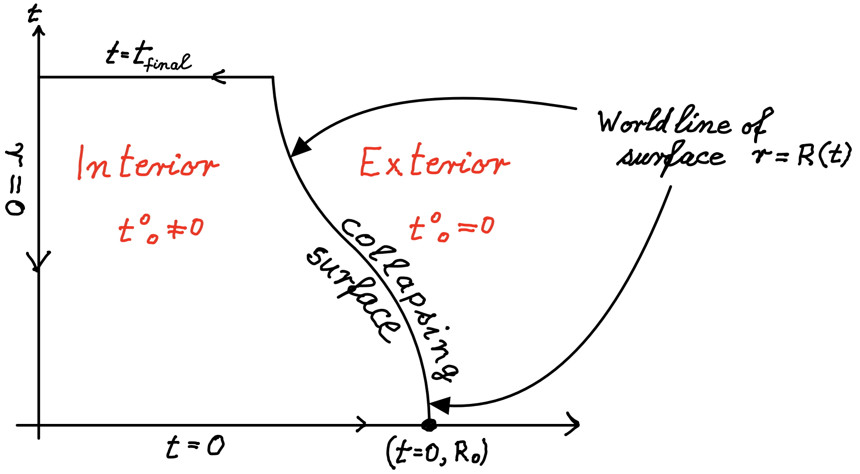


Figure 32.1 A closed contour integral whose initial ($t=0$) and final ($t=t_{final}$) spatial line integral yield the conserved mass M

The integrand vanishes at $x'=r=0$.

It also vanishes beyond the matter-vacuum interface: For $r > R(t)$, where $t_0{}^0=0$.

$$t_0{}^0(t,r)=\begin{cases}\neq 0 & r<R(t) \quad \text{INSIDE}\\ = 0 & R(t)<r \quad \text{OUTSIDE}\end{cases}$$

$$\psi(t=0,\ r=R_0)=\psi\left(t=t_{fin},\ R(t_{fin})\right)$$

$$\approx -\frac{4\pi G}{c^4}\int_0^{R(t)} \epsilon^{r0}\, r^2\, t_0{}^0\, \epsilon_{0r}\, dr$$

The $t_0{}^0$ component of the momenergy tensor is $(-)\rho c^2$, the negative of the mass-energy density,

$$t_0{}^0=(-)\rho(r,t)c^2,$$

where ρ is the mass density. Thus

$$\psi(t,\ R(t)) = \frac{G}{c^2}\int_0^{R(t)} 4\pi r^2 \rho(r)\, dr$$

$$= \frac{G}{c^2}\times\left(\begin{array}{l}\text{Conserved mass}\\ \text{enclosed by a}\\ \text{collapsing sphere}\end{array}\right)$$

$$= \frac{G}{c^2}\, m(t,\ R(t))$$

Conclusions

1. $g^{rr} = 1 - \frac{2G}{c^2} m(t,r)$, the inverse metric coefficient, Eq. (32.11) ⑶②.⑧

where

$$\psi(r;t) = \frac{G}{c^2}\, m(r,t) = \frac{G}{c^2}\int_0^r 4\pi\, \rho(r';t)\, r'^2\, dr'$$

is the mass function in geometrical units, according to Eq. (32.10) units of length. For example,

$$M_\odot = 2\times10^{33}\,gr = 1.5\,km$$

$$M_\oplus = 6\times10^{27}\,gr = .44\,cm$$

2. The mass function ψ is the value of the <u>path independent</u> line integral in M^2:

$$\boxed{\psi(x^0,x^1) = \int^{(x^0,x^1)} d\psi = \frac{G}{c^4}\int_0^{(x^0,x^1)} 4\pi\, r^2(x^C)\, t_0^{\ 0}(x^G)\, \frac{\partial r}{\partial x^E}\, dx^E}$$

Lecture 33 & 34

Non-Euclidean Geometry in
the
Equatorial Plane of a Star

In MTW's chapter 23 read Section 23.8

I. Geometry of spacetime for a static star:

The spacetime geometry for any spherically symmetric system has the form

$$ds^2 = -e^{2\phi(r,t)}dt^2 + \frac{dr^2}{1-\frac{m(r,t)}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

For a system which is also static, there is no time dependence. Its spatial geometry at any fixed time is therefore

$$ds^2\Big|_{t=\text{fixed}} = \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

In the equatorial plane $\theta = \frac{\pi}{2}$ it is

$$ds^2\Big|_{\substack{t=\text{fixed}\\ \theta=\frac{\pi}{2}}} \equiv d\sigma^2 = \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2\, d\varphi^2 \qquad \text{(Non-Euclidean)}$$

which is to be compared with

$$ds^2 = dr^2 + r^2 d\varphi^2 \qquad \text{(Euclidean)}$$

A. Imbedding Space

To obtain a geometrical picture of this non-Euclidean geometry, use the method of the imbedding diagram according to which one views the non-Euclidean plane as a surface of revolution in a 3-d fictitious imbedding space with a Euclidean geometry and spanned by its three coordinates z, r, and φ:

$$dl^2 = dz^2 + dr^2 + r^2 d\varphi^2 \qquad \text{("metric for the imbedding space")}$$

On the to-be-found surface of revolution

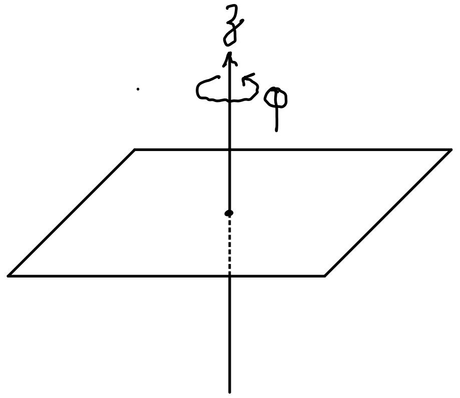


Figure 33.1 Fictitious 3-d imbedding space induces a non-Euclidean surface of revolution.

$z = f(r)$

the ambient Euclidean geometry induces the metric

$$dl^2\Big|_{z=f(r)} = \left[\left(\frac{dz}{dr}\right)^2 + 1\right] dr^2 + r^2 d\varphi^2 .$$

("metric for the surface of revolution")

B. The Imbedding Function

Identify the metric on the to-be-found surface of revolution with the metric on the equatorial plane of the spherically symmetric spacetime. This results in the differential equation

$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1 - \frac{2m(r)}{r}}$$

The solution to this differential equation

$$z(r) = {}_0\!\int^r \left[\frac{2m(r')}{r-2m(r')}\right]^{1/2} dr' + \text{const.} \qquad (33.1)$$

yields a 2-d surface of revolution from a 3-d perspective.

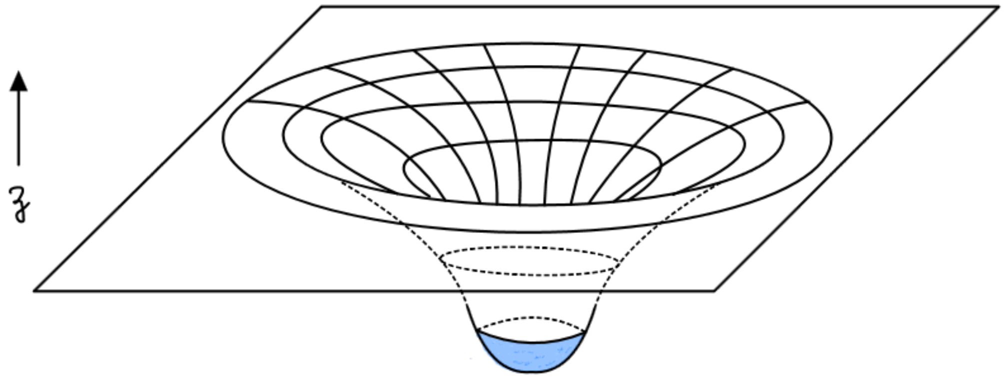


Figure 33.2 Imbedding diagram for the equatorial plane of a homogeneous star.

It allows one to visualize the inner 2-d spatial geometry on the equatorial plane or – because of spherical symmetry – any other rotated plane of the spherically symmetric space.

C. Example

Consider at some fixed time ($t = \text{const.}$) a star with mass density $\rho(r)$ in its interior and vacuum on the outside.

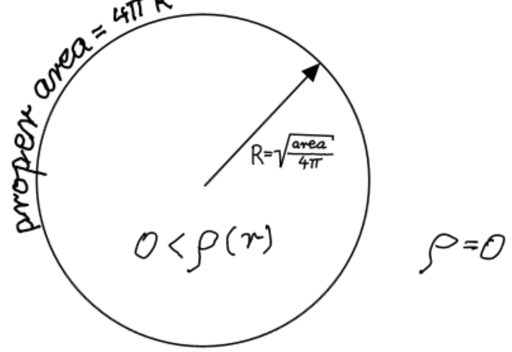


Figure 33.3 The radial parameter R for the concentric spheres of the star's interior and exterior is quantified in term of their proper area: $R = \sqrt{area/4\pi}$.

For such a configuration the mass function and its associated imbedding function are

$$m(r) = \begin{cases} \int_0^r 4\pi r'^2 \rho(r')\,dr' & \text{inside}: \quad r < R \\ M & \text{outside}: \quad r > R \end{cases}$$

and

$$z(r) = \begin{cases} \int_0^r \left[\frac{2m(r')}{r - 2m(r')}\right]^{1/2} dr' + c & \text{inside}: r < R \qquad (33.2) \\ \left[8M(r-2M)\right]^{1/2} + c & \text{outside}: r > R \qquad (33.3) \end{cases}$$

Comment

Here the mass M and the mass density $\rho(r)$ are expressed in term of geometrical units:

$$M = \frac{G}{c^2} M^{\text{conventional}} = \left[\frac{G}{c^2}(\text{mass})\right] = [\text{length}]$$

$$\rho = \frac{G}{c^2}\rho^{\text{conventional}} = \left[\frac{G}{c^2}\frac{(\text{mass})}{(\text{length})^3}\right] = \left[\frac{1}{(\text{length})^2}\right]$$

a) Thus <u>outside</u> the star one has

$$(z-c)^2 = 8M(r-2M)$$

which is a parabola of revolution.

b) <u>Inside</u> the star, near the center

$$m(r) = \frac{4\pi\rho_c}{3} r^3$$

The geometrized mass density has units $\frac{1}{(\text{length})^2}$.
Consequently, the density ρ_c implies a geometrically determined standard of length designated by $\underline{a}$:

$$\frac{8\pi}{3}\rho_c \equiv \frac{1}{a^2}.$$

With this scale factor the imbedding function **brings into sharp focus the essence the nature of the geometry** in the central region inside the star:

$$z = \int_0^r \sqrt{\frac{\left(\frac{r'}{a}\right)^2}{1-\left(\frac{r'}{a}\right)^2}}\, dr' = -a\sqrt{1-\left(\frac{r'}{a}\right)^2}\,\Bigg|_0^r$$

$$= a - \sqrt{a^2 - r^2} \qquad \text{for } r \ll a \text{, near the center}$$

Thus the imbedding function $z(r)$ is part of the circle of revolution:

$$(z-a)^2 + r^2 = a^2$$

c) At the star's boundary

$$\frac{dz}{dr} = \sqrt{\frac{2m(r)}{r - 2m(r)}}$$

is continuous because $m(r)$ is continuous.

The geometry of a star is therefore characterized by a circle of revolution near its center, and a parabola of revolution outside its interior joined to its surface $r = R$

without any kinks. This is because $m(r)$ is continuous there. Figure 33.4 depicts via, the imbedding diagram for a homogeneous star, the equatorial plane with its non-Euclidean geometry.

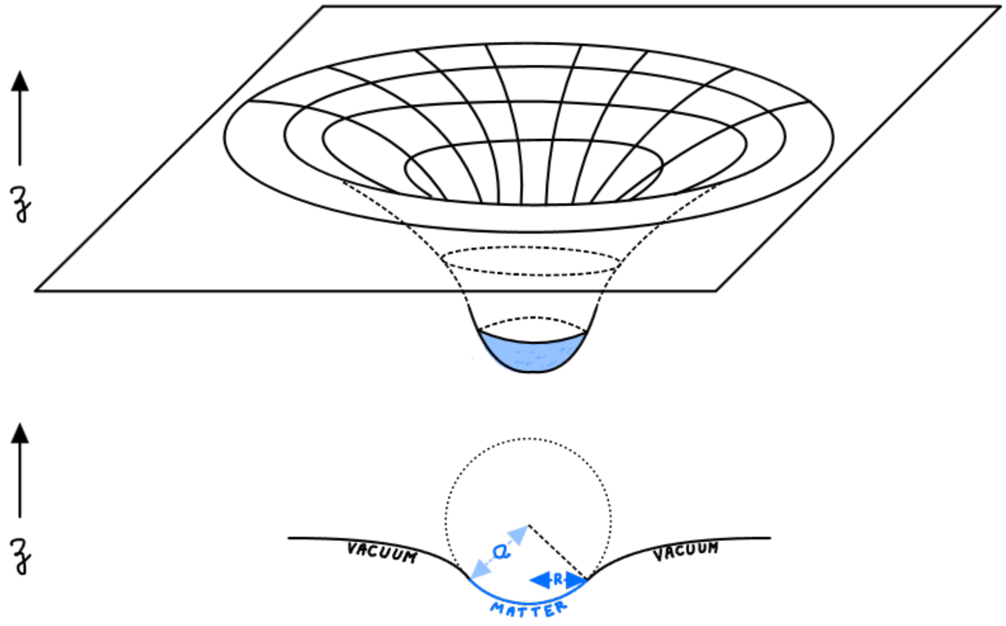


Figure 33.4 The imbedding for the equatorial plane, here the one of a homogeneous star; highlights its non-Euclidean nature both inside and outside the star:

The non-Euclidean nature of the equatorial plane inside the star has its basis in the physical world by comparing two lengths measurements.

1. From the surface of the star drill a hole all the way to its center. Drop a plumb line from the surface to the center. Its proper length ℓ is

$$\ell=\int_0^R \sqrt{g_{rr}}\,dr=\int_0^R \frac{dr}{\sqrt{1-\frac{2m(r)}{r}}} > R$$

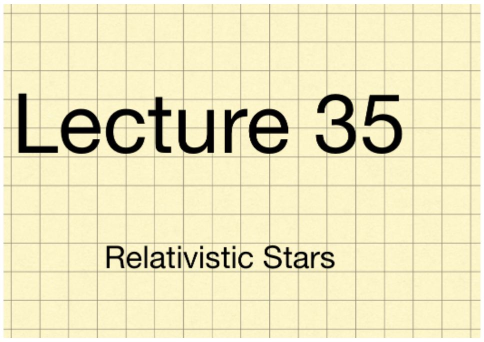
2. Measure the proper equatorial circumference of the star. Given that its surface area is $4\pi R^2$, that circumference $\mathcal{C}$ is

$$\mathcal{C}=\int_0^{2\pi} R\,d\varphi = 2\pi R$$

It is a fact that there is a mismatch between the determination of the circumference based on radial measurements and that based on circum-navigating the star, i.e. that

$$\frac{2\pi\ell}{\mathcal{C}}=\frac{\int_0^R \frac{dr}{\sqrt{1-\frac{2m(r')}{r'}}}}{R} > 1\ ,$$

This is due to the non-Euclidean nature of the equatorial plane of the star.



Lecture 35

Relativistic Stars

In MTW's chapter 23 read Sections 23.2-23.7

I. SPHERICALLY SYMMETRIC SYSTEMS

It is a fact that there exists a multitude of gravitating system which are spherically symmetric.

How does one classify them?

All such systems are governed by the Einstein field equations adapted to spherical symmetry

$$-2r\, r_{,A}{}^{|B} + \delta_A{}^B\left(2r\, r_{,c}{}^{|c} + r_{,c}\, r^{|c} - 1\right) \equiv r^2 G_A{}^B = \frac{8\pi G}{c^4} r^2\, t_A{}^B \qquad (35.1)$$

$$\left(\frac{r_{,c}{}^{|c}}{r} - R\right)\delta_a{}^b \equiv G_a{}^b = \frac{8\pi G}{c^4}\, t\, \delta_a{}^b \qquad (35.2)$$

together with the implied hydrodynamicl Euler equations of motion

$$\left(r^2 t_A{}^B\right)_{|B} - r\, r_{,A}\, t = 0. \qquad (35.3)$$

Here

$$\left[t_\mu{}^\nu\right] = \left[\begin{array}{c:c} t_A{}^B & \bigcirc \\ \hdashline \bigcirc & t\,\delta_a{}^b \end{array}\right],$$

are the components of the momenergy tensor relative to the coordinate frame which reflects the rotational symmetries of the metric tensor field.

How does one distinguish such gravitating systems?

Achieve this task by applying it solving the above equations for a particular spherical star.

II. RELATIVISTIC STAR

Consider a spherical self-gravitating system consisting of a perfect fluid (no viscosity!). The components of its momenergy are (see Lecture 16)

$$t_\mu{}^\nu = (p+\rho)\, u_\mu u^\nu + p\, \delta_\mu{}^\nu = \begin{cases} T_A{}^B = (p+\rho)\, u_A u^B + p\, \delta_A^B \\ T_a{}^b = p_\perp \delta_a^b = \text{xverse pressure} \end{cases}$$

Here p, ρ, and u^μ are the pressure, energy density, and 4-velocity components of the fluid. Their distribution in the star is governed by the law of momenergy conservation

$$t_\mu{}^\nu{}_{;\nu} = 0 \; ; \quad u^\mu t_\mu{}^\nu{}_{;\nu} = 0$$

These are the equations that govern the dynamics of a relativistic fluid.

For a spherically symmetric configuration there is a single vectorial equation on $M^2 = M^4/S^2$:

$$t_c{}^\nu{}_{;\nu} - u_c u^B t_B{}^\nu{}_{;\nu} \equiv$$

$$\equiv u_{c|B} u^B (p+\rho) - \left(\delta_c{}^B + u_c u^B\right) \frac{\partial p}{\partial x^B} = 0 \quad c = 0,1 \qquad (35.4)$$

The metric for any spherically symmetric configuration (by an appropriate choice of coordinates) has the diagonal form

$$g_{\mu\nu}\,dx^\mu dx^\nu = \underbrace{-e^{2\Phi(t,r)}dt^2 + e^{2\Lambda(t,r)}dr^2}_{g_{AB}\,dx^A dx^B} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Focus on a star in equilibrium. Consequently, there is no explicit time dependence is all matter and geometrical variables. In particular

$$p = p(r);\ \rho = \rho(r);\ \{u^A\} = \{u^0, u^1 = 0\}$$

Accordingly, the two components of the vectorial hydrodynamical Eqs. (35.4) yield only one:

For $c=0$ one has $0=0$

For $c=1$ one has $\frac{dp}{dr} = -\frac{d\Phi}{dr}(p+\rho)$ (35.5)

Furthermore, the tensorial Einstein field Eq.(35.1) with its three components yields

$$r^2 G_0{}^0 \equiv -2\frac{\partial m}{\partial r} \qquad = \frac{8\pi G}{c^4} r^2 t_0{}^0 \left(= -\frac{8\pi G}{c^2} r^2 \rho\right) \qquad \{35.6a\}$$

$$r^2 G_{01} \equiv 2r\frac{\partial \Lambda}{\partial t} \qquad = \frac{8\pi G}{c^4} r^2 t_{01} \left(= \quad 0 \quad\right) \qquad \{35.6b\}$$

$$r^2 G_1{}^1 \equiv 2(r-2m)\frac{\partial \phi}{\partial r} - \frac{m}{r} = \frac{8\pi G}{c^4} r^2 t_1{}' \left(= \frac{8\pi G}{c^4} r^2 p\right) \qquad \{35.6c\}$$

For a system in equilibrium these equations imply

$$r^2 G_0{}^0: \quad \boxed{\frac{dm}{dr} = \frac{4\pi G}{c^2}\rho} \qquad (35.7)$$

$$r^2 G_{01}: \quad \dot{\Lambda} = 0 \qquad (35.8)$$

$$r^2 G_1{}^1: \quad \boxed{\frac{d\phi}{dr} = \frac{m + (4\pi G/c^4) r^3 p}{r(r-2m)}} \qquad (35.9)$$

Insert the expression for $\frac{d\phi}{dr}$, Eq. (35.7c) into Eq. (35.5). The result is

$$\boxed{\frac{dp}{dr} = -\frac{m + (4\pi G/c^2) r^3 p}{r(r-2m)}(p+\rho)} = -\frac{G}{c^2}\frac{m^{conv} + (4\pi r^3 p)/c^2}{r^2\left(1 - \frac{2m}{r}\right)}(p+\rho) \qquad (35.10)$$

The three boxed equations form a coupled system of non-linear ordinary differential equations:

a) two for the gravitational degrees of freedom, $m(r)$ and $\phi(r)$,

b) one for the matter degree of freedom. $p(r)$.

These equations govern any static spherically symmetric perfect fluid configuration.

However, in order to keeps one's mathematical connected to the world around us, one must follow the dictum that a diffential equation is never solved until one imposes boundary conditions on its solution. For the determination of the structure of the star the equations (35.7), (35.9), and (35.10) need to be augmented by specifying

(i) the star's central density,

$$\rho(r=0)=\rho_c\,,$$

(ii) an equation of state, $p=p(\rho)$, throughout the star so so that

$$p(r=0)=p(\rho_c),$$

and

(iii) the fact that the star has a center; i.e. that

$$m(r=0)=0\;;$$

Otherwise the pressure gradient, Eq. (35.10), will not be finite at r=0.

III. HOW TO SOLVE THE EQUATIONS OF HYDROSTATIC EQUILIBRIUM

The structure of a star in equilibrium is determined by

$$\boxed{\frac{dp}{dr} = -\frac{(m+4\pi r^3 p)}{r(r-2m)}(p+\rho) \quad \text{with } p(r=0)=p_c}$$

Within a Newtonian framework this equation expresses a balance between a force due to a pressure gradient and the gravitational force acting on a small volume of fluid in the star, namely $\frac{dp}{dr} = -\frac{m}{r^2}\rho$

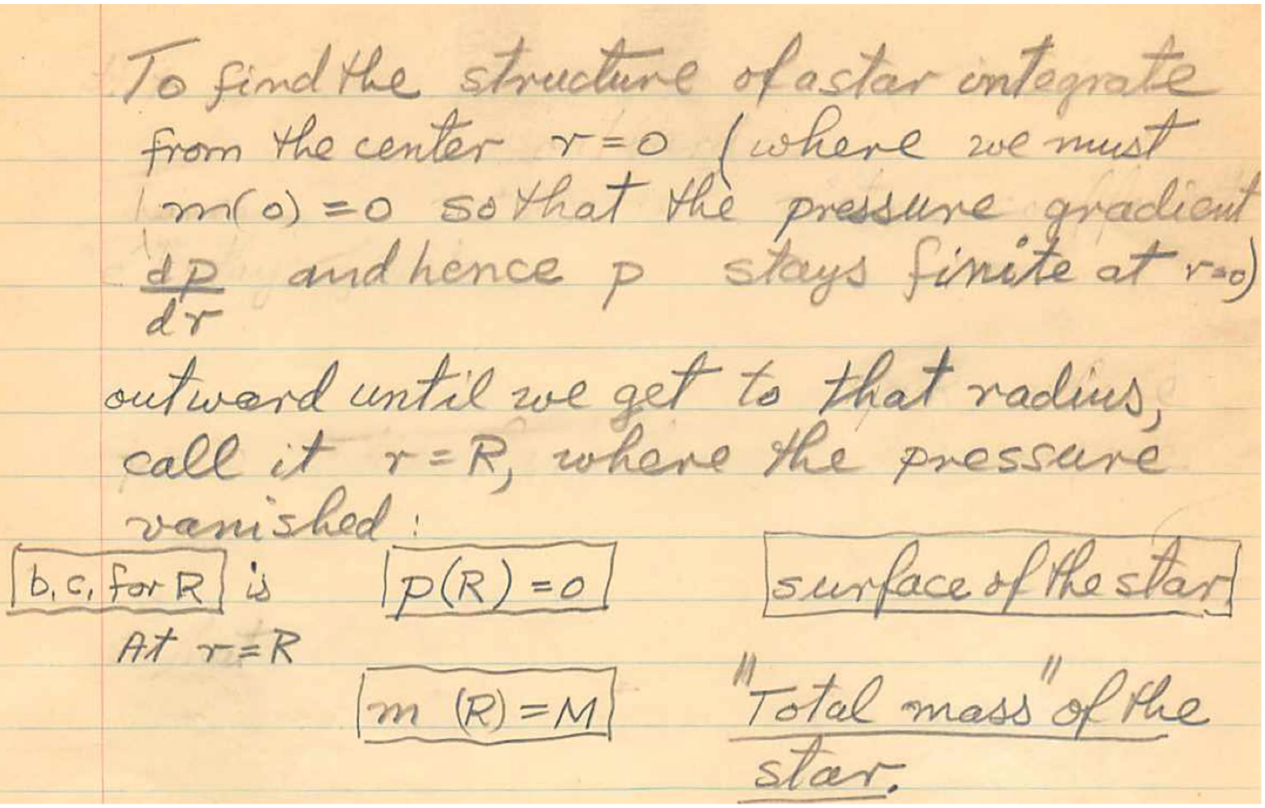
The mass enclosed in a sphere of surface area $4\pi r^2$ is

$$\boxed{m(r) = \int_0^r 4\pi \rho r^2 dr \qquad \text{with } m(0)=0}$$

These two equations together with an equation of state

$$\rho = \rho(p)$$

determine the equilibrium structure of the star.



To find the structure of a star integrate from the center $r=0$ (where we must $m(0)=0$ so that the pressure gradient $\frac{dp}{dr}$ and hence p stays finite at $r=0$)

outward until we get to that radius, call it $r=R$, where the pressure vanished:

$\boxed{\text{b.c. for } R}$ is $\boxed{p(R)=0}$ $\boxed{\text{surface of the star}}$

At $r=R$

$\boxed{m(R)=M}$ "Total mass" of the star.

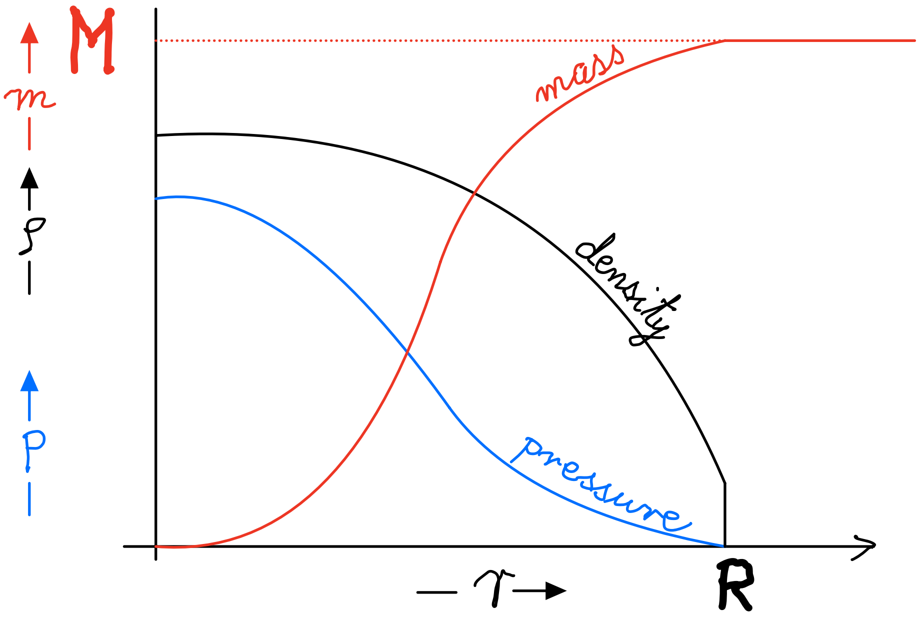
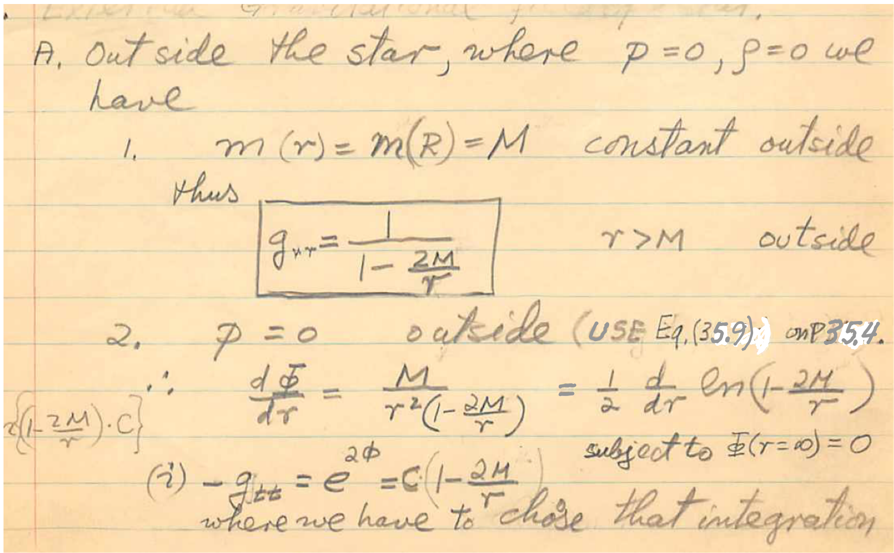


Figure 35.1 Qalitative depiction of a solution to the structure equations, namely, the density $\rho(r)$, pressure $p(r)$, and massfunction $m(r)$ of a star. If the star has

a crust, its surface density would be discontinuous, even though its density is zero at the surface.

IV. EXTERNAL GRAVITATIONAL FIELD OF A STAR



A. Outside the star, where $p=0$, $\rho=0$ we have

1. $m(r)=m(R)=M$ constant outside

thus

$$\boxed{g_{rr}=\frac{1}{1-\frac{2M}{r}}} \qquad r>M \quad \text{outside}$$

2. $p=0$ outside (USE Eq. (35.9) on P 35.4.

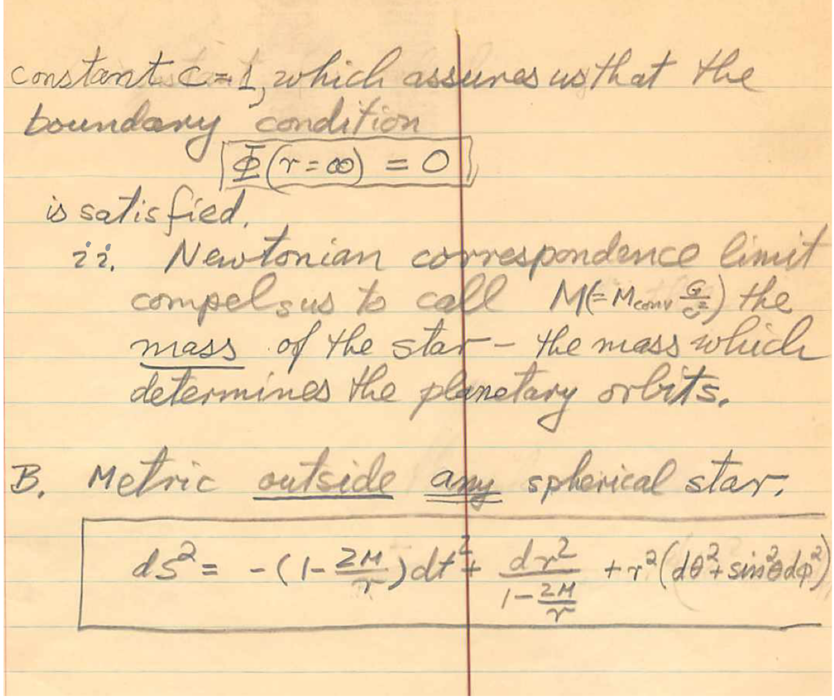
$\{(1-\frac{2M}{r})\cdot C\}$

$$\therefore \quad \frac{d\Phi}{dr}=\frac{M}{r^2(1-\frac{2M}{r})}=\frac{1}{2}\frac{d}{dr}\ln\left(1-\frac{2M}{r}\right)$$

subject to $\Phi(r=\infty)=0$

(i) $-g_{tt}=e^{2\Phi}=C\left(1-\frac{2M}{r}\right)$

where we have to chose that integration



constant $C=1$, which assures us that the boundary condition

$$\boxed{\Phi(r=\infty)=0}$$

is satisfied.

ii. Newtonian correspondence limit compels us to call $M(=M_{conv}\frac{G}{c^2})$ the <u>mass</u> of the star – the mass which determines the planetary orbits.

B. Metric <u>outside</u> <u>any</u> spherical star:

$$\boxed{ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)}$$

Lecture 36

H-J Theory of Particle Mechanics: Asymptotic solutions to Wave equation

In MTW Read BOX 25.3 (The philosophy illustrated)
Peruse Exercise 35.15 (The math)
Read § 25.5 (The application)

I. Wave Equation in the Geometrical Optics Approximation

The geometry of gravition controls the geodesic motion of free particles. The fact that Plank's constant ($\hbar = 6.6 \times 10^{-34}$ mks $= 6.6 \times 10^{-27}$ cgs units) is not zero implies that the laws of particle motion are to be mathematized in terms of the Klein-Gordon (K-G) equation

$$-\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi - \frac{m^2}{\hbar^2}\psi = g^{\mu\nu}\psi_{,\mu;\nu} - \frac{m^2}{\hbar^2} = 0 \qquad (36.1)$$

for relativistic mechanics.* For non-relativistic mechanics let

$$\psi = e^{i m t/\hbar}\varphi$$

and find for $\frac{m}{\hbar}|\varphi| \gg \left|\frac{\partial\varphi}{\partial t}\right|$ that the non-relativistic Schrödinger wave equation in a weak gravitational potential $\left(\frac{\Phi_{grav'l}}{c^2} \ll 1\right.$; see Eq. (5.9)$\left.\right)$ is

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + m\phi_{grav'l}\right]\varphi = i\hbar\frac{\partial\varphi}{\partial t}. \qquad (36.2)$$

* \footnote{For an electron the Compton wave length is $\frac{\hbar}{m_e c} = .4 \times 10^{-10}$cm}

These wave equations govern the manner in which the wave function evolves in spacetime, which in general is not flat.

Both the Schrödinger equation and Newton's equation of motion

$$m\,\frac{d\vec{x}}{dt^2}=\vec{F},$$

which governs the dynamics of a non-relativistic particle, have its mass as an arbitrary parameter. The same observation is true for the K-G equation and the geodesic equation of a relativistic particle.

Hamilton-Jacobi (H-J) theory takes advantage of this observation by furnishing the logical connecting link between (i) wave mechanics and particle mechanics, (ii) physical optics and geometrical optics as well as between (iii) Fourier theory and theory of wave packets.

Both relativistic and non-relativistic wave functions can be reexpressed in terms of a phase function S ("eikonal", "Schrödinger phase", "dynamical phase") and an amplitude $\mathcal{A}$:

$$\psi(x^\alpha) = \mathcal{A}(x^\alpha)\, e^{iS(x^\alpha)/\hbar}.$$

Here, in the asymtodical short wave length / high frequency limit

$\mathcal{A}(x^\alpha)$ = "slowly varying function of x^α"

$e^{iS(x^\alpha)/\hbar}$ = "rapidly " " of x^α"

Introduce such a function into the wave equation (36.1) and find

$$g^{\mu\nu}\left\{\mathcal{A}_{,\mu;\nu} + \frac{2i}{\hbar}\frac{\partial S}{\partial x^\mu}\frac{\partial \mathcal{A}}{\partial x^\nu} + \frac{i}{\hbar} S_{,\mu;\nu}\mathcal{A} - \frac{1}{\hbar^2}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\mathcal{A}\right] - \frac{m^2}{\hbar^2}\mathcal{A}\right\} e^{iS/\hbar} = 0 \qquad (36.2)$$

The wave equation (36.1), and, hence Eq. (36.2), applies to all masses m. Mathematize this observation by introducing the dimensionless variable ϵ into Eq. (36.1) and any of its solutions by letting

$$m = \frac{m_0}{\epsilon}.$$

Thus

$$\epsilon^2 g^{\mu\nu}\psi_{,\mu;\nu} - \frac{m_0^2}{\hbar^2}\psi = 0 \qquad (36.3)$$

and
$$\psi = \left(\mathcal{A}_0 + \mathcal{A}_1\epsilon + \mathcal{A}_2\epsilon^2 + \cdots\right) e^{i\frac{1}{\epsilon}S/\hbar} \qquad (36.4)$$

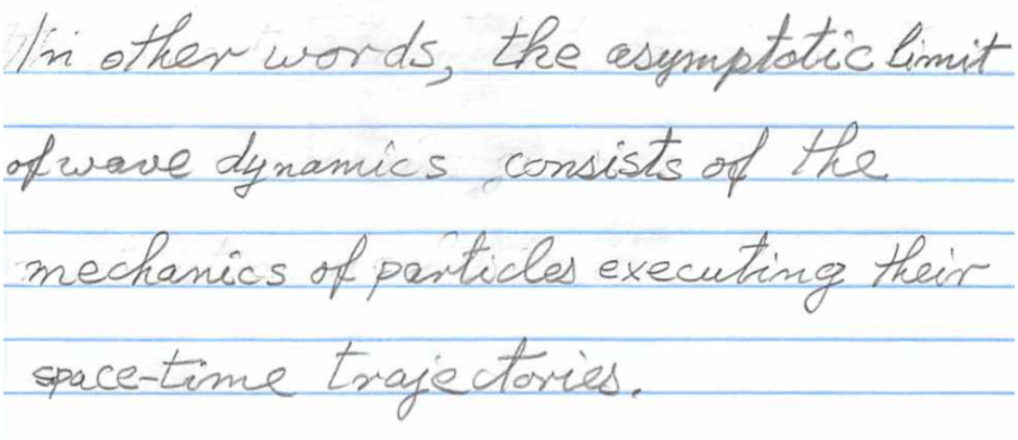
Using H-J theory we shall show that in the asymptotic limit, $\epsilon \to 0$, which corresponds to very heavy particles, (36.4)

$$\frac{m_0}{\epsilon} \to \infty,$$

$\psi(x^\alpha)$ furnishes us with the solution $\{x^\mu(\tau)\}_{\mu=0}^{3}$ to the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \qquad \mu = 0,1,2,3$$

without having to go through the labor of having to solve this system of o.d.e.'s.



In other words, the asymptotic limit of wave dynamics consists of the mechanics of particles executing their space-time trajectories.

Apply Eq.(36.4) to (36.2), collect equal powers of ϵ, and set their coefficients to zero:

36.5

$$\frac{1}{\epsilon^2} \quad : \qquad \frac{\mathcal{R}_0}{\hbar^2}\left\{ g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m_0^2 \right\} = 0$$

$$\frac{1}{\epsilon} \quad : \qquad \frac{i}{\hbar}\left\{ \left(g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\right)_{;\mu} \mathcal{R}_0 + 2 g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\mathcal{R}_{0;\nu} \right\} = 0$$

(Terms of higher order yields "post geometrical optics" corrections)

Dropping the subscript "zero" from m_0, one therefore obtains:

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0 \qquad (\text{H-J eq'n}) \qquad (36.5)$$

$$\left(\mathcal{R}_0^2\, g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\right)_{;\nu} = 0 \qquad \left(\begin{array}{l}\text{Particle}\\ \text{conservation}\end{array}\right) \qquad (36.6)$$

The application of this asymptotic expansion method to the (non-relativistic) Schrödinger equation yields

$$\frac{1}{2m}\vec{\nabla} S \cdot \vec{\nabla} S + U(x) + \frac{\partial S}{\partial t} = 0 \qquad (36.7)$$

$$\frac{\partial(\mathcal{R}_0^2)}{\partial t} + \frac{1}{m}\vec{\nabla}\cdot\left(\mathcal{R}_0^2\vec{\nabla}S\right) = 0 \qquad (36.8)$$

Equations (36.5) and (36.7) are the H-J equations for a relativistic and non-relativistic system respectively.

They are first order partial differential equations whose solutions are the *dynamical phase* S of the given system.

Once it is known for a given system, the task of exhibiting its global spacetime particle trajectories in mathematical form is complete. This is because of the application of the *principle of constructive interference* to the phase function is mathematically trivial (although physically fundamental).

Eqs (36.6) & (36.8), both of which are 4-dimensional divergence conditions, mathematize the law of conservation of particles. Since each particle carries a certain amount of momenergy, one finds that this law plays a key role in mathematizing the law of momenergy conservation in terms of the energy-momentum tensor of an aggregate of particles.

Summary:

In the "geometrical optics" limit of wave mechanics the wave length of a wave is so short that locally the phase fronts have the (straight and parallel) properties of a plane wave.

This is so despite the fact that these phase fronts exhibit curvature outside any local neighborhood.

The shape and spacing of the phase fronts is expressed by the isograms of phase function $S(x^\alpha)$.

It satifies the H-J equation

$$\frac{\partial s}{\partial t} + H\left(\frac{\partial s}{\partial x}, x, t\right) = 0 \quad \text{(non-relativistic)} \quad (9)$$

or

$$\mathcal{H}\left(\frac{\partial s}{\partial x^\alpha}, x^\alpha\right) \equiv g^{\mu\nu}(x^\alpha)\frac{\partial s}{\partial x^\mu}\frac{\partial s}{\partial x^\beta} + m^2 = 0 \quad \text{(relativistic)} \quad (10)$$

If the particle has charge q and is moving in an electro-magnetic field has the electomagnetic vector potential $A_\mu(x^\alpha)$, then its H-J equation is

$$\mathcal{H}\left(\frac{\partial S}{\partial x^\alpha}, x^\alpha\right) \equiv g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - qA_\nu\right) + m^2 = 0$$

Lecture 37

H-J theory: Equation
and the
Principle of Constructive
Interference

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37,1

I. Setting up and solving the H-J equation

Problem. (*Particle in a potential*)
Set up and solve the Hamilton-Jacoby equation for a particle in a one dimensional potential $U(x)$.
Solution. Setting up the H-J equation is a three step process.

(1) Exhibit the Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 - U(x)\,.$$

(2) Determine the momentum and the Hamiltonian:

$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{x}} \\ &= m\dot{x}\,; \\ H &= \dot{x}\frac{\partial L}{\partial \dot{x}} - L \\ &= \frac{1}{2}m\dot{x}^2 + U(x)\,. \end{aligned}$$

(3) Express the Hamiltonian in terms of the momentum:

$$H = \frac{p^2}{2m} + U(x)\,.$$

(4) Write down the H-J equation $-\frac{\partial S}{\partial t} = H\left(x, \frac{\partial S}{\partial x}\right)$:

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x)\,.$$

This a first order non-linear partial differential equation that needs to be solved for the scalar function $S(x,t)$.
This p.d.e. lends itself to being solved by the method of separation of variables according to which one finds solutions of the form

$$S(x,t) = T(t) + X(x)\,. \tag{3.5.7}$$

Introducing this form into the H-J equation, one finds

$$-\frac{dT(t)}{dt} = \frac{1}{2m}\left(\frac{dX(x)}{dx}\right)^2 + U(x)\,.$$

This equation says that the left hand side is independent of x, while the right hand side is independent of t. Being equal, the l.h.s. is also independent of x. Being independent of both t and x, it is a constant. Letting this "separation" constant be equal to E, one obtains two equations

$$\begin{aligned} -\frac{dT(t)}{dt} &= E \\ \frac{1}{2m}\left(\frac{dX(x)}{dx}\right)^2 + U(x) &= E\,. \end{aligned}$$

These are two ordinary equations for T and X. Inserting these equations into Eq. (3.5.7), one obtains the sought after solution to the H-J equation,

$$S(x,t) = -Et + \int^x \sqrt{2m\,(E - U(x'))}\,dx' + \delta(E)\,.$$

Here the "integration constant" $\delta(E)$ is an arbitrary function of E. Furthermore, observe that S depends on E also. This means that one has an E-parametrized

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family of solutions. Thus, properly speaking, separation of variables yields many solutions to the H-J equation, in fact, a one-parameter family of them

$$S(x,t) = S_E(x,t)\,.$$

3.5.2. Several Degrees of Freedom. We shall see in a subsequent section that whenever the H-J for a system with several degrees of freedom, say $\{q^i\}$, lends itself to being solved by the method of the separation of variables, i.e.

$$S(q^i,t) = T(t) + \sum_{i=1}^{s} Q_i(q^i)\,,$$

the solution has the form

$$S = -\int^t E dt + \sum_{i=1}^{s} \int^{q^i} p_i(x^i; E, \alpha_1, \cdots, \alpha_{s-1}) dq^i + \delta\,(E, \alpha_1, \cdots, \alpha_{s-1})$$

Here δ is an arbitrary function of E and the other separation constants that arise in the process of solving the H-J equation. We see that for each choice of $(E, \alpha_1, \cdots, \alpha_{s-1})$ we have a different solution S. Thus, properly speaking, we have $S_{E,\alpha_1,\cdots,\alpha_{s-1}}$, a multi-parametrized family of solutions to the H-J equation.

We shall now continue our development and show that Hamilton-Jacobi Theory is

a) A new and rapid way of integrating the E-L equations

b) The bridge to wave (also "quantum") mechanics.

The virtue of Hamilton's principle is that once the kinetic and potential energy of the system are known, the equations of motion can be set up with little effort. These Euler-Lagrange equations are Newton's equations of motion for the system. Although setting up the equations of motion for a system is a routine process, solving them can be a considerable challenge. This task can be facilitated considerably by using an entirely different approach. Instead of setting up and solving the set of coupled Newtonian ordinary differential equations, one sets up and solves a single partial differential equation for a single scalar function. Once one has this scalar function, one knows everything there is to know about the dynamical system. In particular, we shall see that by differentiating this scalar function (the dynamical phase, the Hamilton-Jacobi function, the eikonal) one readily deduces all possible dynamical evolutions of the system.

3.6. Hamilton-Jacobi Description of Motion

Hamilton-Jacobi theory is an example of the *principle of unit economy*[3], according to which one condenses a vast amount of knowledge into a smaller and smaller number of principles. Indeed, H-J theory condenses all of classical mechanics and all of wave mechanics (in the asymptotic high-frequency/short-wavelength (a.k.a. W.K.B.) approximation) into two conceptual units,, (i) the H-J equation

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[3] The *principle of unit economy*, also known informally as the "crow epistemology", is the principle that stipulates the formation of a new concept

(1) when the description of a set of elements of knowledge becomes too complex,

(2) when the elements comprising the knowledge are used repeatedly, and

(3) when the elements of that set require further study.

Pushing back the frontier of knowledge and successful navigation of the world demands the formation of a new concept under any one of these three circumstances.

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and (ii) the principle of constructive interference. These two units are a mathematical expression of the fact that classical mechanics is an asymptotic limit of wave mechanics.

Hamilton thinking started with his observations of numerous known analogies between "particle world lines" of mechanics and "light rays" of geometric optics. These observations were the driving force of his theory. With it he developed classical mechanics as an asymptotic limit in the same way that ray optics is the asymptotic limit of wave optics. Ray optics is a mathematically precise asymptotic limit of wave optics. Hamilton applied this mathematical formulation to classical mechanics. He obtained what nowadays is called the Hamilton-Jacobi formulation of mechanics. Even though H-J theory is a mathematical limit of wave mechanics, in Hamilton's time there was no logical justification for attributing any wave properties to material particles. (That justification did not come until experimental evidence to that effect was received in the beginning of the 20th century.) The most he was able to claim was that H-J theory is a mathematical method with more unit economy than any other formulation of mechanics. The justification for associating a wave function with a mechanical system did not come until observational evidence to that effect was received in the beginning of the 20th century.

We shall take advantage of this observation (in particular by Davidson and Germer, 1925) implied association by assigning to a mechanical system a wave function. For our development of the H-J theory it is irrelevant whether it satisfies the Schroedinger, the Klein-Gordon, or some other quantum mechanical wave equation. Furthermore, whatever the form of the wave equation governing this wave function, our focus is only on those circumstances where the wave function has the form

$$(3.6.1) \qquad \Psi_E(x,t) = \underbrace{\mathcal{A}(x,t)}_{\textit{slowly}\text{ varying function of } x \text{ and } t} \times \underbrace{\exp\left(\frac{i}{\hbar}S_E(x,t)\right)}_{\textit{rapidly}\text{ varying function of } x \text{ and } t}$$

This circumstance is called the "high frequency" limit or the "semi-classical" approximation. It can be achieved by making the energy E of the system large enough. In that case

$$1 << \frac{S_E(x,t)}{\hbar}$$

with the consequence that the phase factor oscillates as a function of x and t rapidly indeed. The existence of such a wave function raises a non-trivial problem:
If the wave and its dynamical phase, and hence the wave intensity, is defined over all of space-time, how is it possible that a particle traces out a sharp and well defined path in space-time when we are left with three delemas?

(1) The large magnitude ($S \gg \hbar$=1.05×10^{-27}[erg sec]) of the action for a classical particle is certainly of no help.

(2) Neither is the simplicity of the H-J equation

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}, t) = 0$$

which governs the dynamical phase in

$$\Psi = \mathcal{A}\exp\left(i\frac{S}{\hbar}\right),$$

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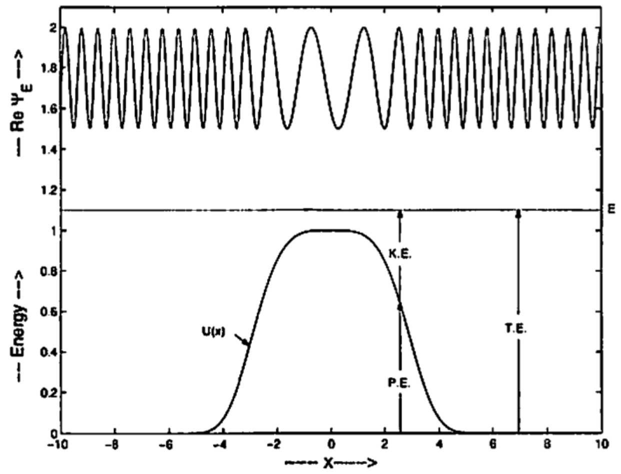

FIGURE 3.6.1. The spatial oscillation rate of the wave function $Re\,\Psi_E$ at $t = const.$ is proportional to its x-momentum, whose square is proportional to the kinetic energy (K.E.=T.E.-P.E.).

(3) Nor is the simplicity of the solution S for a particle of energy E,

$$S(x,t) = -Et + \int_{x_o}^{x} \sqrt{2m(E - U(x))}dx + \delta(E)$$

of any help in identifying a localized trajectory ("world line") of the particle in space-time coordinatized by x and t.

What *is* of help is the basic implication of associating a wave function with a moving particle, namely, it is a linear superposition of monochromatic waves, Eq. (3.6.1), which gives rise to a travelling wave packet – a localized moving wave packet whose history is the particle's world line. To validate this claim we shall give two heuristic arguements (i-ii), one application (iii), a more precise argument (iv) and an observation (v).

(i): The most elementary superposition monochomatic waves is given by the sum wave trains with different wavelengths

$$\Psi(x,t) = \Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots .$$

(ii): In space-time one has the following system of level surfaces for $S_E(x,t)$ and $S_{E+\Delta E}(x,t)$

Destructive interference between different waves comprising Ψ(x,t) occures everywhere except where the phase of the waves agree:

$$S_E(x,t) = S_{E+\Delta E}(x,t)$$

II. Constructive Interference.

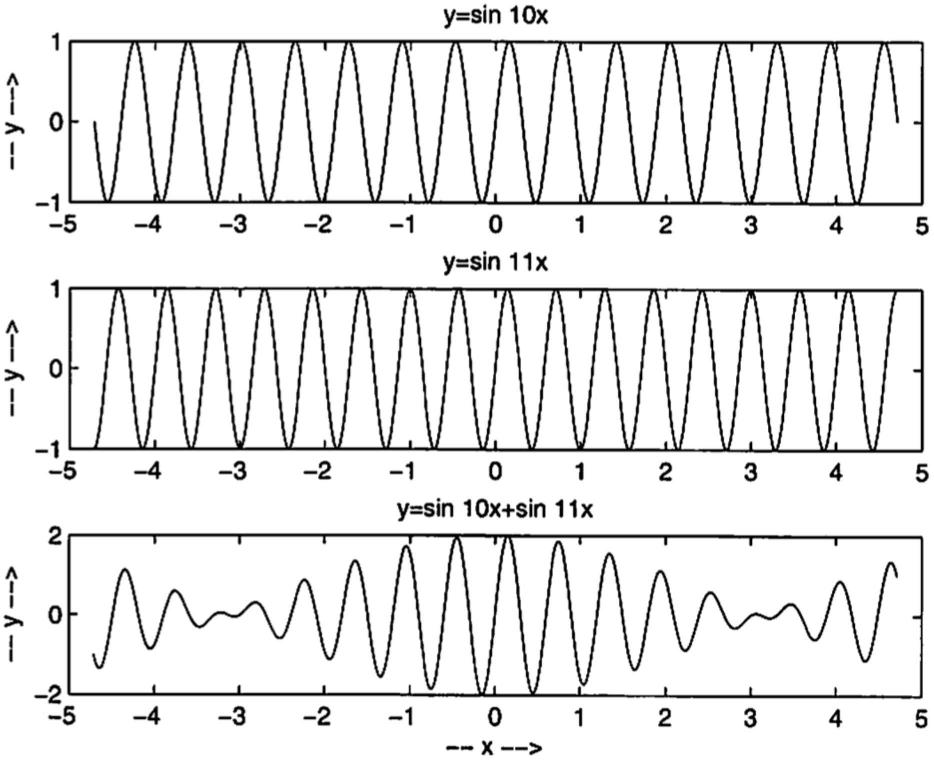

FIGURE 3.6.2. Photographic snapshot in space of two interfering wave trains and their resulting wave packet.

At the locus of events satisfying this condition, the waves interfere constructively and wave packet has non-zero amplitude. The quantum priciple says that this condition of constructive interference

$$0 = \lim_{\Delta E \to 0} \frac{S_{E+\Delta E}(x,t) - S_E(x,t)}{\Delta E} = \frac{\partial S_E(x,t)}{\partial E}$$

yields a Newtonian worldline, i.e. an extremal paths.

(iii): Apply this condition to the action S(x,t) of a single particle. One obtains the time the particle requires to travel to point x,

$$0 = -t + \int_{x_0}^{x} \sqrt{\frac{m}{2}} \left(\frac{1}{E - U(x)} \right)^{\frac{1}{2}} dx + t_0$$

with

$$t_0 \equiv \frac{\partial \delta(E)}{\partial E} .$$

This condition yields the Newtonian worldline indeed. The precise arguement is Lecture 13. The additional observation is on p13 Lecture 13.

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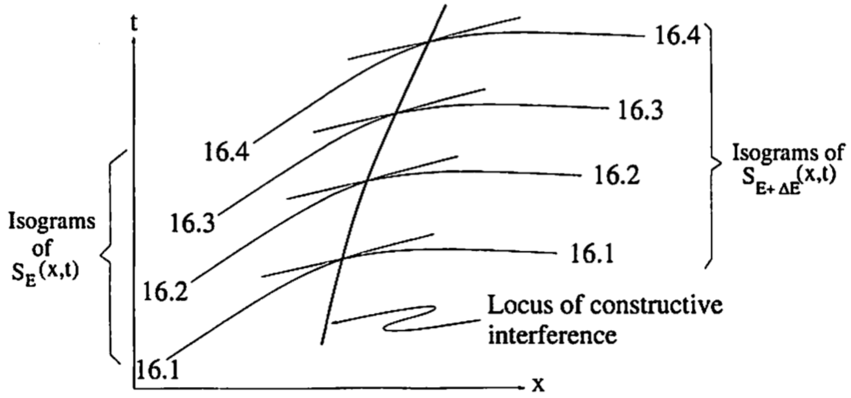

FIGURE 3.6.3. Constructive interference represented in spacetime. The intersection of the respective isograms of $S_E(x,t)$ and $S_{E+\Delta E}(x,t)$ locates the events (x,t) which make up the trajectory of the particle in x-t space – the locus of constructive interference.

Lecture 13

Wave Packets via 3.7. Constructive Interference

Our formulation of constructive interference is based on a picture in which at each time t a superposition of wave trains

$$\Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots \equiv \Psi(x,t)$$

yields a wave packet at time t. The principle of constructive interference itself,

$$\frac{\partial S_E(x,t)}{\partial E} = 0$$

is a condition which at each time t locates where the maximum amplitude of the wave packet is.

It is possible to bring into much sharper focus the picture of superposed wave trains and thereby not only identify the location of the resultant wave packet maximum, but also width of that packet.

3.8. Spacetime History of a Wave Packet

The sharpened formulation of this picture consists of replacing a *sum* of superposed wave amplitudes with an *integral* of wave amplitudes

$$\begin{aligned} \Psi(x,t) &= \text{"}\Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots\text{"} \\ &= \int_{-\infty}^{\infty} f(E)\, e^{\frac{i}{\hbar}S_E(x,t)} dE\,. \end{aligned} \tag{3.8.1}$$

A very instructive example is that of a superpostion of monochromatic ("single energy") wavetrains, each one weighted by the amplitude $f(E)$ of a Gaussian window

III. Wave packet: Structure and Evolution.

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in the Fourier ("energy") domain,

(3.8.2)
$$\boxed{f(E) = Ae^{-(E-E_0)^2/\epsilon^2}}$$

The dominant contribution to this integral comes from within the window, which is centered around the location of E_0 of the Gaussian maximum and has width 2ϵ, which is small for physical reasons. Consequently, it suffices to represent the phase function as a Taylor series around that central point E_0, namely

(3.8.3)
$$S_E(x,t) = S_{E_0}(x,t) + \left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0} (E-E_0) + \frac{1}{2}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} (E-E_o)^2 + \begin{matrix}\textit{higher}\\ \textit{order}\\ \textit{terms}\end{matrix} ,$$

and neglect the higher order terms. Keeping only the first three terms and ignoring the remainder allows an exact evaluation of the Gaussian superposition integral. This evaluation is based on the following formula

(3.8.4)
$$\int_{-\infty}^{\infty} e^{\alpha z^2 + \beta z} dz = \sqrt{\frac{\pi}{-\alpha}} e^{-\frac{\beta^2}{4\alpha}} .$$

Applying it to the superpostion integral, Eq. (3.8.1) together with Eqs. (3.8.2) and (3.8.3), we make the following identification

(3.8.5)
$$\begin{aligned} z =& E - E_0 ; \quad dz = dE , \\ \alpha =& -\frac{1}{\epsilon^2} + \frac{i}{\hbar}\frac{1}{2}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} \equiv -\frac{1}{\epsilon^2}(1 - i\sigma) , \\ -\frac{1}{\alpha} =& \frac{\epsilon^2}{1 - i\sigma} = \epsilon^2 \frac{1 + i\sigma}{1+\sigma^2} , \\ \sigma =& \frac{1}{2}\frac{1}{\hbar}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} \epsilon^2 , \\ \beta =& \frac{i}{\hbar}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} . \end{aligned}$$

Inserting these expressions into the righthand side of the formula (3.8.4), one obtains

$$\begin{aligned} \Psi(x,t) \quad &= \quad A\sqrt{\pi}\epsilon\sqrt{\frac{1+i\sigma}{1+\sigma^2}} \exp\left\{ -\frac{1}{4}\left(\frac{\frac{\partial S(x,t)}{\partial E_o}}{\hbar}\right)^2 \epsilon^2 \left(\frac{1+i\sigma}{1+\sigma^2}\right)\right\} e^{i\frac{S_{E_0}(x,t)}{\hbar}} \\ &\equiv \quad \underbrace{\mathcal{A}(x,t)}_{\substack{\textit{slowly}\\ \textit{varying}}} \; \underbrace{e^{\frac{iS_{E_0}(x,t)}{\hbar}}}_{\substack{\textit{rapidly}\\ \textit{varying}}} . \end{aligned}$$

This is a *rapidly oscillating* function

$$e^{i\,S_{E_0}(x,t)/\hbar}$$

modulated by a *slowly varying* amplitude $\mathcal{A}(x,t)$. For each time t this product represents a wave packet. The location of the maximum of this wave packet is given implicitly by

(3.8.6)
$$\left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0} = 0 .$$

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As t changes, the x-location of the maximum changes. Thus we have curve in x-t space of the locus of those events where the slowly varying amplitude $\mathcal{A}$ has a maximum. In other words, this wave packet maximum condition locates those events (= points in spacetime) where constructive interference takes place.

A wave packet has finite extent in space and in time. This extent is governed by its squared modulus, i.e. the squared magnitude of its slowly varying amplitude,

$$(3.8.7)\quad |\Psi(x,t)|^2 = |\mathcal{A}|^2 = A^2\pi\epsilon^2\frac{1}{\sqrt{1+\sigma^2}}\exp\underbrace{\left\{-\frac{\epsilon^2}{2}\frac{1}{\sqrt{1+\sigma^2}}\frac{\left(\frac{\partial S_E(x,t)}{\partial E}\Big|_{E_0}\right)^2}{\hbar^2}\right\}}_{\mathbf{E}(x,t)}$$

We see that this squared amplitude has nonzero value even if the condition for constructive interference, Eq.(3.8.6), is violated. This violation is responsible for the finite width of the wave packet. More precisely, its shape is controlled by the exponent $\mathbf{E}(x,t)$,

$$\mathbf{E}(x,t) \equiv \left\{-\frac{\epsilon^2}{2}\frac{1}{\sqrt{1+\left(\frac{\epsilon^2}{2\hbar}\frac{\partial^2 S_E(x,t)}{\partial E^2}\Big|_{E_0}\right)^2}}\frac{\left(\frac{\partial S_E(x,t)}{\partial E}\Big|_{E_0}\right)^2}{\hbar^2}\right\} \neq 0\,.$$

The spacetime evolution of this shape is exhibited in Figure 3.8.1 on the next page. Thus the worldline of the particle is not a sharp one, but instead has a slight spread in space and in time. How large is this spread?

The magnitude of the wave mechanical ("non-classical") spread in the world line is the width of the Gaussian wave packet. This spread is Δx, the amount by which one has to move away from the maximum in order that the amplitude profile change by the factor $e^{\frac{1}{2}}$ from the maximum value. Let us calculate this spread under the circumstance where the effect due to dispersion is a minimum, i.e. when σ is neglegibly small. In that case the condition that $\mathbf{E}(x+\Delta x,t) = -1$ becomes

$$\left|\frac{\epsilon}{\hbar}\frac{\partial S_E(x+\Delta x,t)}{\partial E}\Big|_{E_0}\right| = 1\,.$$

Expand the left hand side to first order, make use of the fact that (x,t) is a point in spacetime where the wavepacket profile has a maximum, i.e. satisfies Eq.(3.8.6). One obtains

$$\left|\epsilon\frac{\partial^2 S}{\partial E\partial x}\Delta x\right| = \hbar$$

or, in light of $\partial S_E(x,t)/\partial x \equiv p(x,t;E)$,

$$\left|\epsilon\frac{\partial p}{\partial E}\Delta x\right| = \hbar\,,$$

and hence

$$\boxed{\Delta p\Delta x = \hbar}$$

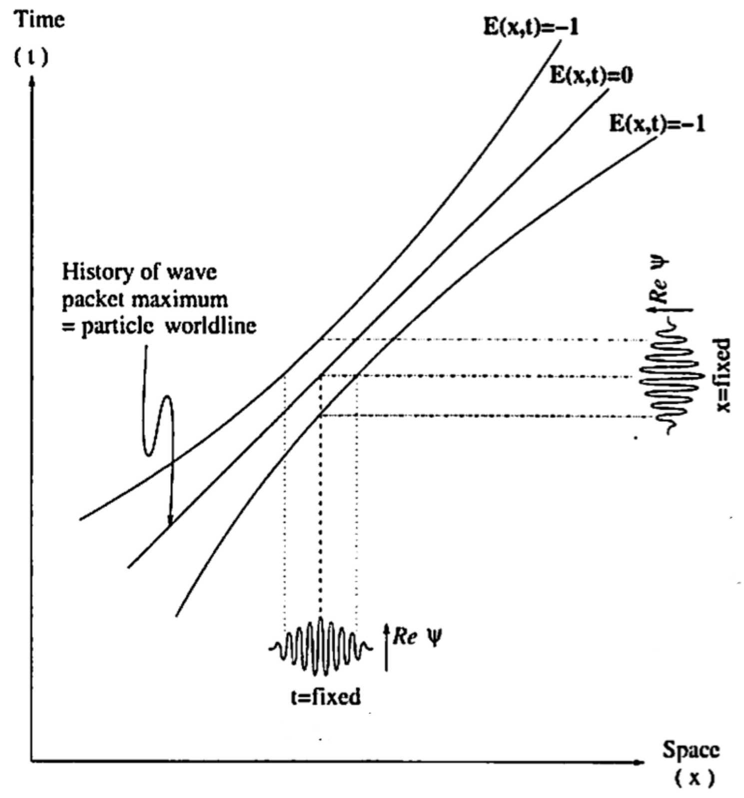

FIGURE 3.8.1. Spacetime particle trajectory ("the $\mathbf{E}(x,t) = 0$ isogram") and the dispersive wave packet amplitude histories surrounding it. The two mutually diverging ones (both characterized by $\mathbf{E}(x,t) = -1$) in this figure refer to the front and the back end of the wave packet at each instant $t = fixed$, or to the beginning and the end of the wave disturbance passing by a fixed location $x = fixed$. The particle and the wave packet maximum are moving with a velocity given by the slope of the $\mathbf{E}(x,t) = 0 = \left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0}$ isogram, which is the locus of constructive interference exhibited in Figure 3.6.3

On the other hand, the convergence and subsequent divergence ("dispersion") of the wave packet is controlled (and expressed mathematically) by the behavior of the second derivative, $\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0}$ of the dynamical phase $S_E(x,t)$. Whereas the behavior of its first derivative characterizes the difference in the motion of particles launched with difference initial conditions, its second derivative characterizes the intrinsically wave mechanical aspects of each of these particles.

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Similarly the temporal extent Δt, the amount by which one has to wait (at fixed x) for the wave amplitude profile to decrease by the factor $e^{-1/2}$ from its maximum value, satisfies the condition

$$\left| \frac{\epsilon}{\hbar} \frac{\partial S_E(x, t+\Delta t)}{\partial E} \Big|_{E_0} \right| = 1$$

which become

$$\left| \epsilon \frac{\partial^2 S_E}{\partial E \partial t} \Big|_{E_0} \Delta t \right| = \hbar$$

$$\left| \epsilon (-) \frac{\partial E}{\partial E} \Big|_{E_0} \Delta t \right| = \hbar$$

or

$$\boxed{\Delta E \Delta t = \hbar}.$$

The two boxed equation are called the Heisenberg indeterminacy relation. Even though we started with the dynamical phase S (see page 38) with $\Psi \sim e^{\frac{iS}{\hbar}}$ to arrive at the extremal path in spacetime, the constant $\hbar$ ("quantum of action") never appeared in the final result for the spacetime trajectory. The reason is that in the limit

$$\frac{S}{\hbar} \rightarrow \infty$$

the location of the wave packet reduces to the location of the wave crest. Once one knows the dynamical phase $S(x, t)$ of the system, the condition of constructive interference gives ***without approximation*** the location of the sharply defined Newtonian world line, the history of this wave crest, an extremal path through spacetime.

Lecture 38

Reconstruction of classical worldlines from the Principle of Constructive Interference

I. Relativistic H-J equation and its solutions

The reconstruction of classical worldlines of particles via the application of the principle of constructive interference can be generalized to any system characterized by an action, and hence by an Hamiltonian.

Consider the H-J equation

$$\mathcal{H}\left(x^\alpha; \frac{\partial S}{\partial x^\mu}\right) \equiv g^{\mu\nu}(x^\alpha)\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0 \qquad (38.1)$$

for a free particle in an environment coordinatized by global rectilinear coordinates. In such an environment the inverse metric is independent of each coordinate, $t, x, y,$ and z. They are termed "cyclic" coordinates, and the H-J equation is simply

$$-\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 + m^2 = 0 \qquad (38.2)$$

Apply the method of the separation of variables* first to x,

$$S = X(x) + S'(t, y, z)$$

$$\left(\frac{dX}{dx}\right)^2 = \left(\frac{\partial S'}{\partial t}\right)^2 - \left(\frac{\partial S'}{\partial y}\right)^2 - \left(\frac{\partial S'}{\partial z}\right)^2 - m^2 = p_x^2 \qquad (= \text{"separation constant"})$$

and find that

$$S = p_x x + S'(t, y, z) + \text{const.}$$

The resulting principle is this:

Whenever the H-J equation has cyclic coordinate, its solution is a linear function of this coordinate.

Applying this principle to the y and z coordinates results in

$$S = p_x x + p_y y + p_z z + T(t) + \text{const.}$$

Thus

$$p_x^2 + p_y^2 + p_z^2 + m^2 = \left(\frac{dT}{dt}\right)^2$$

which implies that** the dynamical phase is

$$S = p_x x + p_y y + p_z z - \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}\; t + \beta(p_x, p_y, p_z) \tag{38.3}$$

* \footnote{ Had one started by first separating t,

$$S = T(t) + S''(x, y, z),$$

one would have found that the dynamical phase is

$$S = -p_0 t + p_x x + p_y y \pm \sqrt{p_0^2 - p_x^2 - p_y^2 - m^2}\; z + \gamma(p_0, p_x, p_y).\}$$

** \footnote{ The minus sign in front of the square-root has been chosen in order that the phase velocity 4-vector

$$\left\{\eta^{\nu\mu}\frac{\partial S}{\partial x^\mu}\right\}_{\nu=0}^{3} = \left\{-\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}, p_x, p_y, p_z\right\}$$

points into the future.}

Consider the H-J equation in the static environment of a spherical system,

$$-\frac{1}{1-\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)^2+\left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{r^2}\,\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2+m^2=0. \qquad (38.4)$$

For this dynamical system t and φ are cyclic coordinates, while θ and r are not. This H-J equation is soluble by the method of the separation of variable. Solutions such as these,

$$S=S\,(x^0,x^1,x^2,x^3;\alpha_1,\alpha_2,\alpha_3)+\beta(\alpha_1,\alpha_2,\alpha_3)\equiv S(x^\mu;\alpha_i)+\beta(\alpha_i), \qquad (38.5)$$

if one can find them, always have three separation/integration constants — constants that refer to the essential properties of the dynamical system:

$$(\alpha_1,\alpha_2,\alpha_3)=\begin{cases}(p_x,\,p_y,\,p_z)\\ \left(\sqrt{p_x^2+p_y^2+p_z^2+m^2},\,p_y,\,p_z\right)\\ \left(\text{energy},\ \begin{matrix}\text{total}\\ \text{angular}\\ \text{momentum}\end{matrix},\ \begin{matrix}\text{azimuthal}\\ \text{angular}\\ \text{momentum}\end{matrix}\right)\end{cases}$$

II. Constructive Interference.

Mathematically, constructive interference is based on the condition that

$$\psi(x^\mu)=\iiint_{\{(\alpha_1,\alpha_2,\alpha_3)\}}\underbrace{A(x^\mu;\alpha_i)}_{\substack{\text{slowly}\\ \text{varying}}}\ \underbrace{e^{\,i\,S(x^\mu;\alpha_i)/\hbar}}_{\substack{\text{radidly}\\ \text{varying}}}\ d\alpha_1\,d\alpha_2\,d\alpha_3 \qquad (38.6)$$

represent a localized wavepacket whose maximum intensity $|\psi|^2_{max}$ traces out a worldline in spacetime. This maximum occurs (38.4) wherever the phase of $e^{iS/\hbar}$ is stationary in the α_1-α_2-α_3-space.

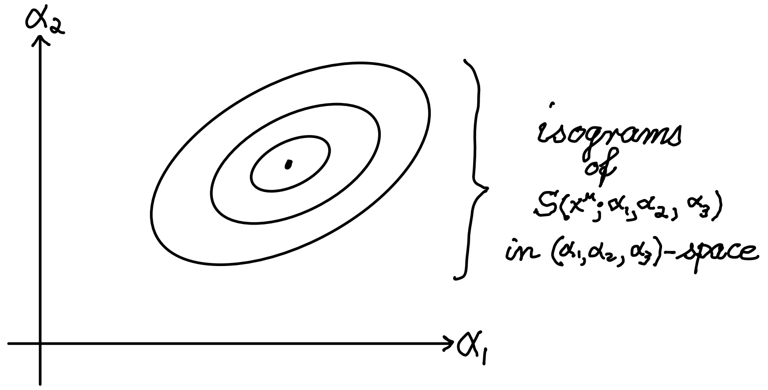


Figure 38.1 The integral $\psi(x^\mu)$ gets it dominant contribution from the neighborhood surrounding the critical point of S in $(\alpha_1, \alpha_2, \alpha_3)$-space.

The conditions which guarantee this are

$$\left.\begin{aligned}
0 &= \frac{\partial S}{\partial \alpha_1} \equiv \frac{\partial \mathcal{S}(x^0, x^1, x^2; \alpha_i)}{\partial \alpha_1} + \frac{\partial B}{\partial x_1} \\
0 &= \frac{\partial S}{\partial \alpha_2} \equiv \frac{\partial \mathcal{S}(x^0, x^1, x^2; \alpha_i)}{\partial \alpha_2} + \frac{\partial B}{\partial x_2} \\
0 &= \frac{\partial S}{\partial \alpha_3} \equiv \frac{\partial \mathcal{S}(x^0, x^1, x^2; \alpha_i)}{\partial \alpha_3} + \frac{\partial B}{\partial \alpha_3}
\end{aligned}\right\} \qquad (38.7)$$

Each of these conditions is the equation for a 3-d manifold in the 4-d spacetime. Their intersection is a 1-d trajectory, a geodesic in spacetime.

III. Geodesic Equations

The tangent $\mathbf{u}$ to this 1-d trajectory lies in the intersection of these three manifolds of stationary phase.

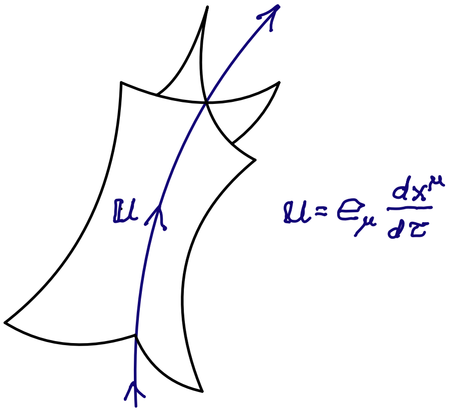


Figure 38.2 Geodesic as the intersection of the surface of stationary phase

Consequently, $\frac{\partial S}{\partial \alpha_i}$ is constant along this 1-d trajectory

$$\frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_1}\right)=0 \qquad \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_2}\right)=0 \qquad \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_3}\right)=0$$

Equivalently one has

$$\begin{bmatrix} \frac{\partial^2 S}{\partial x^0\,\partial \alpha_1} & \frac{\partial^2 S}{\partial x^1\,\partial \alpha_1} & \frac{\partial^2 S}{\partial x^2\,\partial \alpha_1} & \frac{\partial^2 S}{\partial x^3\,\partial \alpha_1} \\ \frac{\partial^2 S}{\partial x^0\,\partial \alpha_2} & \frac{\partial^2 S}{\partial x^1\,\partial \alpha_2} & \frac{\partial^2 S}{\partial x^2\,\partial \alpha_2} & \frac{\partial^2 S}{\partial x^3\,\partial \alpha_2} \\ \frac{\partial^2 S}{\partial x^0\,\partial \alpha_3} & \frac{\partial^2 S}{\partial x^1\,\partial \alpha_3} & \frac{\partial^2 S}{\partial x^2\,\partial \alpha_3} & \frac{\partial^2 S}{\partial x^3\,\partial \alpha_3} \end{bmatrix} \begin{bmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix} = [0] \qquad (38.8)$$

On the other hand the H-J Eq.(38.1) is

$$\mathcal{H}\Big(x^\nu;\underbrace{\frac{\partial S}{\partial x^\mu}(x^\nu;\alpha_1,\alpha_2,\alpha_3)}_{\substack{\equiv\\ p_\mu}}\Big)=0 \quad \text{for all } \alpha_1,\alpha_2,\alpha_3.$$

Thus

$$\frac{\partial}{\partial\alpha_i}\mathcal{H}=\frac{\partial}{\partial\alpha_i}\left(\frac{\partial S}{\partial x^\mu}\right)\frac{\partial\mathcal{H}}{\partial p_\mu}=0 \qquad i=1,2,3$$

or equivalently, since mixed partial are equal,

$$\begin{bmatrix}\frac{\partial^2 S}{\partial x^0\partial\alpha_1} & \frac{\partial^2 S}{\partial x^1\partial\alpha_1} & \frac{\partial^2 S}{\partial x^2\partial\alpha_1} & \frac{\partial^2 S}{\partial x^3\partial\alpha_1}\\ \frac{\partial^2 S}{\partial x^0\partial\alpha_2} & \frac{\partial^2 S}{\partial x^1\partial\alpha_2} & \frac{\partial^2 S}{\partial x^2\partial\alpha_2} & \frac{\partial^2 S}{\partial x^3\partial\alpha_2}\\ \frac{\partial^2 S}{\partial x^0\partial\alpha_3} & \frac{\partial^2 S}{\partial x^1\partial\alpha_3} & \frac{\partial^2 S}{\partial x^2\partial\alpha_3} & \frac{\partial^2 S}{\partial x^3\partial\alpha_3}\end{bmatrix}\begin{bmatrix}\frac{\partial\mathcal{H}}{\partial p_0}\\ \frac{\partial\mathcal{H}}{\partial p_1}\\ \frac{\partial\mathcal{H}}{\partial p_2}\\ \frac{\partial\mathcal{H}}{\partial p_3}\end{bmatrix}=[0] \qquad (38.9)$$

As before,
we assume that the set $\{\alpha_1,\alpha_2,\alpha_3\}$ is a <u>complete</u> set of integration constants, i.e. that there is no functional relation between them. This fact is mathematized by the statement that the 3×4 matrix $\left[\frac{\partial^2 S}{\partial x^\mu\partial\alpha_i}\right]$ has <u>maximal rank</u>, i.e. its null space is 1-dimensional

$$\dim\mathcal{N}\left(\left[\frac{\partial^2 S}{\partial x^\mu\partial\alpha_i}\right]\right)=1.$$

It follows that the nullspace solution to Eq.(38.9) has a unique direction also, namely

$$\frac{\partial \mathcal{H}}{\partial p_\mu} = M \epsilon^{\mu\alpha\beta\gamma} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial S}{\partial \alpha_1}\right) \frac{\partial}{\partial x^\beta}\left(\frac{\partial S}{\partial \alpha_2}\right) \frac{\partial}{\partial x^\gamma}\left(\frac{\partial S}{\partial \alpha_3}\right)$$

(39.10)

where M is a proportionality factor.

Combining Eqs. (38.7) and (38.8) results in

$$\boxed{\frac{dx^\mu}{d\tau} = N(\tau) \frac{\partial \mathcal{H}}{\partial p_\mu}} \quad \left(= N(\tau)\, \epsilon^{\mu\alpha\beta\gamma} (S_{,\alpha_1})_{,\alpha} (S_{,\alpha_2})_{,\beta} (S_{,\alpha_3})_{,\gamma}\right)$$

which is the 1st half of the Hamilton's equations of motion. The arbitrariness in the τ-dependent proportionality factor expresses the indeterminateness in the parametrization of the curve.

Having established the direction of the tangent at one point, we now ask and answer about changes in the momentum $p_\mu = \frac{\partial S}{\partial x^\mu}$ as one proceeds along the worldline,

The fact that the H-J holds everywhere implies

$$0 = \frac{\partial}{\partial x^\nu}\mathcal{H}\Big(\overbrace{\frac{\partial S(x^\sigma, \alpha_i)}{\partial x^\mu}}^{\equiv p_\mu}, x^\nu\Big)$$

$$= \left.\frac{\partial}{\partial x^\nu}\left(\frac{\partial S}{\partial x^\mu}\right)\frac{\partial \mathcal{H}}{\partial p_\mu}\right)_{x^\nu} + \left.\frac{\partial \mathcal{H}}{\partial x^\nu}\right)_{p_\mu}$$

With the help of

$$\frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{1}{N}\frac{dx^\mu}{d\tau}$$

we obtain

$$0 = \frac{\partial}{\partial x^\mu}\underbrace{\left(\frac{\partial S}{\partial x^\nu}\right)}_{p_\nu}\frac{1}{N}\frac{dx^\mu}{d\tau} + \left.\frac{\partial \mathcal{H}}{\partial x^\nu}\right)_{p_\mu} = 0$$

or

$$\boxed{\frac{dp_\nu}{d\tau} = -N(\tau)\frac{\partial \mathcal{H}}{\partial x^\nu}}$$

This equation together with

$$\boxed{\frac{dx^\mu}{d\tau} = N(\tau)\frac{\partial \mathcal{H}}{\partial p_\mu}}$$

are Hamilton's equations of motion.

Applied to

$$\mathcal{H} = g^{\alpha\beta} p_\alpha p_\beta + m^2$$

they imply that

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \frac{1}{N}\frac{dN}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

which is the equation for a geodesic if we choose a parametrization $d\lambda = N\,d\tau$.

Consistency demands that $\mathcal{H} = 0$ be satisfied along the whole worldline. This can be verified from the fact that

$$\frac{d}{d\tau}\mathcal{H} = 0$$

i.e. $\mathcal{H} = \text{const}$, is a consequence of the Hamilton's equations of motion.

Lecture 39

Particle orbits in a static, spherically symmetric spacetime environment

Read in MTW Box 25.4, and Section 25.5

I. The ubiquity and depth of H-J theory in physics

In physics momenergy manifests itself in the form of the dynamics of particles and fields. Considering their diverse manifestation, it is difficult to find a perspective that provides a wider conceptual unification than the mathematical physics perspective of H-J theory. The commonality in its manifestations can be summarized by the symbolic equation

$$\text{H-J theory} = \begin{pmatrix}\text{particle}\\ \text{mechanics}\end{pmatrix} \cap \begin{pmatrix}\text{wave}\\ \text{mechanis}\end{pmatrix} \cap \begin{pmatrix}\text{geometrical}\\ \text{optics}\end{pmatrix} \cap \begin{pmatrix}\text{wave}\\ \text{optics}\end{pmatrix} \cap \begin{pmatrix}\text{classical}\\ \text{relativistic}\\ \text{mechanics}\end{pmatrix} \cap \begin{pmatrix}\text{relatistic}\\ \text{quantum}\\ \text{mechanics}\end{pmatrix}$$

II. H-J theory for the mechanics of a particle in the Schwartzschild geometry.

A. H-J equation and its solution

The metric for the Schwarzschild geometry is

$$g_{\mu\nu}\, dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right) dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{39.1}$$

The corresponding H-J equation is

$$0 = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = -\frac{1}{1-\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)^2 + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + m^2 \tag{39.2}$$

Its prominent feature is that t and φ are cyclic coordinates. Consequently, separation of variables yields

$$\frac{\partial S}{\partial t} = const \equiv -E$$

$$\frac{\partial S}{\partial \varphi} = const. \equiv p_\varphi,$$

and therefore

$$S = -Et + p_\varphi \varphi + S'(r,\theta) .$$

It follows that the H-J equation reduces to a p.d.e. of two variables only

$$-\frac{E^2}{1-\frac{2M}{r}} + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S'}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial S'}{\partial \theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right] + m^2 = 0 .$$

Continuing with the separation of variables process, set

$$S'(r,\theta) = R(r) + \Theta(\theta)$$

and isolate the expression which depends on θ only:

$$\left[\left(\frac{d\Theta}{d\theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right] = \frac{r^2}{\left(1-\frac{2M}{r}\right)}\left\{E^2 - m^2\left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^2\left(\frac{dR}{dr}\right)^2\right\} .$$

Both sides of this reworked H-J equation are equal to the same (obviously) non-negative constant $\ell^2 \geq 0$.

It follows that

$$\frac{d\Theta}{d\theta} = \pm\sqrt{\ell^2 - \frac{p_\varphi^2}{\sin^2\theta}}$$

$$\frac{dR}{dr} = \frac{\pm 1}{\left(1-\frac{2M}{r}\right)}\left\{E^2 - \left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right\}^{1/2}$$

The complete solution to the H-J equation is therefore

$$S(t,r,\theta,\varphi) = \underbrace{\int^{t} -E\,dt'}_{p_t} \pm \underbrace{\int^{r}\left\{E^2-\left(\frac{\ell^2}{r'^2}+m^2\right)\left(1-\frac{2M}{r'}\right)\right\}^{1/2}\frac{1}{\left(1-\frac{2M}{r'}\right)}}_{p_r}\,dr' \pm \int^{\theta}\underbrace{\sqrt{\ell^2-\frac{p_\varphi^2}{\sin^2\theta'}}}_{p_\theta}\,d\theta' + \int^{\varphi}\underbrace{p_\varphi}_{p_\varphi}\,d\varphi' + \beta(E,\ell^2,p_\varphi^2) \qquad (39.4)$$

This globally defined dynamical phase is in the form of a path-independent line integral. Its gradient is the 4-momentum covector

$$dS = \frac{\partial S}{\partial x^\mu}\underset{\sim}{d}x^\mu = p_\mu(x^\alpha)\underset{\sim}{d}x^\mu .$$

B. The conditions for constructive interference.

A dynamical phase function is characterized by three separation/integration constants:

$\alpha_1 = -E$ (– "mass-energy")
$\alpha_2 = \ell^2$ ("[angular momentum]^2")
$\alpha_3 = p_\varphi$ ("z-component of the angular momentum")

Constructive interference applied to a dynamical

$\boxed{39.4}$

phase function, Eq. (39.4), yields

$$0=\frac{\partial S}{\partial E}=-t+\int\frac{E}{\pm\left[E^2-\left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}}\frac{dr}{\left(1-\frac{2M}{r}\right)}+\frac{\partial\beta}{\partial E} \qquad (39.5)$$

$$0=\frac{\partial S}{\partial \ell^2}=\int\frac{\frac{1}{2}\,d\theta}{\pm\sqrt{\ell^2-\frac{p_\varphi^2}{\sin^2\theta}}}-\int\frac{1/2}{\pm\left[E^2-\left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}}\frac{dr}{r^2}+\frac{\partial\beta}{\partial \ell^2} \qquad (39.6)$$

$$0=\frac{\partial S}{\partial p_\varphi}=\int\frac{p_\varphi}{\pm\sqrt{\ell^2-\frac{p_\varphi^2}{\sin^2\theta}}}\frac{d\theta}{\sin^2\theta}+\varphi+\frac{\partial\beta}{\partial p_\varphi} \qquad (39.7)$$

For a given set of integration (= separation) constants

$$E,\ \ell^2,\ p_\varphi;\ \frac{\partial\beta}{\partial E},\ \frac{\partial\beta}{\partial \ell^2},\ \frac{\partial\beta}{\partial p_\varphi} \qquad (39.8)$$

each of these three interference conditions defines a 3-dimensional submanifold in the ambient 4-d spacetime spanned by its (t, r, θ, φ) coordinate system.

The intersection of these submanifolds is a specific 1-d submanifold, the globally defined particle worldline. Each one is uniquely identified by the six parameters, Eq. (39.8)

The tangents to these worldlines are determined by

$$\frac{dx^\mu}{d\tau}=\frac{\partial\mathcal{H}}{\partial p_\mu}, \qquad \mu=0,1,2,3 \qquad (39.9)$$

where $\mathcal{H}=\frac{1}{2}g^{\mu\nu}p_\mu p_\nu$

$$\frac{dt}{d\tau}=\frac{E}{1-\frac{2M}{r}}=-\frac{1}{1-\frac{2M}{r}}\frac{\partial S}{\partial t} \qquad (39.10)$$

$$\frac{dr}{d\tau}=\left[E^2-\left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+m^2\right)\right]^{1/2}=\left(1-\frac{2M}{r}\right)\frac{\partial S}{\partial r} \qquad (39.11)$$

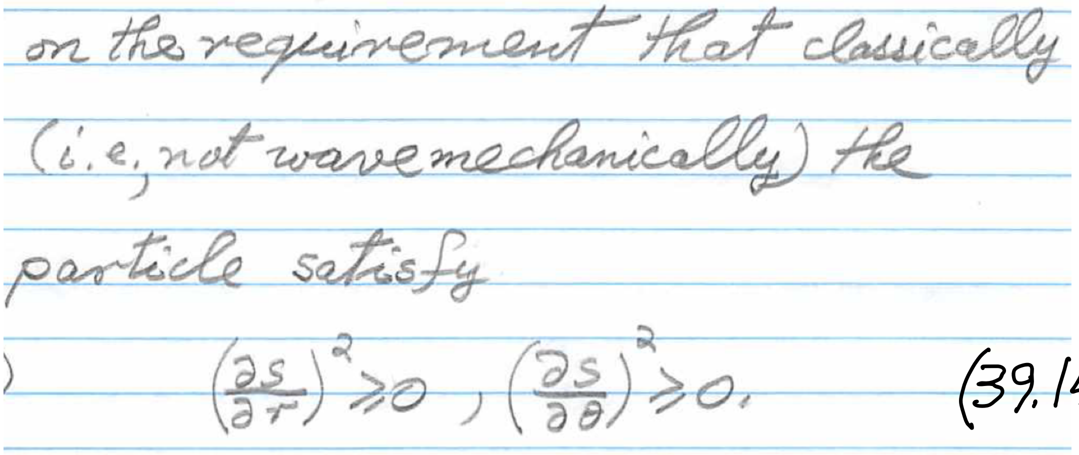
$$\frac{d\theta}{d\tau}=\frac{1}{r^2}\left[\ell^2-\frac{P_\varphi^2}{\sin^2\theta}\right]^{1/2}=\frac{1}{r^2}\frac{\partial S}{\partial\theta} \qquad (39.12)$$

$$\frac{d\varphi}{d\tau}=\frac{1}{r^2}\frac{P_\varphi}{\sin^2\theta}=\frac{1}{r^2\sin^2\theta}\frac{\partial S}{\partial\varphi} \qquad (39.13)$$

The constructive interference conditions Eqs. (39.5)-(39.7) on page 39.4 do not lack any geometrical and physical information about the dynamics of free particle in the Schwarzschild geometry represented relative to the metric as represented by

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2\left[d\theta^2+\sin^2\theta\, d\varphi^2\right].$$

However, instead of giving a mathematically completed analysis based on Eqs (39.5)-(39.7) one can already draw important conclusions based



on the requirement that classically (i.e, not wavemechanically) the particle satisfy

$$\left(\frac{\partial S}{\partial r}\right)^2 \geq 0\ ,\ \left(\frac{\partial S}{\partial \theta}\right)^2 \geq 0. \qquad (39.14)$$

C. Classically allowed vs classically forbidden regions

Because of inequalities Eqs.(39.14), space is divided into regions which a <u>classically allowed</u> vs. those which are <u>classically forbidden</u>. There

$$\left(\frac{\partial S}{\partial r}\right)^2 < 0 \text{ and } \left(\frac{\partial S}{\partial \theta}\right)^2 < 0,$$

meaning that the momentum components become imaginary!

The boundary between the regions is located where

$$\boxed{\left(\frac{\partial S}{\partial r}\right)^2 = 0} \quad \boxed{\left(\frac{\partial S}{\partial \theta}\right)^2 = 0}$$

The significance of this boundary one infers from the Hamilton's equations of motion Eqs. (39.9). They imply that

$$\frac{dr}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_r} = g^{rr}\frac{\partial S}{\partial r} = \pm\left[E^2 - \left(\frac{\ell^2}{r^2}+m^2\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}$$

$$\frac{d\theta}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\theta} = g^{\theta\theta}\frac{\partial S}{\partial \theta} = \pm\frac{1}{r^2}\sqrt{\ell^2 - \frac{p_\varphi^2}{\sin^2\theta}}$$

Thus the boundary between what is classically allowed and what is forbidden is the locus of points where the radial and polar angle motion, comes to a momentary halt:

$$\frac{dr}{d\tau}=0$$

and

$$\frac{d\theta}{d\tau}=0.$$

This is the location of <u>turning points</u>, where particle motions $\frac{dr}{d\tau}$ and $\frac{d\theta}{d\tau}$ must reverse sign.

D. The effective potential for classically allowed motion.

From this locus of turning points one can infer major qualitative aspects such as bounded vs unbounded motion, stable vs unstable motion. As an example, consider the radial motion as determined by its locus of turning points:

$$\frac{dr}{d\tau}=0 \Rightarrow E^2-V^2_{eff}(r)=0$$

Upon considering equatorial motion $\theta=\frac{\pi}{2}$ one has $\ell^2=p_\varphi^2$ so that

$$V^2_{eff}=m^2-\frac{2M}{r}m^2+\frac{p_\varphi^2}{r^2}-\frac{2M}{r}\frac{p_\varphi^2}{r^2}$$
$$=m^2\left(1-\frac{2M}{r}\right)\left(1+\frac{p_\varphi^2}{r^2}/m^2\right)$$

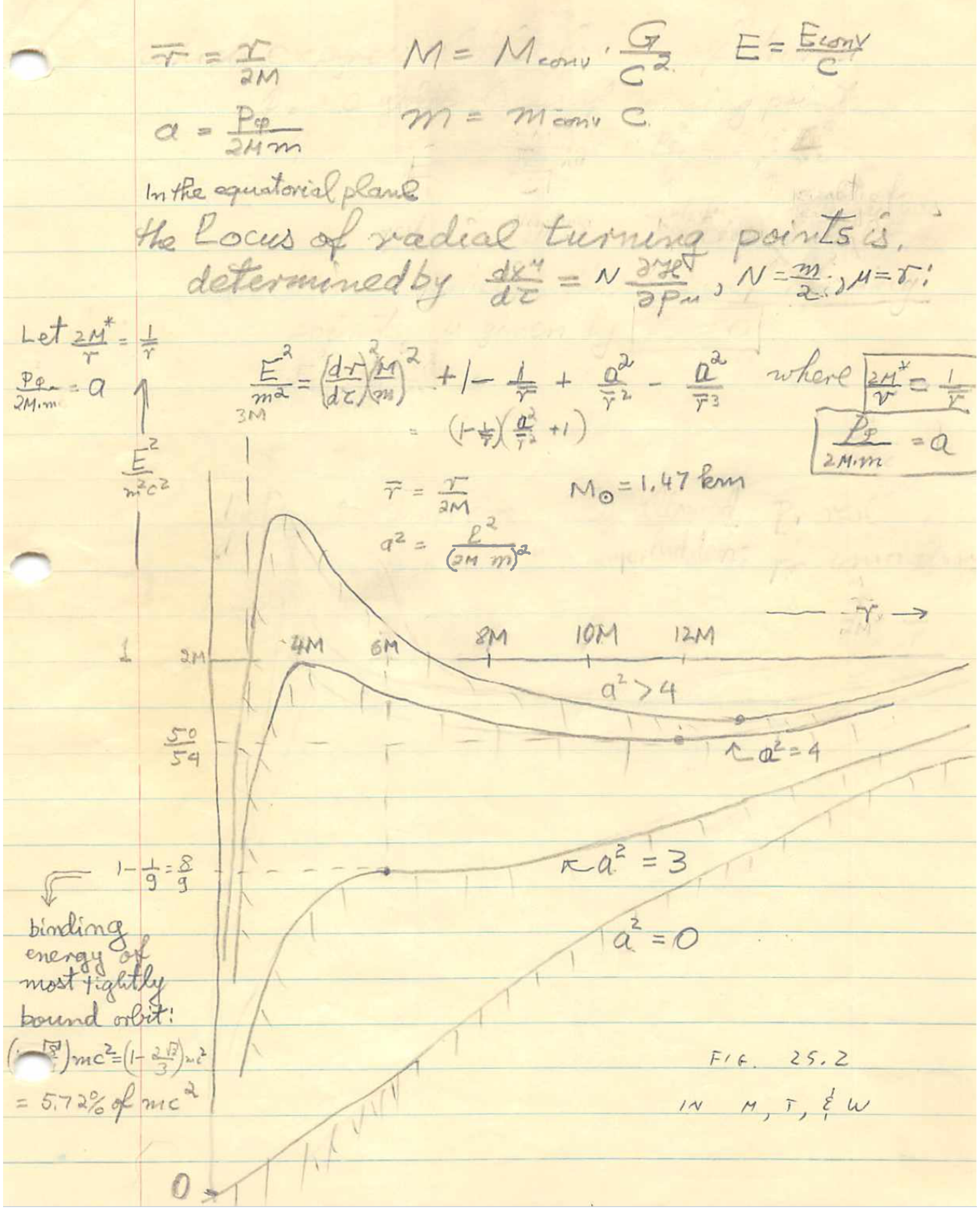
Upon introducing dimensionless quantities

$$\frac{2M}{r}=\frac{1}{\bar r}\ ,\quad \frac{p^2}{2Mm}=a$$

we obtain the following contributions to the radial potential:

$$\frac{E^2}{m^2}=\underbrace{1}_{\substack{\text{rest}\\ \text{mass}}}-\underbrace{\frac{1}{\bar r}}_{\substack{\text{Newtonian}\\ \text{attraction}}}+\underbrace{\frac{a^2}{\bar r^2}}_{\substack{\text{centrifugal}\\ \text{repulsion}}}-\underbrace{\frac{a^2}{\bar r^3}}_{\substack{\text{Angular}\\ \text{kinetic}\\ \text{energy has}\\ \text{weight.}}}=\left(1-\frac{1}{\bar r}\right)\left(1+\frac{a^2}{\bar r^2}\right)$$

which expresses the locus of turning points that seperates a classically allowed from a classically forbidden region.



$\bar{r} = \frac{r}{2M}$ $M = M_{conv} \cdot \frac{G}{c^2}$ $E = \frac{E_{conv}}{c}$

$a = \frac{p_\varphi}{2Mm}$ $m = m_{conv}\, c$

In the equatorial plane the locus of radial turning points is, determined by $\frac{dx^\mu}{d\tau} = N \frac{\partial \mathcal{H}}{\partial p_\mu}$, $N = \frac{m}{2}$; $\mu = r$:

Let $\frac{2M^*}{r} = \frac{1}{\bar{r}}$

$\frac{p_\varphi}{2M \cdot m} = a$

$$\frac{E^2}{m^2} = \left(\frac{dr}{d\tau}\right)^2 \left(\frac{M}{m}\right)^2 + 1 - \frac{1}{\bar{r}} + \frac{a^2}{\bar{r}^2} - \frac{a^2}{\bar{r}^3}$$

$$= \left(1 - \frac{1}{\bar{r}}\right)\left(\frac{a^2}{\bar{r}^2} + 1\right)$$

where $\boxed{\frac{2M^*}{r} = \frac{1}{\bar{r}}}$ $\boxed{\frac{p_\varphi}{2M \cdot m} = a}$

$\bar{r} = \frac{r}{2M}$ $M_\odot = 1.47$ km

$a^2 = \frac{\ell^2}{(2M\,m)^2}$

$1 - \frac{1}{9} = \frac{8}{9}$

binding energy of most tightly bound orbit:

$\left([illegible] \sqrt{\frac{8}{9}}\right) mc^2 = \left(1 - \frac{2\sqrt{2}}{3}\right) mc^2$

$= 5.72\%$ of mc^2

FIG. 25.2

IN M, T, & W

Figure 39.1 Locus of turning pts of particles have angular momenta $p_\varphi = 0, \sqrt{3}(2Mm), 2(2Mm); p_\varphi > 2(2Mm)$.

We note that for large enough angular momentum ($p_\varphi > \sqrt{3}\, 2M \cdot m$)

(i) there is bounded motion $E < m$

unbounded motion $E > m$

as well as motion in which the particle disappears into the black hole ($r < 2M$).

(ii) there exist stable ("Newtonian") as well as unstable ("relativistic") circular orbits.

They are determined by

$$\frac{dE^2(r)}{dr} = 0 ,$$ which implies

$$\frac{r}{2M} = a^2\left(1 \pm \sqrt{1 - \frac{3}{a^2}}\right) \qquad a = \frac{p_\varphi}{2M m}$$

From the catalogue of circular orbits

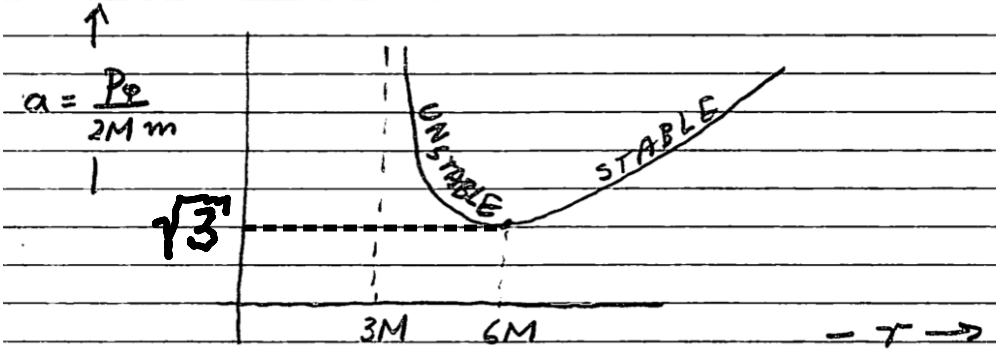


Figure 39.2 Circular orbits of particles catalogued by their angular momentum p_φ.

39,10

one can see that there exist no circular orbits, stable or unstable, for $r < 3M$.

and that the most tightly bound stable circular orbit has radius

$$r = 6M,$$

Lecture 40

Schwarzschild Spacetime:
It's topological and
causal structure

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{In MTW read \S 31.6, Figure 31.5}\\ \text{'' \quad '' \quad do Exercise 31.7}\end{array}\right\}\begin{array}{l}\text{Topological}\\ \text{structure}\end{array}\\ \left.\begin{array}{l}\text{'' \quad '' \quad read Box 31.2}\\ \text{'' \quad '' \quad read Section 32.3}\end{array}\right\}\begin{array}{l}\text{Causal}\\ \text{structure}\end{array}\end{array}\right\}\begin{array}{l}\text{For a}\\ \text{spherical}\\ \text{Vacuum}\\ \text{Configuration}\end{array}$$

I. The Schwarzschild Spacetime

In 1923 G. Birkoff showed that

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2(d\theta^2+\sin^2\theta\, d\varphi^2), \qquad (40.1)$$

which was first exhibited by K. Schwarzschild, is the only solution to the E.F.E.s in vacuum which is spherically symmetric. The Schwarzschild coordinates relative to which the Schwarzschild solution is a mathematical package deal: a combination of good and non-good; it directs attention to the spatial topological structure while at the same time lacking information about its causal structure. The latter is exposed relative the Eddington-Finkelstein coordinates. But to exhibit both structures requires the representation by means of the globally defined Kruskal-Szekeres coordinates.

A. Schwarzschild Coordintes: Spatial Topological Structure.

The most eye-catching aspect of the Schwarzschild metric, Eq. (40.1), is its mathematical behavior at $r=2M$. This, however, is not a signal for some sort of extreme physical behavior.

1. Indeed, consider using a plumb line to measure the proper distance from some fixed radius ($r=\sqrt{\frac{\text{area}}{4\pi}}$) down to $r=2M$:

$$\int_{r}^{r=2M} \frac{dr'}{\sqrt{1-\frac{2M}{r'}}} = \text{finite}.$$

2. Second, consider the amount of proper time it takes for a particle to plunge from some finite radius down to $r=2M$. From Eq. (39.11) (in previous lecture) one has

$$\tau = \int_{r}^{r=2M} \frac{dr'}{\left\{E^2-\left(\frac{\ell^2}{r'^2}+m^2\right)\left(1-\frac{2M}{r'}\right)\right\}^{1/2}} = \text{finite}$$

3. Third, consider the physical (O.N.) curvature components

$$\hat{R}_{\alpha\beta\gamma\delta} = \begin{Bmatrix} \pm\frac{2M}{r} \\ \pm\frac{M}{r} \end{Bmatrix} = \text{finite} \quad \text{(Relive to the Schsch. frame)}$$

$$\hat{R}_{\bar\alpha\bar\beta\bar\gamma\bar\delta} = \begin{Bmatrix} \pm\frac{2M}{r} \\ \pm\frac{M}{r} \end{Bmatrix} = \text{finite}$$

(Relative to any frame with arbitrary radial velocity relative to the Schsch frame.)

Fourth, the ~~4. The~~ imbedding diagram of the equatorial plane $\theta = \frac{\pi}{2}$ at $t = \text{const}$ in a 3-d imbedding space for

$$d\sigma^2 = \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\varphi^2$$

$$= dz^2 + dr^2 + r^2 d\varphi^2$$

yields

$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1-\frac{2M}{r}} \Rightarrow z = \pm\sqrt{8M(r-2M)}$$

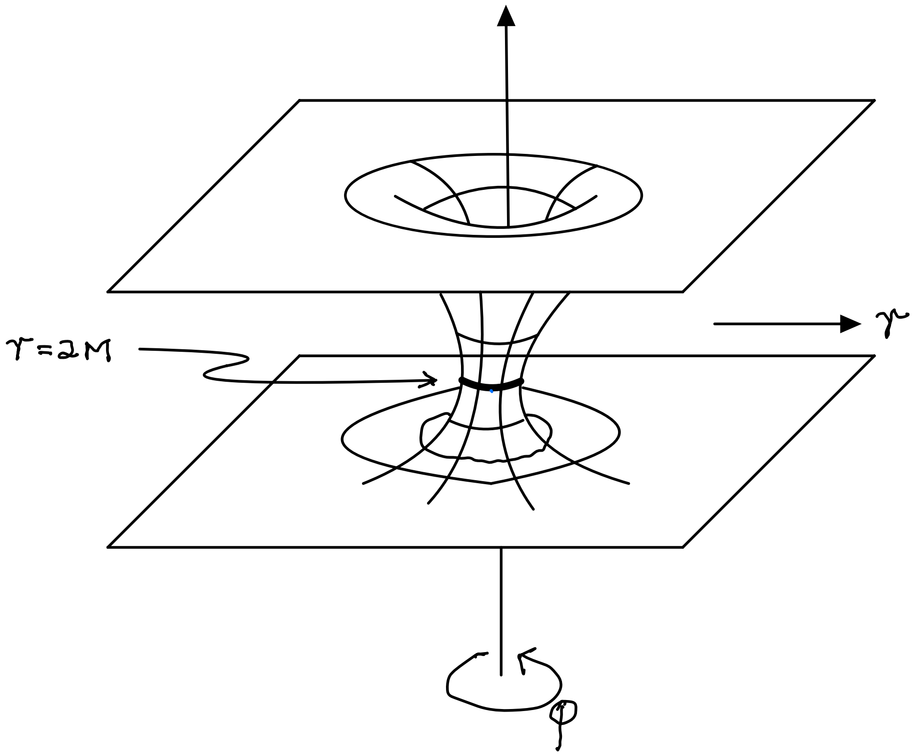


Figure 40.1 The topology of the spatial Schsch. equatorial plane is that of two asymptotically flat domains connected by a throat whose minimal radius is 2M.

The two asymptotically flat 3-d spaces are isometric copies of each other. Indeed, this isometry becomes evident when one changes from the radial Schsch coordinate to the radial conformal coordinate,

$$r \rightarrow \rho,$$

as determined by the condition that

$$d\sigma^2 = -\frac{dr^2}{1-\frac{2M}{r}} + r^2 d\varphi^2 = f^2(\rho)\left[d\rho^2+\rho^2 d\varphi^2\right].$$

By equating coefficients one obtains

$$\left.\begin{array}{r} r = \rho f(\rho) \\ \dfrac{1}{\sqrt{1-\frac{2M}{r}}} = f(\rho)\dfrac{d\rho}{dr} \end{array}\right\} \frac{dr}{d\rho} = \frac{r}{\rho}\sqrt{1-\frac{2M}{r}}$$

Consequently,

$$r = \rho\left(1+\frac{M}{2\rho}\right)^2 \text{ and } f(\rho) = \left(1+\frac{M}{2\rho}\right)^2.$$

(i) The graph of the Schsch radial coordinate

$$r \equiv \frac{\text{circumference}}{2\pi}$$

in terms of the conformal coordinate ρ is depicted in Figure 40.2

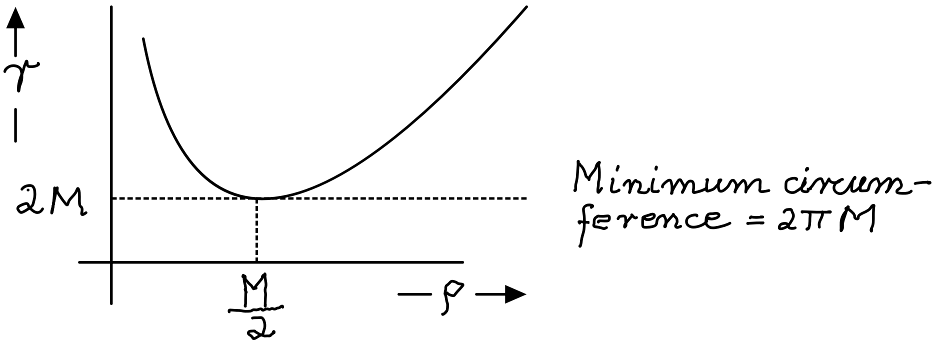


Figure 40.2

(ii) Relative to this conformal radial coordinate the Schsch metric is

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2} dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \left[d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)\right] \qquad (40.2)$$

(iii) This geometry is symmetric around $\rho = \frac{M}{2}$ under the <u>isometric</u> transformation

$$\frac{2\rho}{M} \longrightarrow \frac{M}{2\bar\rho} \qquad \left(\rho = \left(\frac{M}{2}\right)^2 \frac{1}{\bar\rho}\right) \qquad (40.3)$$

This transformation maps the entire manifold $\rho > 0$ onto itself with the same metric:

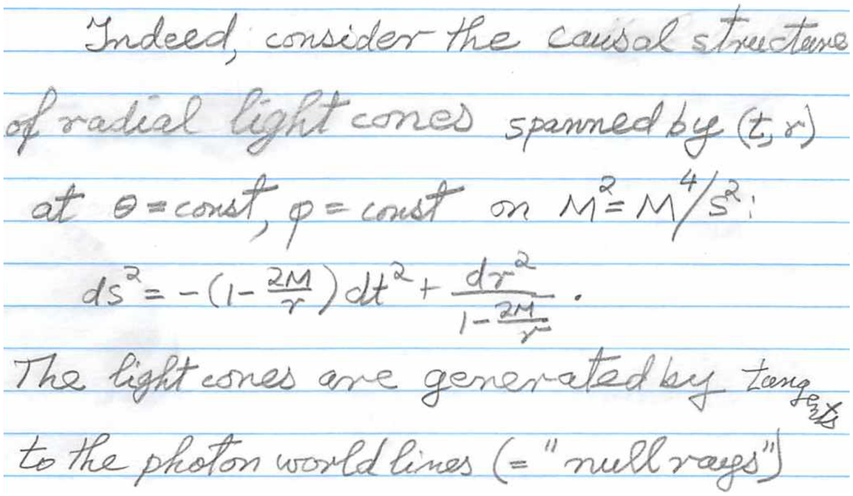
$$ds^2 = -\frac{\left(1-\frac{M}{2\bar\rho}\right)^2}{\left(1+\frac{M}{2\bar\rho}\right)^2} dt^2 + \left(1+\frac{M}{2\bar\rho}\right)^4 \left[d\bar\rho^2 + \bar\rho^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)\right]$$

(iv) Conclusion:

The global static Schsch solution to the EFEs is a mathematization of two asymptotically flat isometric 3-d spaces depicted in Figure 40.1.

B. Causal Structure: The Eddington-Finkelstein coordinatization

The time invariant representation of the Schsch metric, Eq. (40.1) on page 40.1, is deficient. This is because the $r=2M$ singularity in the time component of the metric hides the causal structure of Schsch spacetime.



Indeed, consider the causal structure of radial light cones spanned by (t,r) at $\theta=\text{const}$, $\varphi=\text{const}$ on $M^2=M^4/S^2$:

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}.$$

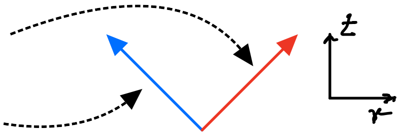
The light cones are generated by tangents to the photon world lines (= "null rays")

$$(\Delta s)^2=0: \quad \frac{dr}{dt}=\left(1-\frac{2M}{r}\right) \qquad (40.4)$$

outgoing

$$\frac{dr}{dt}=-\left(1-\frac{2M}{r}\right) \qquad (40.5)$$

ingoing



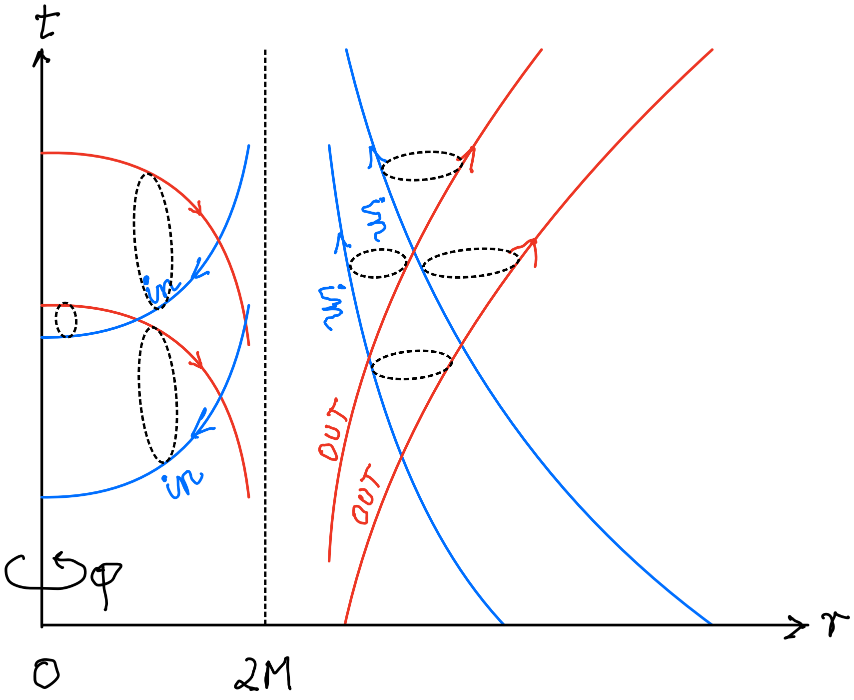
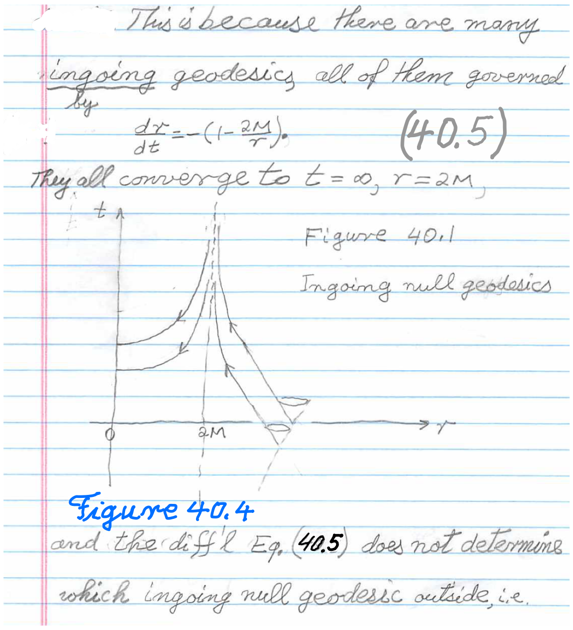


Figure 40.3 Ingoing and outgoing world lines of classical photons

The metric $r=2M$ singularity exposes the fact that t-coordinate is a bad coordinate at $r=2M$.



This is because there are many ingoing geodesics all of them governed by

$$\frac{dr}{dt}=-\left(1-\frac{2M}{r}\right). \qquad (40.5)$$

They all converge to $t=\infty$, $r=2M$,

Figure 40.1

Ingoing null geodesics

Figure 40.4

and the diff'l Eq. (40.5) does not determine which ingoing null geodesic outside, i.e.

$2M<r$, goes with which ingoing null geodesic inside, i.e. $r<2M$. Furthermore, is $(t=\infty, r=2M)$ a single event or is a set of distinct events?

The same ambiguity holds for outgoing radial null geodesics which are mathematized by the d.e.

$$\frac{dr}{dt}=\left(1-\frac{2M}{r}\right).$$

They all diverge from $(t=-\infty, r=2M)$

Figure 40.5

These ambiguities have been resolved by Eddington (1924) and Finkelstein (1958). They integrated Eq. (40.5) and introduced the integration constant as a new coordinate function that replaces the "bad" Schsch time coordinate t: The integrals of

$$0=dt+\frac{dr}{1-\frac{2M}{r}}\equiv dt+dr^*$$

are

$$\tilde{V}=t+r+2M\ln\left(\frac{r}{2M}-1\right)=\tilde{V}$$

The isograms of

$$\tilde{V}(t,r) = t + r + 2M \ln\left(\frac{r}{2M} - 1\right) = \text{const.}$$
$$= t + r^{*} = \text{const} \qquad (40.6)$$

are the ingoing null geodesics:

$\tilde{V}$ is their new coordinate function. It called the advanced Schsch time coordinate,

Similarly, the isograms of

$$\tilde{U}(t,r) = t - r^{*} = \text{const} \qquad (40.7)$$

are the outgoing null geodesics;

and $\tilde{U}$ is called the retarded Sch sch time coordinate.

The radial variable

$$r^{*} \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right) \qquad (40.8)$$

is called the "tortoise coordinate"

Introduce $(\tilde{V}, r)$ as the (new) ingoing Eddington-Finkelstein coordinates for the Sch sch geometry:

$$\left.\begin{aligned} d\tilde{V} &= dt + \frac{dr}{1-\frac{2M}{r}} \\ dr &= dr \end{aligned}\right\} \quad dt = d\tilde{V} - \frac{dr}{1-\frac{2M}{r}} \qquad (40.9)$$

Relative to these new coordinates

the Schsch metric becomes non-diagonal;

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2$$ 40.10

$$= -\left(1-\frac{2M}{r}\right)\left(dt + \frac{dr}{1-\frac{2M}{r}}\right)\left(dt - \frac{dr}{1-\frac{2M}{r}}\right) + "$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}\left(d\tilde{V} - \frac{2\,dr}{1-\frac{2M}{r}}\right) + "$$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)d\tilde{V}^2 + 2\,dV\,dr + r^2 d\Omega^2},$$ (40.10)

but it is nonsingular at $r=2M$.

Thus ingoing radial null geodesics are characterized by

$$\frac{dt}{dr} + \frac{1}{1-\frac{2M}{r}} = \boxed{\frac{d\tilde{V}}{dr} = 0}$$ (40.11)

while outgoing radial null geodesics are characterized by

$$0 = \frac{dt}{dr} - \frac{1}{1-\frac{2M}{r}} = \frac{d\tilde{V}}{dr} - \frac{2}{1-\frac{2M}{r}}$$

or $\boxed{\frac{d\tilde{V}}{dr} = \frac{2}{1-\frac{2M}{r}}}$ (40.12)

The two boxed Eqs. (40.11) - (40.12) are the tangents to the two sets of null geodesics

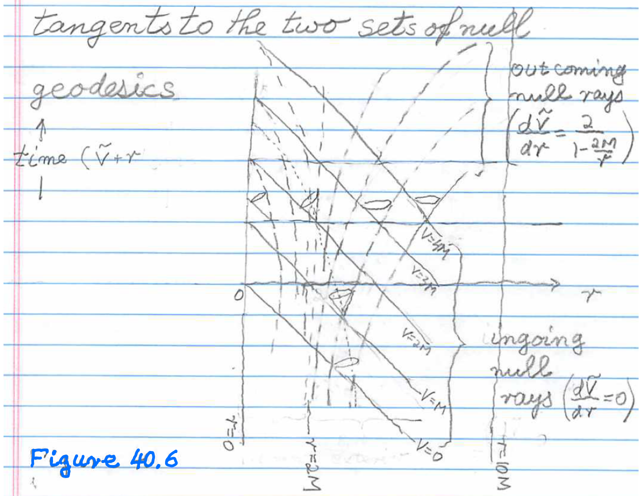


Figure 40.6

Ingoing Eddington-Finkelstein coordinate system $(\tilde{V}, r)$.

b) Outgoing E-F ("Retarded time") coordinates.

These coordinates are based on outgoing null geodesics

$$\frac{dr}{dt} = 1 - \frac{2M}{r},$$

the integrals of

$$0 = dt - \underbrace{\frac{dr}{1-\frac{2M}{r}}}_{dr^*}$$

are

$$\tilde{U} = t - \overbrace{\left(r+2M\ln\left(\frac{r}{2M}-1\right)\right)}^{r^*}$$

The isograms of

$$\tilde{U}(t,r) = t - \left(r+2M\ln\left(\frac{r}{2M}-1\right)\right)$$

are the outgoing null geodesics, and $\tilde{U}$ is called the retarded Schsch. time coordinate.

Relative to the $(\tilde{U}, r)$ coordinate system the Schsch metric has the form

$$ds^2 = -(1-\frac{2M}{r})\,d\tilde{U}^2 - 2\,d\tilde{U}\,dr + r^2\,d\Omega^2$$

The outgoing null geodesics obey

$$\frac{d\tilde{U}}{dr} = 0$$

while the ingoing null geodesics obey

$$\frac{d\tilde{U}}{dr} = \frac{-2}{1-\frac{2M}{r}}$$

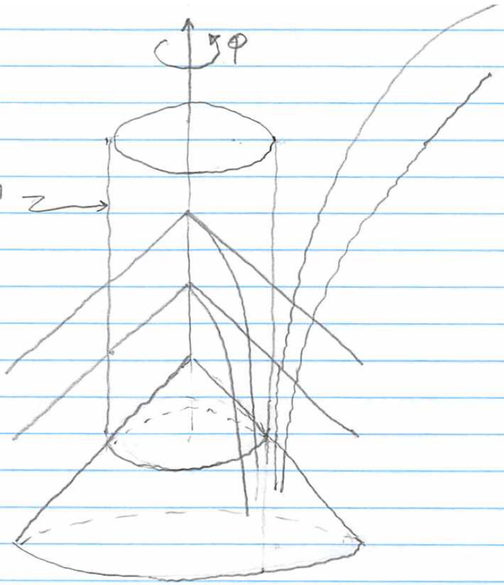


Figure 40.7
OUTGOING Eddington-Finkelstein coordinates system $(\tilde{V}, r)$.

Lecture 41

Spacetime geometry of a
spherical black hole and
“white hole”

In MTW peruse Box31.2

Inspite of the fact that the black hole metric representation

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (41.1)$$

exhibits an obvious pathology at $r=2M$, physical and geometrical scrutiny indicate that the opposite is the case (finite proper distances, proper times, curvature, smoothness of the spatial geometry): $r=2M$ is <u>not</u> a physical singularity.

The cause and the cure of this mathematical pathology is found once one takes cognizance of the central role of radial light pulses in representing the black hole metric.

The spacetime trajectories (a.k.a. "null rays" or "null geodesics") of these light pulses are solutions to the differential equations obtained by inspecting Eq.(41.1),

$$dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2} = 0 \Rightarrow \begin{cases} dt + \dfrac{dr}{1-\frac{2M}{r}} \equiv d\tilde{V} = 0 \rightarrow t + r + 2M\ln\left(\frac{r}{2M}-1\right) = \tilde{V} & (41.2) \\ dt - \dfrac{dr}{1-\frac{2M}{r}} \equiv d\tilde{U} = 0 \rightarrow t - r - 2M\ln\left(\frac{r}{2M}-1\right) = \tilde{U} & (41.3) \end{cases}$$

It is a fact that for $r>2M$ an observer distinguishes between ingoing and outging light pulse world lines. They are mathematized by Eqs.(41.2) and (41.3) and their depiction relative to Schsch coordinates are given in Figures 41.1 and 41.3

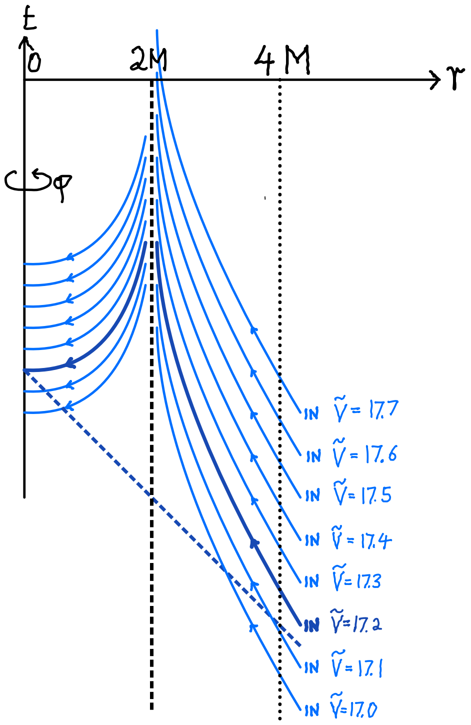


Figure 41.1 World lines of ingoing light pulses travelling towards $r=2M$ of a black hole. Every such world line **has its own value of $\tilde{V}$ as a constant of motion.** Equation (41.2) necessitates that the Schsch slope of every world line depicted in the figure is

$$\frac{dt}{dr} = -\frac{1}{1-\frac{2M}{r}},$$

and that their graphs in the Schsch plane are

$$t + r + 2M \ln\left|\frac{r}{2M} - 1\right| = \tilde{V}.$$

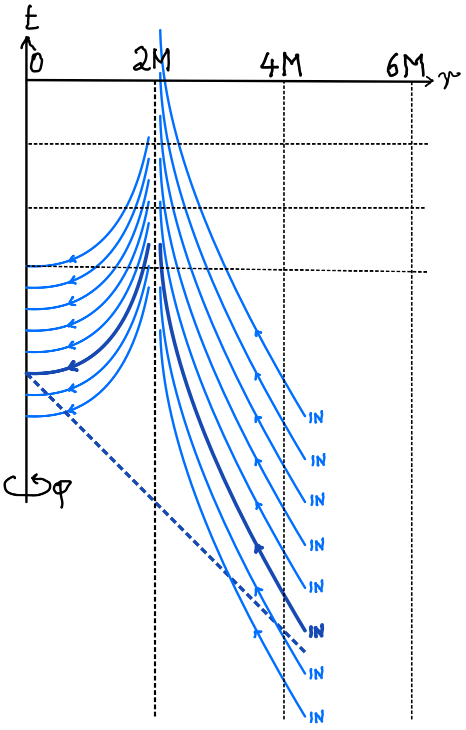
The exterior domain of a black hole is $r > 2M$. Its interior is $r < 2M$.

But this partitioning, with the $r = 2M$ boundary between the two, and knowing which event on one side is close to an event on the other, requires a representation of the black hole metric which is continuous across both domains. Transitioning from the Schsch to the <u>ingoing</u> Eddington-Finkelstein coordinates fulfills this requirement. As shown in the text, the black hole metric is indeed continuous across both domains. As a consequence each ingoing light pulse world continues it evolution uniquely into the interior $r < 2M$. The heavy dark trajectory in the interior is the continuous evolution of the ingoing one in the exterior:

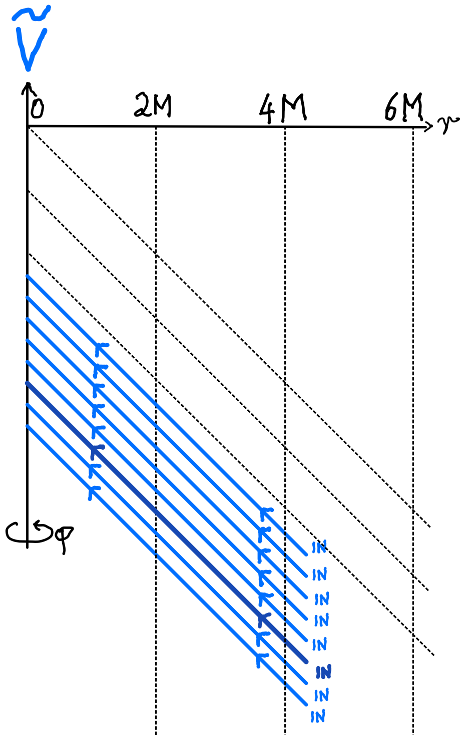
Even though $r=2M$ is not a physical singularity (i.e. all measurable attributes involving $r=2M$ are finite and well-defined), the Schsch coordinate representation of the spacetime metric of a black hole is singular at $r=2M$. According to this mathematical representation, no ingoing entity, including light pulses, can pass through $r=2M$.

This contradiction is removed by changing to a coordinate system relative to which the black hole metric is non-singular at $r=2M$.

The coordinate transformation $(t,r)\rightarrow(\tilde{V},r)$ of Eq. (41.2)



A



B

Figure 41.2 Standard (A) and light pulse-induced (B) coordinatizations of the exterior and the interior of a black hole.

Panel A: World lines of ingoing light pulses relative to Schsch cordinates (t, r).

Panel B: World lines of ingoing light pulses are straightened out by the ingoing Eddington-Finkelstein coordinates $(\tilde{V}, r)$. They terminate mathematically at $r=0$.

yields a representation of the black hole metric which is non-singular across $r=2M$,

$$ds^2 = -\left(1-\frac{2M}{r}\right) d\tilde{V}\left(d\tilde{V} - 2\,\frac{dr}{1-\frac{2M}{r}}\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}^2 + \quad 2\, d\tilde{V}\, dr \quad + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

The world lines of radial ingoing light pulses trace out the isograms of the coordinate function $\tilde{V}$ of the oblique $\tilde{V}$-r coordinate system. They terminate mathematically at $r=0$.

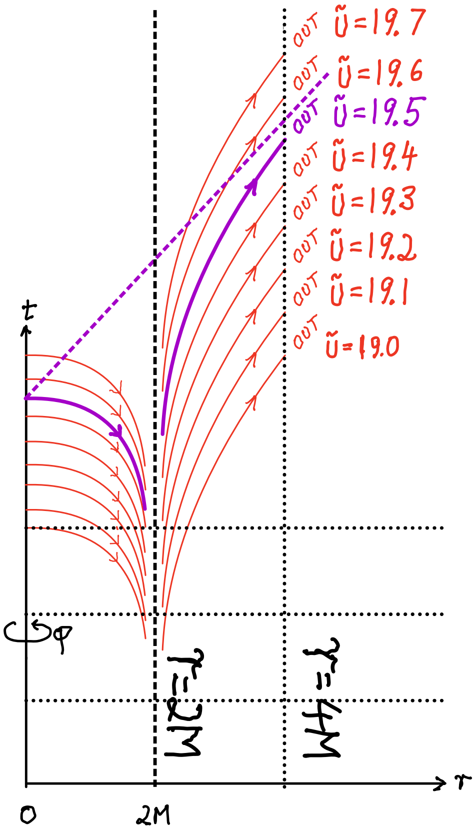


Figure 41.3 World lines of outgoing light pulses coming from $r=2M$ of a "white hole". Every such world line **has its own value of $\tilde{U}$ as a constant of motion.** Equation (41.3) necessitates that the

Schsch slope of every world line depicted in the figure is

$$\frac{dt}{dr}=\frac{1}{1-\frac{2M}{r}},$$

and that their graphs in the Schsch plane are

$$t-r-2M\ln\left|\frac{r}{2M}-1\right|=\tilde{U}.$$

The exterior domain of a "white hole" is $r>2M$. Its interior is $r<2M$.

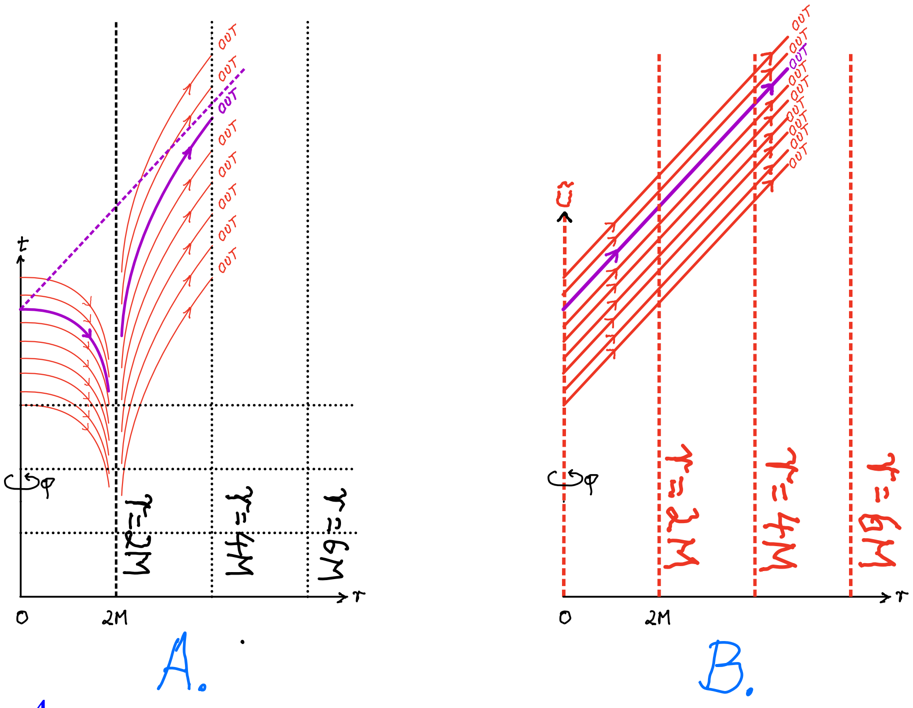


Figure 41.$\overset{4}{\text{x}}$ <---41.4

In the Schsch coordinate representation (panel A) of the "white hole" spacetime, $r=2M$ is not part of the manifold. In any nbhd surrounding $r=2M$ there is no causal connection between radial light pulse worldlines created in $r<2M$ and those in $r>2M$. However, relative to the (outgoing) E-F coordinate representation (panel B), every outgoing radial light pulse in $r>2M$ evolves continuosly from a unique light pulse worldline created in $r<2M$.

In particular, the world line depicted by the dark heavy curve in panel B connects (into a single world line) what appears as two dark and heavy disjoint worldlines relative to the Schsch coordinatized spacetime depicted in panel A.

The representation of the "white hole" metric relative to the outgoing light pulse induced E-F coordinates is

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[d\tilde{U}\cdot\left(d\tilde{U} + \frac{2}{1-\frac{2M}{r}}dr\right)\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$= -\left(1-\frac{2M}{r}\right)d\tilde{U}^2 - 2\, d\tilde{U}\, dr \quad + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Unlike the Schsch representation, the E-F representation of the metric is non-singular, even at $r=2M$.