MATH 5757

MODERN
MATHEMATICAL
METHODS
IN
RELATIVITY THEORY II

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Modern Mathematical Methods in Relativity Theory II

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Lecture 1

Imprints of gravitation on the states of motion of particles. A comparison of Galileo's, Newton's, and Einstein's formulation of these imprints.

Read §1.1 in MTW
1.1 gravity in terms of the characteristic

The purpose of this course is to grasp

the nature of the world, in particular

that of gravitation.

To do this, one's thinking must start

with information received from the

world. In the case of gravitation

this information is in the specific form

of the laws of motion of bodies.

In fact it is precisely in terms of the

observed motion of bodies that gives

rise to the existence and definition

of the concept gravitation. Indeed

the most important way of identifying

imprints it leaves on the motion of

bodies. In fact, historically gravita-
tion has been the chief motivating

force for the mathematical formulation

of the laws of motion because they

constitute the premier tool for identifying

what gravitation is.

(cont'd on p. 13)
The foot had the mechanical effect of...
With this law, Newton accomplished two feats:

1. He introduced a new concept, the inertial mass of a body, and
2. He gave a local definition of acceleration by means of a double limiting process applied to differential quotients.

It is the local nature of this law which allows it to capture the common aspect of the state of motion of all particles found in Nature. [There is only one proviso: the “action” of the particle’s state of motion must be large compared to Planck’s quantum of action \(h\).]

If this action is comparable to \(h\) (as it is the case, for example, for the bound motion of an electron in hydrogen atom) then Newton’s local framework does not apply, but must be replaced by a quantum-mechanical framework. In that case, the particle’s state of motion is expressed by its non-local wave function, which satisfies an appropriate wave equation.

(End of parenthetical comment)

Newton used his laws of motion to capture the imprints of gravitation. He did this by using the observational evidence expressed by Kepler’s three laws and identifying its gravitational force field so that his equations of motion took the form:

\[
\frac{d^2}{dt^2} = \frac{-GMm}{r^3}
\]

From this law one can infer the states of motion of planets, comets, satellites, and so on. One can thereby identify gravitation by its characteristic imprints on the trajectories of these bodies. Three of the best known examples of these imprints are Kepler’s three laws of planetary motion.
(1) The radius vector sun–planet sweeps out equal areas in equal times.

(2) Planets travel in an ellipse, with the sun at one focus.

(3) \( G (\text{mass of sun}) \cdot (\frac{2\pi}{\text{period}})^2 \cdot (\text{major axis})^3 = GM^2 = \omega^2 R^3 \)

These three laws constitute three particularly important imprints because they imply and are implied by Newton's law of gravitation when it is combined with Newton's second law of motion, i.e.,

\[ \{ (1), (2), \& (3) \} \Rightarrow \text{gravitational force} = \frac{GM^2}{r^3} \]

Exercise: Show that the implication \( \Rightarrow \) is indeed the case. This is much easier than showing the implication \( \Leftarrow \) which is done in most texts on mechanics. \( \Leftarrow \) entails integration, while \( \Rightarrow \) entails only differentiation. The latter is developed in the ensuing article 25-10.
How Newton was led to his universal law of gravitation: a road map

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Abstract

We identify the roots and the fundamental premise of Newton's scientific achievements: to grasp the nature of the world, one's thinking must begin with information received from the world. Adopting it, we apply elementary calculus to three pieces of information, Kepler's three laws, to obtain Newton's universal law of gravitation.

1 Newton's fundamental premise and its origin

1.1 The fundamental premise

Q: What was the fundamental premise which paved the way towards Newton's unprecedented achievement? Why was he successful, while others (like Descarte) were not?

A: Newton stated it thusly:

\[
\ldots \text{ I frame no hypotheses; \ldots} \\
\text{The word "hypothesis" is here used by me to signify only} \\
\text{(i) such a proposition as is not a phenomenon} \\
\text{(ii) nor deduced from any phenomena,} \\
\text{but assumed or supposed – without any experimental proof [whatsoever].}
\]
To be more explicit\textsuperscript{1}, he used “hypothesis” to refer to an arbitrary statement, i.e. a claim unsupported by any observational evidence. Here are some examples:

(i) The works of Plato are being studied by a reading group of gremlins on the planet Venus (to pick an obvious example).

(ii) Colored light is produced by rotating particles and white light is less produced by nonrotating particles (Descarte).

(iii) White light is a symmetrical wave pulse (Robert Hooke).

(iv) Quarks are composed of strings in a 26-dimensional space (20th century string theorist), etc.\textsuperscript{2}

As Wolfgang Pauli would say, none of these statements is right; they aren't even wrong. Following Newton, what Pauli was directing attention to was that there are three types of claims:

1. Right ones, which are true because they have a positive relationship to reality,

2. False ones, which are untrue because they have a negative relationship to reality, and

3. Arbitrary ones, for which there is no evidence whatsoever: they are detached from reality.

and it is the arbitrary ones “which aren't even wrong”.

Q: What cognitive value, if any, did Newton see in one's contemplation of arbitrary claims?

A: Newton must be credited with being the first one to identify what in 20th century vernacular is called Garbage In Gargage Out (G.I.G.O.). Writing to a friend, he said:

\textsuperscript{1}Newton did not mean to reject out of hand all hypotheses that lacked full experimental proof.

\textsuperscript{2}A continuation of this list would include astrology, intelligent design, ESP, God, an afterlife, reincarnation, ... .
"If anyone may offer conjectures about the truth of things from the mere possibility of hypotheses, I do not see by what stipulation anything certain can be determined in any science; since one or another set of hypotheses may always be devised which will appear to supply new difficulties. Hence I judged that one should abstain from contemplating hypotheses, as one does from improper argumentation."

In other words, one’s thinking ("contemplation") should not start with Garbage, i.e. arbitrary claims ("hypotheses") because the output, "conjectures about the truth of things," will also be Garbage, just as one gets "from improper argumentation".

Furthermore, as David Harriman puts it\textsuperscript{3}, one cannot even achieve the misguided goal of disproving an arbitrary idea. Such a claim can always be shielded by further arbitrary assertions ("one or another set of hypotheses") There is only one way out of such a proliferating web of arbitrary conjectures, and that is to dismiss them outright as uncognitive and unworthy of attention.

This is why Newton insisted that the arbitrary be rejected without contemplation.

With this Newton introduced a new epistemological principle into the theory of knowledge: The outright dismissal of arbitrary claims, without contemplation!

For this principle alone Newton deserves to be regarded as the greatest epistemologist of his era.

Q: What, then, is Newton’s fundamental premise stated positively?

A: To grasp the nature of the world one’s thinking has to start with information received from the world.

a) What is the nature of the world? The world is a causal realm ruled by natural law. It is not a realm of inexplicable miracles ruled by a supernatural power, nor is it an unintelligible chaos ruled by chance. Instead, the nature of the world is expressed by the observation that

“Things are what they are because they were what they were, and things will be what they will be because they are what they are”

This is the law of causality, Aristotle’s law of identity (everything has a specific nature; things are what they are; A is A) applied to actions.

b) That one’s thinking “start with information received from the world” is the starting point of Aristotle’s epistemology, the evidence of the senses, be they aided or unaided by specialized instruments.

1.2 Its origin

Newton did not arrive at his fundamental premise in a cultural vacuum.

Q: What was the frame of reference – the context – that led to the achievements of Newton, those before him, and those after him?

A: Here it is essential to realize that Aristotle, whose works Newton had studied as a college student, may be considered as the cultural barometer of Western History. Whenever his influence dominated the scene it paved the way towards histories most brilliant eras, whenever it fell so did mankind.

Aristotle’s revival in the 13th century brought men to the Renaissance, and the Renaissance led to the Age of Reason, the Enlightenment. Indeed, Galileo was born in the year that Michelangelo died (1564), and Newton was born on the day that Galileo died (1642).

2 Newton’s universal law of gravitation

The Enlightenment was ushered in by Newton’s unprecedented achievements. There were three of them:

1. His Opticks, an inspiration and exemplar of Induction and the Experimental Method.

2. His infinitesimal calculus.
3. His universal law of gravitation.

All three of them illustrate Aristotle’s dictum which Newton adopted as his basic premise:

“To grasp the nature of the world one’s thinking has to start with information received from the world”.

To grasp the nature of gravitation, Newton’s thinking started with information about the dynamics of moving bodies,

\[ m \times \text{acceleration} = \text{Force}, \]

applied to the motion of planets as given by Kepler’s three laws.

2.1 Kepler’s three laws

(1) The radius vector sun-planet sweeps out equal areas in equal times.

(2) The trajectory of each planet is an ellipse with the sun located at one focus,

\[ r = \frac{p}{1 - \epsilon \cos \theta}. \]

(The parameter \( p \) is called the semi-latus rectum of the ellipse. It is the vertical distance from the focus to ellipse. The parameter \( \epsilon \) is the eccentricity of the ellipse.)

(3) The square of the planets’ orbital periods vary as the third power of the major axes of their ellipses:

\[ \frac{T^2}{a^3} = \text{same const. for all planets}. \]

2.2 Newton’s first step: acceleration of a moving body

Using his second law of motion,

\[ m \frac{d^2 \vec{R}}{dt^2} = \vec{F}, \]

Newton determined \( \vec{F} \) by evaluating the acceleration along the trajectory of a moving body.
a) Location: $\mathbf{R} = r \mathbf{u}_r$; 
\[
\mathbf{u}_r = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta \quad \mathbf{u}_\theta = -\mathbf{i} \sin \theta + \mathbf{j} \cos \theta
\]

b) Velocity: 
\[
\frac{d\mathbf{R}}{dt} = \frac{dr}{dt} \mathbf{u}_r + r \frac{d\mathbf{u}_r}{dt} + r \frac{d\mathbf{u}_\theta}{dt} \quad \frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta} \quad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r
\]

c) Acceleration:
\[
\frac{d^2 \mathbf{R}}{dt^2} = \frac{d^2 r}{dt^2} \mathbf{u}_r + 2 \frac{dr}{dt} \frac{d\mathbf{u}_r}{dt} - r \frac{d^2 \mathbf{u}_r}{dt^2} + r \frac{d\mathbf{u}_\theta}{dt} \left( \frac{d\theta}{dt} \right)^2 + r \frac{d\mathbf{u}_\theta}{dt} \frac{d^2 \theta}{dt^2} - a_r \left[ \frac{d^2 r}{dt^2} - \frac{1}{2} \left( \frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \frac{1}{r} \frac{d}{dt} \left( \frac{r^2}{a_\theta} \frac{d\theta}{dt} \right) \mathbf{u}_\theta
\]

2.3 Newton's second step: use Kepler's laws

Applying $\mathbf{F} = m \mathbf{a}$ to Kepler's three laws yields both the direction and the magnitude of the gravitational force on a planet.

2.3.1 Kepler's first law

Equal areas in equal times, $\Delta (\text{area}) \propto \Delta t$, implies $\frac{d(\text{area})}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{c}{2}$. Consequently,
Thus the acceleration, Eq.(2), is purely radial along the direction sun-planet:

\[ \frac{d^2 \vec{R}}{dt^2} = a_r \vec{u}_r. \]

This is the first key result.

### 2.3.2 Kepler's second law

Next consider one of Kepler's ellipses having semi-latus rectum \( p \) and eccentricity \( \epsilon \). Calculate the radial acceleration \( a_r \) using Kepler's first and second laws:

\[
\begin{align*}
  r &= \frac{p}{1 - \epsilon \cos \theta} \quad \Leftarrow Kepler \ (2) \\
  \frac{dr}{dt} &= -\epsilon \frac{p \sin \theta}{(1 - \epsilon \cos \theta)^2} \frac{d\theta}{dt} \\
  &= -\epsilon \frac{\sin \theta}{p} r^2 \frac{d\theta}{dt} \quad \Leftarrow Kepler \ (1) \\
  &= -\epsilon \frac{\sin \theta}{p} c \quad \Leftarrow Kepler \ (1) \\
  \frac{d^2 r}{dt^2} &= -\epsilon \frac{p \epsilon \cos \theta d\theta}{dt} \\
  &= \frac{p}{p} \left( \frac{p}{r} - 1 \right) \frac{d\theta}{dt} \\
  &= \frac{c}{p} \left( \frac{p}{r} - 1 \right) \frac{c r^2}{dt} \quad \Leftarrow Kepler \ (2) \\
  r \left( \frac{d\theta}{dt} \right)^2 &= \frac{c^2}{p} \quad \Leftarrow Kepler \ (1)
\end{align*}
\]

Subtracting the last two lines, one finds that the radial acceleration \( a_r \) in Eq.(2) is

\[
\frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 = -\frac{c^2}{p} \frac{1}{r^2}.
\]

Thus, not only is the acceleration, Eq.(2), of a moving planet purely radial, but its magnitude is inversely proportional to its squared distance, with a constant of proportionality constant \( (c^2/p) \) that depends on the square of the planet's areal velocity and the shape of the planetary ellipse.

**Question:** Is this acceleration the same for all planets? The answer depends on the orbital periods of the planets.
2.3.3 Kepler's third law

Kepler's first law implies that the orbital period is proportional to the planetary ellipse. This ellipse has major and minor axes $a$ and $b$. Consequently,

$$\frac{c}{2} = \frac{d(\text{area})}{dt} \Rightarrow \frac{c}{2}T = \text{area} = \pi ab$$

Hence

$$c = \frac{2\pi ab}{T}$$

Q: What is the relation between the semi-latus rectum $p$ and the two axes $a$ and $b$?

A: Passing through the two foci of the ellipse are its two lati recti ("straight sides"), the vertical chords through the two focal points located at $\pm e a = \pm \sqrt{a^2 - b^2}$. The size of each latus rectum is $2p$. Thus one has a right triangle whose two sides are $2ea$ and $p$, and whose hypotenuse is $2a - p$. Pythagoras tells us that

$$p^2 + (2ea)^2 = (2a - p)^2.$$ 

Using $(ea)^2 = a^2 - b^2$ one finds that the semi-latus rectum is

$$p = \frac{b^2}{a}.$$ 

By applying the two boxed expressions to the radial acceleration $a_r$
\[
a_r = -\frac{c^2 a}{p \, r^2} = \left(\frac{2\pi ab}{T}\right)^2 \frac{a}{b^2} \frac{1}{r^2}
= -4\pi^2 \frac{a^3}{T^2} \frac{1}{r^2}.
\]

Using Kepler’s third law, one obtains

\[
\boxed{a_r = -\frac{\gamma}{r^2}}
\]

where \(\gamma = \gamma(M)\) is a constant which is the same for all planets, but which depends on the mass \(M\) of the sun in an as-yet-unspecified way.

2.4 Newton’s third step: use his 2nd and 3rd law of motion.

Q: What is the value of that planet-independent constant \(\gamma\)?
Newton answers this question by resorting to his second and third law of motion.

(i) Applying his second law to a planet of mass \(m\),

\[
m \times \text{(acceleration)} = \overrightarrow{F},
\]

one obtains the purely radial gravitational force,

\[
\boxed{F_r^{SP} = -\frac{m\gamma(M)}{r^2}}
\]

acting on the planet. This is the force with which the sun attracts the planet.

(ii) On the other hand, by his third law there is an equal but opposite force acting on the sun,

\[
F_r^{PS} = -F_r^{SP},
\]
which is to say that the weight of the planet towards the sun is equal to the weight of the sun towards the planet. Consequently,

\[
\frac{M \Gamma(m)}{r^2} = \frac{m \gamma(M)}{r^2}.
\]

This equality holds for all pairs of masses \( m \) and \( M \). Consequently,

\[
F_r = -\kappa \frac{Mm}{r^2}.
\]

Here \( \kappa \), Newton's gravitational constant, is a universal constant independent of \( M \) and \( m \). The boxed equation is a mathematical statement of Newton's universal law of gravitation.

### 2.5 The Cavendish Experiment (1789)

The universal constant \( \kappa \) has the value

\[
\kappa = \frac{1}{15 000 000} \left[ \frac{cm^3}{gr \ text{ sec}^3} \right]
\]

in c.g.s. units. This constant is determined by measuring the attractive force between masses separated by a known distant \( r \). One suspends two small masses \( m \) from a torsional balance.

By bringing large masses \( M \) to each of the masses \( m \), Cavendish measured the gravitational force \( F_r \) by measuring the angular deflection of the pendulum. Newton's third law of motion, "For every action there is an equal and opposite reaction", applied to his law of gravitation, implies that the weight of an apple
attracted by the earth's gravity equals the weight of the earth attracted by the gravity of the apples. This equality determines the mass of the earth once the weight of the apple and its distance from the (center of the) earth have been measured.
The time interval between Newton and Einstein (18th and 19th century) was marked by the development of the "Hamilton's Principle" of least action by Euler, Lagrange, and Hamilton.

This principle used the calculus of variations to replace Newton's vectorial equation of motion with the requirement that the scalar integral, the "action" of the mechanical system, be an extremum

\[ \int \delta (KE-PE) \, dt = 0 \]

independent of the coordinates chosen for KE & PE.

The main virtue of this formulation of the classical laws of motion is that the action of a mechanical system is a scalar and that the extremum of this scalar is independent of the choice of coordinates used to describe the mechanical system. If one reexpresses the Lagrangian KE - PE in terms of different coordinates.

Then the resulting Lagrange's equations of motion (whose solution extremizes the action) still describe the same mechanical system, but relative to that new set of coordinates.

**Question 1:** What is the physical origin of Hamilton's Principle as formulated by Lagrange?

**Answer 1:** The observation-based reasoning leading to this principle is given on PI-7 of the ensuing article.

**Question 2:** Is it possible to give a non-trivial application of this principle?

**Answer 2:** Yes, Pages 9-25 of the ensuing article develop the theory of the "Restricted 3-Body Problem."
Lagrangian Mechanics and the Three-body Problem

Agenda:

1. What facts of reality give rise to Lagrangian Mechanics?
2. The restricted planar three-body problem.
   a) dynamics relative to a rotating frame via dynamics in a combined magnetic and electric field.
   b) Jacobi's integral of motion.
   c) forbidden regions and the topology of mathematically unquantified regions.
   d) the five libration points of Lagrange.

The theme which unites 1. and 2. is "transformations". They form a key connecting link which allows a mathematician to think like a physicist and a physicist like a mathematician, to the advantage of both. The transformations are to a curvilinear coordinate frame, i.e. to an accelerated frame in 1. and to a rotating frame in 2.

1 Lagrange's Equations of Motion

What is their physical origin?

A. Launch a particle vertically from $x_1$ at time $t_1$, watch it reach its maximum height, and then catch it at time $t_2$ at the instant it is located at $x_2$. 

![Diagram showing particle motion from $x_1$ to $x_2$ at times $t_1$ and $t_2$.]
Figure 1: Spacetime trajectory of a particle thrown into the air.

From Galileo we learned that in its travel from \((t_1, x_1)\) to \((t_2, x_2)\) the particle traces a space-time trajectory which is given by a parabola. Why so? Answer:

1. Newton’s 1st Law: Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

2. The principle of equivalence.

B. Simpler case: Free Particle.

Consider the motion of a particle moving freely in a free float (“inertial”) frame. This particle moves with constant velocity, i.e. its space-time trajectory is a straight line.

\[
\begin{align*}
\Delta x_2 \\
x \\
\quad x_1 \\
\hline \\
\hline \\
\hline
\quad t_1 \\
\hline \\
\hline \\
\quad t_2 \\
\hline \\
\hline \\
\hline
\end{align*}
\]

Figure 2: Spacetime trajectory of a free particle is a straight line.

The implication of this fact is that for such a curve the integral

\[
\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left( \frac{dx(t)}{dt} \right)^2 dt \equiv (v^2) = \text{min}
\]

as compared to other curves having the same starting and termination points.

Q: Why?
Figure 3: Straight line \( x(t) \) and its variant \( \bar{x}(t) \) have the same average velocity: \( \langle \bar{v} \rangle = v \) (= const.).

A: All such curves have the same end points,

\[
\begin{align*}
\bar{x}(t_1) &= x(t_1) \\
\bar{x}(t_2) &= x(t_2).
\end{align*}
\]

Thus they all have the same average velocity,

\[
\langle \bar{v} \rangle = \frac{\bar{x}(t_2) - \bar{x}(t_1)}{t_2 - t_1} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \langle v \rangle.
\]

Consequently,

\[
\langle \bar{v} \rangle = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \frac{dx}{dt} \, dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \frac{dx}{dt} \, dt = \langle v \rangle,
\]

which means that the area under the curves \( \bar{v}(t) \) and \( v(t) = v \) are the same.

Applying this fact to the positivity of the averaged squared deviation (away from the average),

\[
0 \leq \langle (\bar{v} - \langle \bar{v} \rangle)^2 \rangle = (\bar{v}^2) - (\langle \bar{v} \rangle)^2 = (\bar{v}^2) - (\langle v \rangle)^2,
\]

one has

\[
\langle \bar{v}^2 \rangle \geq (\langle v \rangle)^2 = v^2,
\]

or

\[
\int_{t_1}^{t_2} \left( \frac{dx(t)}{dt} \right)^2 \, dt \geq \int_{t_1}^{t_2} \left( \frac{\bar{x}(t)}{dt} \right)^2 \, dt.
\]
This says that a free particle moves so that the integral of its kinetic energy is a minimum:

\[ \int_{t_i}^{t_f} K.E. \, dt \equiv \int_{t_i}^{t_f} \frac{1}{2} m \left( \frac{dx(t)}{dt} \right)^2 \, dt = \min! \]

C. Free particle in an accelerated frame.

Consider the motion of the same particle moving freely in a frame accelerated uniformly with acceleration \( g \).

A point \( \xi \) fixed in the accelerated frame will move relative to the free float frame according to

\[ x = \xi + \frac{1}{2} gt^2. \]

It follows that, relative to the accelerated frame, the spacetime trajectory of the particle, \( \xi(t) \), is given by

\[ x(t) = \xi(t) + \frac{1}{2} gt^2. \]  \hspace{1cm} (1)

Here \( x(t) \) is the linear trajectory in Figure 2.
Figure 4: Minimizing trajectory $\xi(t)$ and one of its variants $\bar{\xi}(t)$.

The to-be-minimized integral takes the form

$$
\min = \int_{t_1}^{t_2} \left( \frac{dx(t)}{dt} \right)^2 dt = \int_{t_1}^{t_2} \left( \frac{d\xi}{dt} + gt \right)^2 dt \\
= \int_{t_1}^{t_2} \left\{ \left( \frac{d\xi}{dt} \right)^2 + 2gt \frac{d\xi}{dt} + g^2 t^2 \right\} dt \\
= \int_{t_1}^{t_2} \left\{ \left( \frac{d\xi}{dt} \right)^2 - 2g\xi \right\} dt + 2gt\xi|_{t_1}^{t_2} + \frac{1}{3}gt^2|_{t_1}^{t_2}
$$

The last line is the result of an integration by parts. The last two terms are the same for all trajectories passing through the given points $(t_1, x_1)$ and $(t_2, x_2)$. Consequently,

$$
\int_{t_1}^{t_2} \frac{1}{2} m \left( \frac{dx(t)}{dt} \right)^2 dt = \min \iff \int_{t_1}^{t_2} \left\{ \frac{m}{2} \left( \frac{d\xi}{dt} \right)^2 - mg\xi \right\} dt = \min
$$

D. Free particle in an equivalent gravitational field.

The equivalence principle is an observation of the fact that in an accelerated frame the laws of moving bodies are the same as those in a homogeneous gravitational field.
Figure 5: Trajectories in an accelerated frame are indistinguishable from those in a gravitational field. In particular the motion of particles of different composition (gold, aluminum, snakewood, etc.) is independent of their composition.

Recall that in a gravitational field

\[ mg\xi = P.E. \]

represents the potential energy of a mass \( m \) at a height \( \xi \). Consequently, the trajectory of a particle in a gravitational field is determined by

\[ \int_{t_i}^{t_f} (K.E. - P.E.) \, dt \equiv \int_{t_i}^{t_f} L(\dot{x}, x, t) \, dt = \text{min}. \]

In fact, the trajectory of a particle which satisfies this minimum condition satisfies the Euler-Lagrange

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \]

which is Newton's second law of motion

\[ ma = F \]
for the one-dimensional motion of a particle.

Nota bene:

1. The same minimum principle holds even if \( g \), and hence the potential energy
   \( P.E. \), depends explicitly on time.

2. This principle is a special case of what is known as Hamilton's principle
   of least action. The difference is that the latter also accommodates motion
   which are subject to constraints.

E. Extension to multi dimensions and generic potentials.

The Lagrangian formulation opens new vistas on the notion of bodies. It can
be fruitfully implemented for more general motions and potentials. These generalizations are alternate but equivalent formulations of Newtonian mechanics.
They are simply expressed by the statement that

\[
\int_{t_i}^{t_2} (K.E. - P.E.) \, dt = \text{min}
\]

with

\[
K.E. = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{x}_i \cdot \ddot{x}_i
\]

\[
P.E. = U(t, \mathbf{x}_i)
\]

on the class of all system trajectories having fixed endpoints.

The advantage of Lagrangian Mechanics becomes evident in the process of setting up Newton's equations of motion. In Newtonian Mechanics one must do
this for each force component separately, a task which becomes non-trivial relative to curvilinear coordinate frames (spherical, cylindrical, etc.). By contrast, in the Lagrangian approach one merely identifies the two scalars \( K.E \) and \( P.E. \) relative to the coordinate frame of one's choice. The remaining task of setting up the differential equations of motion is done automatically by merely writing down the Euler-Lagrange equations.

2 The Three-body Problem

Taking advantage of the road paved by Newton and Euler, Lagrange asked the
following question: Does there exist a configuration of gravitationally interacting
bodies which, when launched with appropriate velocities, will execute the motion
of three rigidly connected points?
He demonstrated that the answer is “yes”. Considered three masses, $M_1$, $M_2$, and $M_3$, configured into an equilateral triangle with equal sides $a$.

![Equilateral triangle diagram](image)

Figure 6: Equilateral planar three-body system in a state of rigid rotation around its center of mass.

If these masses are launched so that their angular velocity around their center of mass satisfies

$$\omega^2 = \frac{G(M_1 + M_2 + M_3)}{a^3}$$

("Keppler's third law")

then they will continue to rotate uniformly about their center of mass as if they form a rigid triangle. In other words, in the co-rotating frame the three masses are in a state of equilibrium: Newton's laws permit a perfect balance between the attractive gravitational force and the repulsive centrifugal force.

The second question is: Is this equilibrium stable or unstable? The answer is given by the inequality

$$(M_1 + M_2 + M_3)^2 > 27(M_1 M_2 + M_2 M_3 + M_3 M_1).$$  \(2\)

If the three masses satisfy this inequality then they form a stable\(^1\) configuration: the triangle will oscillate by changing its area and/or its shape, but it will

\(^1\)To be precise, the equilibrium is *linearly* stable. This means that non-linear perturbations have been ignored. As far as I know, whether it remains stable when one does not ignore these nonlinearities, is a nontrivial open question. However, in the case of the *restricted* three-body problem, where $M_3 \ll M_1, M_2$, so that the gravitational influence of $M_3$ is negligible, one can give criteria for *absolute* stability. They are found near the end of this section. Astronomically, the difference between linear and absolute stability is a question of time. The former refers to stability at least in the intermediate future (many orbital revolutions/librations), the latter refers to the whole future.
not disintegrate. The configuration behaves like cosmic rotating and vibrating molecule. On the other hand, if this inequality is reversed, the equilibrium is unstable.

2.1 The Restricted Three-Body Problem.

We shall consider the restricted planar three-body system. Examples are

1. Sun-Jupiter-Asteroid/Space-probe
2. Sun-Earth-Spacecraft

Each system consists of two heavy masses $M_1$ and $M_2$, and a third body having such small mass, say $m$, that its gravitational influence on $M_1$ and $M_2$ is negligible.

2.1.1 The Starting Point: The Nature of Things in the Inertial Frame of the Fixed Stars

The mathematical formulation of these three-body problems rests on four interrelated properties:

1. The systems under consideration are those where the orbits of $M_1$ and $M_2$ are circular and the motion of $m$ is co-planar with that of $M_1$ and $M_2$. This implies that the mass $m$ is subjected to the gravitational force of two bodies whose separation vector

   \[
   \overrightarrow{M_1 M_2} \equiv \vec{a}(t) = a \vec{\alpha}(t) = a \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}
   \]  

   rotates, but does so rigidly with constant length

   \[ |\vec{a}(t)| = \text{constant} \equiv a. \]

2. By choosing to put the origin at the center of mass of $M_1$ and $M_2$, the location of the two masses is given by

   \[
   M_1 : \quad \vec{R}_1(t) = -\frac{M_2}{M_1 + M_2} \vec{a}(t) \equiv -\mu a \vec{a}(t)
   \]

   \[
   M_2 : \quad \vec{R}_2(t) = +\frac{M_1}{M_1 + M_2} \vec{a}(t) \equiv (1 - \mu) a \vec{a}(t)
   \]
Here we have introduced the fractional masses

\[ \mu = \frac{M_2}{M_1 + M_2} \quad ("\text{Planet}"") \quad (6) \]

\[ 1 - \mu = \frac{M_1}{M_1 + M_2} \quad ("\text{Sun}"") \quad (7) \]

of the two respective masses \( M_2 \) and \( M_1 \).

3. The angular velocity with which \( M_1 \) and \( M_2 \) orbit their center of mass is determined by applying mass×(centripetal acceleration) = gravitational force to either mass. Newton’s Third Law together with Figure 8 guarantee that the results will be the same. One obtains

\[ M_1 \omega^2 \frac{M_2}{M_1 + M_2} a = \frac{GM_1M_2}{a^2}. \]

and hence

\[ G(M_1 + M_2) = \omega^2 a^3. \quad (8) \]

This is the 1-2-3 law, also known as Kepler’s generalized Third Law. It says that once one has measured the period \( \frac{2\pi}{\omega} \) and the size \( a \) of a binary system, its total mass \( M_1 + M_2 \) is known and determined.
4. The time-periodic potential energy of a mass \( m \) located at \( \vec{r}(x, y, z) \) is

\[
P.E. (\vec{r}) = -\frac{GM_2 m}{|\vec{R}_2(t) - \vec{r}|} - \frac{GM_1 m}{|\vec{R}_1(t) - \vec{r}|}
\]

\[
= -\frac{G(M_1 + M_2)m}{a} \left( \frac{\mu}{(1 - \mu)\vec{a}(t) - \vec{r}/a} \right) + \frac{1 - \mu}{(1 - \mu)\vec{a}(t) - \vec{r}/a} \right)
\]

\[
= -\omega^2 a^2 m \left( \frac{\mu}{\rho_2} + \frac{1 - \mu}{\rho_1} \right)
\]

where we used Eqs. (4-5), (6-7), and (8) respectively.

2.1.2 Transformation into the Corotating Coordinate Frame

We now focus on the motion of the body \( m \) relative to the frame corotating with \( M_1 \) and \( M_2 \). A point \( \vec{x}_{\text{rot}} = \left( \begin{array}{c} x_{\text{rot}} \\ y_{\text{rot}} \end{array} \right) \) fixed in this frame will move relative to the fixed stars according to

\[
\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x_{\text{rot}} \\ y_{\text{rot}} \end{pmatrix} \equiv T(t) \vec{x}_{\text{rot}}.
\]

The task of applying this transformation to the to-be-minimized Lagrangian integral

\[
\int (K.E. - P.E.) \, dt
\]

consists of the five steps below. The final result is given by Eq. (17)

Step (i)
The trajectory of the body relative to the corotating frame, \( \vec{x}_{\text{rot}}(t) \), is related to its trajectory \( \vec{r}(t) \) relative to the fixed stars by

\[
\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = T(t) \vec{x}_{\text{rot}}(t).
\]

Step (ii)
The separation vector \( \vec{M}_1\vec{M}_2 \), Eq. (3), has the same relation between its fixed-stars and its corotating frame representatives,

\[
\vec{a}(t) = \begin{pmatrix} a \cos\omega t \\ a \sin\omega t \end{pmatrix}_\text{fixed} = T(t) \begin{pmatrix} a \\ 0 \end{pmatrix}_\text{rot}.
\]
Thus, while in the fixed frame the separation vector $\overrightarrow{M_1M_2}$ rotates, in the rotating frame it remains fixed lying along the $x_{rot}$-axis. In other words, in the rotating coordinate frame the bodies $M_1$ and $M_2$ remain statically situated along the $x_{rot}$-axis.

The static nature of the bodies $M_1$ and $M_2$ in the rotating coordinate frame is depicted in Figure 9. There, for subsequent mathematical efficiency, the coordinates have been scaled in terms of the constant $M_1M_2$-separation $a$.

$$\eta \left( = \frac{y_{rot}}{a} \right)$$

$$m_{0} \left( \xi, \eta \right)$$

$$\rho_1$$

$$\rho_2$$

$(1 - \mu, 0)$

Figure 9: Rotating coordinate frame in which bodies $M_1$ and $M_2$ remain situated along its horizontal axis. In this frame they provide a static gravitational field for the dynamics of the body $m$.

Step (iii)
Whereas relative to the fixed-stars frame the potential energy function $P.E.$, Eq.(10), is a periodic function of time, in the rotating frame it is static. From a physics perspective this is obvious\(^2\). From a mathematical perspective this is a consequence of the orthogonality of the time-dependent point transformation $T$, Eqs.(12) and (14),

$$T(t) : \begin{cases} \overrightarrow{x_{rot}} \rightarrow T(t)\overrightarrow{x_{rot}} = \overrightarrow{r} \\ \vec{a}_0 \rightarrow T(t)\vec{a}_0 = \vec{a}(t) \end{cases}$$

\(^2\)In a merry-go-round corotating with two masses about their center of mass the laws of physics during one interval of time are the same as those during a later interval of time. For example, the spinning earth with mountain masses on opposite hemispheres make up such a merry-go-round.
Applying it to the arguments of the distance from \( m \) to \( M_1 \) and \( M_2 \), namely

\[
\begin{align*}
\rho_1 \left( a(t), \tilde{r} \right) &= \left| -\mu \tilde{a}(t) - \tilde{r} \right| \\
\rho_2 \left( a(t), \tilde{r} \right) &= \left| (1 - \mu) \tilde{a}(t) - \tilde{r} \right|
\end{align*}
\]

yields new functions\(^3\). Their domain is the rotating frame, and they are given by

\[
\begin{align*}
\rho_1 \circ T(t) (\tilde{a}_0, \tilde{x}_{rot}) &= \left| -\mu T(t) \tilde{a}_0 - T(t) \tilde{x}_{rot} \right| \\
&= \left| -\mu T(t) \begin{pmatrix} a \\ 0 \end{pmatrix} - T(t) \begin{pmatrix} x_{rot} \\ y_{rot} \end{pmatrix} \right| \\
&= \left| -\mu \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} x_{rot} \\ y_{rot} \end{pmatrix} \right| \\
&= a \sqrt{\frac{(\mu + x_{rot})^2 + (y_{rot})^2}{a}} \\
&= a \sqrt{(\mu + \xi)^2 + \eta^2}
\end{align*}
\]

and

\[
\begin{align*}
\rho_1 \circ T(t) (\tilde{a}_0, \tilde{x}_{rot}) &= \left| (1 - \mu) T(t) \tilde{a}_0 - T(t) \tilde{x}_{rot} \right| \\
&= a \sqrt{\xi + \eta^2}.
\end{align*}
\]

These new functions depend only on the dimensionless rotating frame coordinates

\[
\begin{align*}
\xi &= \frac{x_{rot}}{a} \\
\eta &= \frac{y_{rot}}{a}
\end{align*}
\]

Thus relative to the rotating frame the gravitational potential energy, Eq.(11), is

\[
P.E. = -\omega^2 a^2 m \left( \frac{\mu}{\sqrt{(\xi + \mu)^2 + \eta^2}} + \frac{1 - \mu}{\sqrt{(\xi + 1 - \mu)^2 + \eta^2}} \right).
\]

(15)

It is independent of time and depends only on the position \((\xi, \eta)\) of the body \( m \).

Step (iv)

\(^3\)the "pull backs" of \( \rho_i \) by \( T \)
Consider the kinetic energy of \( m \) in the fixed-stars (inertial) frame:

\[
K.E. = \frac{m}{2} \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} \\
= \frac{m}{2} (T\dot{\mathbf{x}}_{\text{rot}}) \cdot (T\dot{\mathbf{x}}_{\text{rot}}) \\
= \frac{m}{2} \left( \dot{\mathbf{x}}_{\text{rot}} T^4 + \dot{\mathbf{x}}_{\text{rot}} T^4 \right) \left( T\dot{\mathbf{x}}_{\text{rot}} + T\dot{\mathbf{x}}_{\text{rot}} \right)
\]

Here the superscript "\( t \)" indicates transpose. For the rotation matrix \( T \), Eq.(12), one has

\[
T^t T = I \\
\dot{T}^t \dot{T} = \omega^2 I
\]

\[
\dot{\mathbf{x}}_{\text{rot}} T^t \dot{T} \dot{\mathbf{x}}_{\text{rot}} + \dot{\mathbf{x}}_{\text{rot}} T^t \dot{T} \dot{\mathbf{x}}_{\text{rot}} = 2\dot{\mathbf{x}}_{\text{rot}} T^t \dot{T} \dot{\mathbf{x}}_{\text{rot}} \\
= 2\omega \dot{\mathbf{x}}_{\text{rot}} \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \dot{\mathbf{x}}_{\text{rot}} \\
= 2\omega a^2 (\xi \dot{\eta} - \eta \dot{\xi})
\]

Consequently, relative to the rotating frame, the kinetic energy decomposes into three parts:

\[
K.E. = \frac{1}{2} m a^2 (\dot{\xi}^2 + \dot{\eta}^2) + m \omega a^2 (\xi \dot{\eta} - \eta \dot{\xi}) + \frac{1}{2} m \omega a^2 (\xi^2 + \eta^2). \tag{16}
\]

From the perspective of physics it is worthwhile to identify them individually:

1. The rotational kinetic energy, \( \text{rot.K.E.} \), relative to the rotating frame.

2. The Coriolis energy, \( \text{C.E.} \), which is the amount of energy in the inertial frame necessary to speed up \( m \)'s angular velocity\(^4\) from zero to \( \omega \) in the inertial frame.

\(^4\)Obtained by introducing polar coordinates \( a \xi = r \cos \theta, a \eta = r \sin \theta \). For pure \( \theta \)-motion consider the rotational version of Newton's Second Law,

\[
\text{torque} = m r^2 \ddot{\theta}.
\]

The work performed my this torque as it acts over an angular displacement \((\omega + \dot{\theta}) \Delta t\) increases the energy of an orbiting mass \( m \) by an amount

\[
d(\text{energy}) = m r^2 \dot{\theta} (\omega + \dot{\theta}) \Delta t.
\]

Consequently, the total amount of inertial energy imparted to \( m \) is

\[
\int (\text{torque}) (\Delta \text{sweped out by } m \text{ in the inertial frame during time } \Delta t) = m r^2 \dot{\theta} \omega + m r^2 \dot{\theta} \Delta \theta/2.
\]

This is the sum of two partial energies, namely, (i) the Coriolis Energy C.E. \( = m r^2 \dot{\theta} \omega = ma^2 (\xi \dot{\eta} - \eta \dot{\xi}) \), which is the middle term of Eq.(16), and (ii) \( \text{rot. K.E.} = m r^2 \dot{\theta} \Delta \theta/2 \), which is the additional inertial work necessary to give \( m \) non-zero angular velocity \( \dot{\theta} \) in the rotating frame.
3. The kinetic work function, $W_{\text{rot}}$, which is the amount of inertial kinetic energy necessary to move $m$ from the origin to the location $(\xi, \eta)$ in the rotating frame.

The Lagrangian integral to be minimized is therefore

$$\int (K.E. - P.E.) \, dt = \int (\text{rot.K.E. + C.E. - rot.P.E.}) \, dt$$

(17)

where

$$\text{rot.P.E.} = ma^2 \omega^2 \Phi(\xi, \eta)$$

and

$$\Phi(\xi, \eta) = \frac{1}{2} (\xi^2 + \eta^2) + \frac{1 - \mu}{\sqrt{(\xi + \mu)^2 + \eta^2}} + \frac{\mu}{\sqrt{(\xi + \mu + 1)^2 + \eta^2}}$$

(18)

is the dimensionless scalar function in the rotating frame. It includes also the kinetic work function as an additive contribution to the (negative of the) two gravitational potentials.

2.1.3 The Equations of Motion

A necessary condition for the Lagrange integral to be minimized by $(\xi(t), \eta(t))$ is that the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \quad i = \xi, \eta$$

be satisfied. They are with the help of Eq.(18)

$$i = \xi: \quad \ddot{\xi} = +2\omega \dot{\eta} + \omega^2 \partial_\xi \Phi$$

$$= +2\omega \dot{\eta} - \omega^2 \left[ \xi \left( 1 + \frac{1}{\rho_1^2} + \frac{\mu}{\rho_3} \right) + \mu (1 - \mu) \left( \frac{1}{\rho_1^2} - \frac{1}{\rho_3^2} \right) - \eta \left( 1 + \frac{1}{\rho_2^2} + \frac{\mu}{\rho_3} \right) \right]$$

and

$$i = \eta: \quad \ddot{\eta} = -2\omega \dot{\xi} + \omega^2 \partial_\eta \Phi$$

$$= -2\omega \dot{\xi} - \omega^2 \left[ \eta \left( 1 + \frac{1}{\rho_2^2} + \frac{\mu}{\rho_3} \right) \right]$$

(19)

(20)
Here, for the sake of notational economy, we write

\[ \rho_1 = \sqrt{(\xi + \mu)^2 + \eta^2} \]
\[ \rho_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2} \]

as the rescaled distance of \( m \) from \( M_1 \) and \( M_2 \) in the rotating frame.

This pair of coupled differential equations is mathematically equivalent to the equations of motion of a negatively charged particle with charge-to-mass ratio \( \frac{q}{m} = -\omega^2 \).

moving in a planar electric field

\[ \vec{E} = - \left( \vec{i} \cdot \partial_\Phi + \vec{j} \cdot \partial_\Phi \right) = -\vec{\nabla} \Phi \]

combined with a constant magnetic field

\[ \vec{B} = \vec{i} \cdot 0 + \vec{j} \cdot 0 + \vec{k} \cdot \frac{-2}{\omega} \]

which is perpendicular to the electric field and to the orbital plane. Identifying the particle trajectory as the moving vector

\[ \vec{x}(t) = \vec{i} \cdot \xi(t) + \vec{j} \cdot \eta(t) + \vec{k} \cdot 0 \]

one rewrites Eqs.(19)-(20) as

\[ \ddot{\vec{x}} = 2\omega \dot{\vec{x}} \times \vec{k} + \omega^2 \vec{\nabla} \Phi \]
\[ = \frac{q}{m} \dot{\vec{x}} \times \vec{B} + \frac{q}{m} \vec{E}. \]  

(21)

In physics these equations are recognized as the Lorentz equations of motion, Newton's equations in the context of electromagnetic forces. In engineering they are recognized as governing the operation of a magnetron, the heart of radar transmitters and microwave ovens.

- Hot cathode emits electrons which travel outward
- Stable magnetic field B

Electrons from a hot filament would travel radially to the outside ring if it were not for the magnetic field. The magnetic force deflects them in the sense shown and they tend to sweep around the circle. In so doing, they "pump" the natural resonant frequency of the cavities. The currents around the resonant cavities cause them to radiate electromagnetic energy at that resonant frequency.
The advantage of such a recognition is that it leads directly to an energy type integral. Indeed, multiply Eq.(21) by $\dot{z}$ and obtain
\[
\frac{d}{dt} \left( \frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) \right) = 0 + \frac{d}{dt} (\omega^2 \Phi)
\]

It follows that
\[
\frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) - \omega^2 \Phi(\xi, \eta) = H(\xi, \eta, \xi, \eta) \quad "\text{Jacobi's integral}"
\]
is an integral of motion, which was identified by Jacobi in 1836. The function $H$ is a constant along any given trajectory that satisfies the equations of motion.

### 2.1.4 Energy Conservation in the Rotating Frame

The constancy of $H$ along a trajectory is a statement of the conservation of energy of the third mass $m$ in the rotating frame,

\[
H(\xi, \eta, \xi, \eta) = (K.E.)_{rot} + (P.E.)_{rot} = const \equiv (T.E.)_{rot}.
\]

Here
\[(K.E.)_{\text{rot}} = \frac{1}{2} (\xi^2 + \eta^2)\]

and

\[(P.E.)_{\text{rot}} = -\Phi(\xi, \eta) = -\frac{1 - \mu}{\sqrt{(\xi + \mu)^2 + \eta^2}} - \frac{\mu}{\sqrt{(\xi + \mu + 1)^2 + \eta^2}} = \frac{1}{2} (\xi^2 + \eta^2).\]

The roles of the energies which make up the energy law Eq. (22) are most efficiently captured by the simultaneous rendering of the two graphs, \((P.E.)_{\text{rot}}(\xi, \eta)\) and \((T.E.)_{\text{rot}}\) in Figure 11.

Figure 11: Graphs of \((P.E.)_{\text{rot}}(\xi, \eta)\) restricted to \(\eta = 0\) and of \((T.E.)_{\text{rot}}\), which is constant. The classically allowed (forbidden) region is the one where the difference between \((T.E.)_{\text{rot}}\) and \((P.E.)_{\text{rot}}\), namely the kinetic energy \((K.E.)_{\text{rot}}\), is positive (negative). The allowed region is called the Hill region.

An unrestricted rendition of these graphs would be a 3-d graph, which, as in Figure 12, would extend over the whole \((\xi, \eta)\)-plane. The motion of the particle, although in general quite irregular and even chaotic, would be deterministic and would be confined strictly to those regions for which \((K.E.)_{\text{rot}} > 0\). Their shape and topology depend on the value of \((T.E.)_{\text{rot}}\), and they are the shaded/green regions in Figure 13.

\(^5\)The potential energy function \((P.E.)_{\text{rot}}(\xi, \eta)\) differs by a “mere” minus sign from the mathematical function \(\Phi(\xi, \eta)\). However, in the hierarchy of concepts (i.e., from the perspective of epistemology, see e.g., “Introduction to Objectivist Epistemology” by Ayn Rand) \((P.E.)_{\text{rot}}\) as introduced in physics with its minus sign – logically precedes \(\Phi\). Before one can understand the meaning of \(\Phi\), one first has to understand the meaning of \((P.E.)_{\text{rot}}\). Physicists such as K.R. Symon (in his book “Mechanics”) uses \((P.E.)_{\text{rot}}\), which he designates by \(V(x, y)\). By contrast, mathematicians like J. Moser (in his “Lectures on Hamiltonian Dynamics”) and V.I. Arnold et al (in their “Mathematical Aspects of Classical Mechanics”) work with \(\Phi\), which they designate by \(V(x, y)\) and \(V(\xi, \eta)\), and which is easier to manipulate mathematically.
The latter is, of course, just a constant. The difference between the two is the kinetic energy \( (K.E.)_{\text{rot}} = \frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) \), which must never be negative. It follows that the motion of the particle \( m \) with a given total energy \( (T.E.)_{\text{rot}} \) is restricted to only those \((\xi, \eta)\)-regions which satisfy

\[
(P.E.)_{\text{rot}} (\xi, \eta) \leq (T.E.)_{\text{rot}}
\]

(23)

These regions are called Hill regions. For every \((T.E.)_{\text{rot}}\) there is one or more of these allowed regions. Those \((\xi, \eta)\)-regions which violate the energy condition, Eq.(23) are inaccessible. They are classically forbidden\(^6\). These regions are the unshaded ones in Figure 13.

### 2.1.5 The Jacobi Integral

The restricted planar three-body problem could be solved completely, if besides \( H(\dot{\xi}, \dot{\eta}, \xi, \eta) \) one could identify another integral, say \( F(\dot{\xi}, \dot{\eta}, \xi, \eta) \), which is functionally independent of \( H \). If that were the case, then one could find the trajectories in a way that led to analytical solubility of the two-body problem.

However, even by itself, \( H \) does give very useful information. In particular, as we shall see, \( H \) identifies which regions are dynamically accessible and which are forbidden by classical mechanics.

The constancy of \( H \), say

\[
H(\dot{\xi}, \dot{\eta}, \xi, \eta) = \text{constant} \equiv h,
\]

implies that the only accessible regions are those that satisfy

\[
0 \leq \frac{1}{2\omega^2} (\dot{\xi}^2 + \dot{\eta}^2) = h + \Phi(\xi, \eta)
\]

\[
= h + \left[ \frac{1}{2} (\xi^2 + \eta^2) + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right]
\]

(24)

These regions are known as Hill's regions. They are important for exploration by space probes which have limited amount of fuel.

Those regions where the inequality (24) is violated are forbidden by classical mechanics. Such regions are characterized by

\[
h < - \left[ \frac{1}{2} (\xi^2 + \eta^2) + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right].
\]

\(^6\)If the classical mechanics formulation of the particle motion is replaced by one in terms of wave (quantum) mechanics, then the allowed regions are those where the wave function of the particle oscillates. On the other hand, the classically forbidden regions are still accessible, but the wave function is decreasing exponentially so that the expectation value for of measuring the particle as present is also decreasing exponentially.
An unpropelled body may not penetrate such regions. The boundary(s) of these regions is the locus of points where the value $h$ of the Jacobi integral is such that the velocity vanishes:

$$\dot{\xi}^2 + \dot{\eta}^2 = 0.$$ 

This condition is equivalent to a relation between $h$ and the boundary points,

$$h = -\left[\frac{1}{2} (\xi^2 + \eta^2) + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2}\right],$$

whose graph is exhibited in Figure 12.
Figure 12: Boundary between forbidden and allowed regions parametrized by the value \( h \) of the Jacobi integral. At this boundary the velocity (in the rotating coordinate frame) of the body vanishes. The regions below the surface are forbidden, those above are allowed.

The intersection of this graph with each horizontal plane \( h = \text{constant} \) consists of a boundary between the allowed and the forbidden region(s). In fact, these intersections form a parametrized family of boundaries. They are exhibited in Figure 13. Each of its panels is a horizontal slice through surface in Figure 12.

Figure 13: One parameter family of boundaries that separate the allowed (shaded/green) regions from those forbidden (unshaded/white) to an unpropelled body. The horizontal and vertical axes are those of \( \xi \) and \( \eta \) respectively. The numbers are the values of the Jacobi integral, which is the parameter.

The shape of the boundary(s) and the topology of the forbidden regions is determined by the most important attributes of the dynamical system, the location and the nature of its equilibrium trajectories. For these the body has zero velocity and zero acceleration,

\[
\xi^2 + \eta^2 = 0 \quad \text{and} \quad \dot{\xi} = \dot{\eta} = 0.
\]
This occurs where the effective potential $\Phi(\xi, \eta)$, Eq.(18), has its critical points,

$$\partial_\xi \Phi = \partial_\eta \Phi = 0.$$

In other words, because of Eqs.(19)-(20) one has the equations for static equilibrium in the rotating frame

$$0 = \partial_\xi \Phi \equiv \xi f + \mu(1 - \mu) \left( \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right),$$

$$0 = \partial_\eta \Phi \equiv \eta f,$$

here

$$f = -1 + \frac{1 - \mu}{\rho_1^2} + \frac{\mu}{\rho_2^2},$$

and again,

$$\rho_1 = \sqrt{(\xi + \mu)^2 + \eta^2},$$

$$\rho_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2}.$$

2.1.6 Five Lagrange Points

The critical points of the effective potential fall into two classes, those for which $\eta = 0$ and those for which $\eta \neq 0$. Of the critical points with $\eta = 0$ there are three, $L_1$, $L_2$, and $L_3$. As one can see from Figure 14, they lie along the line connecting $M_1$ and $M_2$, and they are saddle points of $\Phi(\xi, \eta)$. Of the critical points with $\eta \neq 0$ there are two, $L_4$ and $L_5$. From the figures one sees that they form two equilateral triangles with $M_1$ and $M_2$, and they are maxima of $\Phi(\xi, \eta)$. 

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Figure 14: Isograms of equal \( h \) in the rotating coordinate frame spanned by \( \xi \) and \( \eta \). \( L_1, L_2 \) and \( L_3 \) are the three unstable collinear Lagrange point. \( L_4 \) and \( L_5 \) are the two triangular point.

(i) Triangular Critical Points (\( L_4 \) and \( L_5 \)).

For \( \eta \neq 0 \) the equations for static equilibrium are solved by

\[
f = 0 \quad \text{and} \quad \rho_1 = \rho_2 = 1,
\]

and hence by

\[
\xi = \frac{1}{2} - \mu, \quad \eta = \pm \frac{\sqrt{3}}{2}.
\]

The location of the two critical points is therefore

\[
L_4 = a \left( \frac{1}{2} - \mu, \frac{\sqrt{3}}{2} \right)
\]

\[
L_5 = a \left( \frac{1}{2} - \mu, -\frac{\sqrt{3}}{2} \right).
\]

The masses \( M_1 \) and \( M_2 \) are located at

\[
M_1 : \quad (\xi, \eta) = (-a\mu, 0)
\]

\[
M_2 : \quad (\xi, \eta) = (a(1 - \mu), 0)
\]
It follows that \( M_1 M_2 L_4 \) and \( M_1 M_2 L_6 \) form two equilateral triangles whose sides equal \( a \), the separation between \( M_1 \) and \( M_2 \).

The motion of particles (asteroids, spacecraft, etc.) in the neighborhood of \( L_4 \) and \( L_6 \) raises a number of key problems, namely the question of stability, of the existence of periodic solutions, and of integrals of motion. Answering these questions requires pushing back the frontier of mathematics. Juergen Moser has reported on this in his "Lectures on Hamiltonian Systems" (Memoirs of the American Mathematical Society, Number 81 (1968), P1-60, QA1 A527 No.80-81).

By specializing the linear stability criterion Eq.(2) on page 8 to the restricted \( (M_3 \to 0) \) three-body problem, one obtains

\[
(1 - \mu)\mu < \frac{1}{27}; \quad \mu = \frac{M_2}{M_1 + M_2}.
\]

This inequality implies that one has linear stability whenever

\[ 0 < \mu < \mu_1 = .0385 \]  \hspace{1cm} (25)

If the second mass, say Jupiter (in the the Sun-Jupiter-Achilles system) were to lie outside this interval, then the libration points \( L_4 \) and \( L_6 \) would be unstable. However, for the Sun-Jupiter system one has

\[
\mu = \frac{M_2}{M_1 + M_2} = .000954.
\]

Consequently, one concludes that the S-J-A is stable in the linear approximation. But linear stability ignores the nonlinear (quadratic, cubic, etc) contributions to the evolution of Achilles perturbed away from the critical point \( L_4 \) (or \( L_6 \)) of \( \Phi(\xi, \eta) \). If one takes all these contributions into account, one finds that the interval, Eq.(25), gets shrunk to

\[ 0 < \mu < \mu_{1c} = .0109. \]

In spite of this, Jupiter with its \( \mu = .000954 \) still satisfies the stronger inequality. Consequently, Achilles is not only linearly stable, but absolutely (non-linearly) stable as well.

(ii) Collinear Critical Points \( (L_1, L_2, L_3) \)

For \( \eta = 0 \) the only nontrivial equation for static equilibrium is the condition

\[
\partial_\xi \Phi(\xi, \eta = 0) = 0.
\]

This is a one-dimensional critical point problem for \( \Phi \) evaluated along the \( \xi \)-axis. There one has

\[
\Phi(\xi, \eta = 0) = -\frac{1}{2} \xi^2 - \frac{1 - \mu}{\xi + \mu} - \frac{\mu}{\xi + \mu - 1}.
\]
Finding the critical point of this function amounts to finding the roots of a quintic equation. That there are only three real roots \((L_1, L_2, L_3)\) is evident from Figure 14. Alternatively, one arrives at the same conclusion by interpolating between the asymptotic behaviors of \(\Phi\) as

\[
\begin{align*}
\xi &\rightarrow -\infty \\
\xi &\rightarrow -\mu \\
\xi &\rightarrow 1 - \mu \\
\xi &\rightarrow +\infty
\end{align*}
\]

By inspection one sees that \(\partial_\xi \partial_\xi \Phi < 0\) whenever \(\Phi\) is defined. Consequently, \(\Phi\) behaves as shown in Figure: there are only three critical point corresponding to \(L_1, L_2\) and \(L_3\).

\[(-)\Phi\]

\[
\begin{align*}
L_3 &\sim -\frac{1}{2} \xi^2 \\
L_2 &\sim \frac{-\mu}{|\xi + \mu|} \\
L_1 &\sim \frac{-(1-\mu)}{|\xi + \mu - 1|}
\end{align*}
\]

Figure 15: The three collinear unstable Lagrange libration points \(L_1, L_2\) and \(L_3\). They lie on the straight line passing through the two masses \(M_1\) ("Sun") and \(M_2\) ("Jupiter").
By the time Einstein examined the nature of nature, he had at his disposal, and thus made excellent use of, the highly developed art of analytical mechanics as formulated by Lagrange and Hamilton.

In 1913 he took a key step. Using Hamilton’s principle of least action, he equated the Hamilton’s action integral to the (coordinate) invariant length

$$ S = \int \sqrt{\text{g}^{ij} dx^i dx^j} $$

of a world line between two events, and then pointed out that the metric tensor field

$$ g^{ij} dx^i dx^j $$

where gravitation puts its imprints (characterized the gravitational field) this means that the geodesic worldlines i.e. the solutions to the Lagrangian equations obtained from Hamilton’s principle, carry the imprints of gravitation.

The fact that gravitation manifests itself through the geodesic states of motion of particles, introduced a qualitatively new viewpoint about gravitation: it is to be viewed as the manifestation of the geometry of spacetime.

This is very different from Newton’s viewpoint of gravitation as a force field in a (flat) three-dimensional Euclidean space.
Let us summarize the difference in the way in which Galileo, Newton and Einstein expressed how gravitation leaves its imprints on the states of motion of matter.

Galileo \[ \text{gravitation} \quad \text{(speed)} = g \text{ (time)} \quad \text{leading} \quad \frac{d^2r}{dt^2} = -\frac{GMm}{r^3} \]

Newton \[ \text{gravitation} \quad \text{"descriptive mechanics"} \quad \text{"Newtonian mechanics"} \]

Einstein \[ \text{gravitation} \quad \text{small velocity} \quad \text{weak gravitational} \]

Is it possible to express the imprints of gravitation in quantum mechanics in a way that is consistent with the laws of quantum mechanics? The answer is that it is possible to express the imprints of gravitation in quantum mechanics in a way that is consistent with the laws of quantum mechanics.

\[ \frac{d^2x}{dt^2} + \Gamma^x_{ab} \frac{dx^a}{dt} \frac{dx^b}{dt} = 0 \]

\[ \Psi = e^{ie} \]
It is possible to state this question also from the viewpoint of mathematics.

As we shall see, the equations of geodesic motion arise (via "constructive interference") from the short wave length (i.e. "high frequency" or "W.K.B") approximation to a wave equation defined on the spacetime manifold; in other words, in the short wavelength approximation the maximum of a wave packet traces out a geodesic trajectory.

Thus, fundamentally, the straight lines and circles upon which geometry is founded, should be replaced by wave mechanical scalar functions. This means that, fundamentally, the study of geometry should be based on the properties of scalar fields, Hilbert space, noncommuting operators, etc., and not on geodesics.

In fact, if Riemann or Cartan had appreciated the fundamental significance of wave mechanics, one might wonder how different the Riemannian geometry of a manifold would look like today if it were based strictly on wave mechanical consideration instead of geodesics.