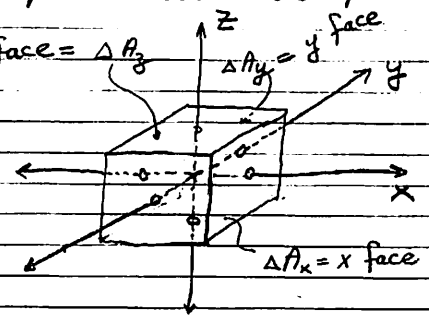
# Lecture 9b (Appendix)

## Symmetry of the stress tensor: Why?

The spatial stress components form a symmetric matric $T^{ij} = T^{ji}$

One arrives at this conclusion with the help of Newton's equation applied to the rotational motion of a small cube of matter of volume $L^3$.




The mass in such a cube is $T^{00} L^3$. The moment of inertia of this cube is $\sim T^{00} L^5$. Newton's equation of motion applied to the cube's rotation around the z-axis is

$$\dot{\Omega}\left(T^{00}L^5\right) = (\text{torque})^z = \left(\vec{r} \times \Delta\vec{P}\right)^z$$

$$= \Big[ \underbrace{\tfrac{L}{2}}_{\substack{\text{lever arm}\\ \text{to x face}}} \underbrace{(-)\,T^{yx}L^2}_{\substack{\text{y-force on}\\ \text{x face}}} + \underbrace{(-)\tfrac{L}{2}}_{\substack{\text{lever arm}\\ \text{to -x face}}} \underbrace{T^{yx}L^2}_{\substack{\text{y-force on}\\ \text{-x face}}} \Big]$$

$$- \Big[ \underbrace{\tfrac{L}{2}}_{\substack{\text{lever arm}\\ \text{to y face}}} \underbrace{(-)\,T^{xy}L^2}_{\substack{\text{x force on}\\ \text{y-face}}} + \underbrace{(-)\tfrac{L}{2}}_{\substack{\text{lever arm}\\ \text{to -y face}}} \underbrace{T^{xy}L^2}_{\substack{\text{x force on}\\ \text{-y face}}} \Big]$$



Comments:

a) Note that at $x = \frac{L}{2}$

$$\Delta F^y = T^{yx}\,\Delta A_x = T^{yx}L^2$$

is a y force exerted by the x-face on the outside medium ($\frac{L}{2} \leq x$). This is because this y force expresses a flow of y-momentum into the +x direction at the +x face.

b) Equivalently

$$(-)\,T^{yx}L^2.$$

is a y force exerted on the x-face by the outside medium. This expresses a flow of y-momentum into the −x direction at the +x face. In other words, it is the direction of the flow of momentum that that gets reversed when one reverses the origin and the destination of the application of a force.

c) At the $-x$ face $T^{yx}L^2$ also expresses a flow of $y$ momentum into the $x$ direction. But there it represents a $y$ force exerted on the $-x$ face by the outside medium ($x \leq -\frac{L}{2}$); in other words, the momentum flow $T^{yx}$ is from the outside to the inside of the cube across the $-x$ face.

d) As an aside, we note that the $y$-force on the $+x$ face can be represented in terms of the vector valued 3-form ${}^*\Pi$ by ${}^*\Pi(A', B, C)$ where

$$\begin{array}{lcccc} & t & x & y & z \\ A' : & (1, & 0, & 0, & 0) \\ B : & (0, & 0, & L, & 0) \\ C : & (0, & 0, & 0, & L) \end{array}$$

as follows

$$(-)T^{yx}\Delta A_x = (-)T^{yx}L^2 = (-)T^{yx}(-)\epsilon_{x0yz}\, 1 \cdot L \cdot L \qquad \boxed{\epsilon_{0xyz} > 0}$$

$$= T^{y\mu}\epsilon_{\mu|\alpha\beta\gamma|}\,\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma(A', B, C)$$

Flow of $y$ momentum across $\Delta A_x$ into negative direction (into the cube in the picture on P10)



e) In fact, more generally we note that the spatial components of the force $\Delta\vec{F}$ together with the energy rate ("power")

$$\sum_i T^{0i}\Delta A_i = T^{0x}\Delta A_x + T^{0y}\Delta A_y + T^{0z}\Delta A_z$$

make up the components of the 4 momentum

$$\epsilon_\mu T^{\mu i}\Delta A_i = (-)\sum_A p_A N_A u_A^\nu \epsilon_{\nu 0ij}\, 1\, B^i C^j = (-)\,{}^*\Pi(A', B, C)$$

crossing the element of area $\Delta\vec{A} = \vec{B}\times\vec{C}$ during the time of the vector $A' : (1, 0, 0, 0)$

By contrast the 4-momentum in the future directed 3-volume $L^3 = \vec{A}\cdot\vec{B}\times\vec{C}$ is

$$T^{\mu 0}L^3 = \sum_A p_A N_A u^\nu \epsilon_{\nu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma = +\,{}^*\Pi(A, B, C)$$

where $A : (A^0, \vec{A})$
$B : (B^0, \vec{B})$
$C : (C^0, \vec{C})$

are spacelike.

Back to Newton's eq'n on p ~~10~~ 1:

After simplifying Newton's equations for cube's rotation around the z axis (p 10) one obtain

$$(\dot{\Omega})^z = \frac{T^{xy} - T^{yx}}{T^{00}} \frac{L^3}{L^5}$$

We see that if it were true that that $T^{xy} \neq T^{yx}$, then

$$(\dot{\Omega})^z \to \infty \quad \text{as} \quad L \to 0 .$$

In other words, the cube of matter would be spinning with arbitrarily large angular velocities if we would consider the cube to be of sufficiently small volume $L^3$. The fact that matter does not behave that way demands that

$$T^{xy} = T^{yx}$$

or more generally

$$T^{ij} = T^{ji}$$



Summary

The stress ~~energy~~ tensor $T$ has the following symmetric array of components

$$T^{\mu\nu} = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{c} \overbrace{\phantom{T^{00}}}^{\text{Energy density}} \\ \left[\begin{array}{c|ccc} T^{00} & T^{01} & T^{02} & T^{03} \\ \hline T^{10} & & & \\ T^{20} & & T^{ij} & \\ T^{30} & & & \end{array}\right] \\ \underbrace{\phantom{T^{10}}}_{\text{Momentum density}} \end{array} \begin{array}{l} \\ \left.\vphantom{T^{00}}\right\} \text{Energy flux} \\ \left.\vphantom{T^{10}}\right\} \text{Stress} \end{array}$$

where $T^{\mu\nu} = T^{\nu\mu}$