Lecture 96 (Appendix)

Symmetry of the stress tensor: Why?
The spatial stress components form a symmetric
matrix \( T^{ij} \).

One arrives at this conclusion with the help of Newton's equation applied to the rotational
motion of a small cube of matter of volume \( V^3 \).

\( \Delta \alpha = \Delta x \)  
\( \Delta y = \Delta y \)

\[ \begin{align*}
\Delta x &= x \text{ face} \\
\Delta y &= y \text{ face} \\
\Delta z &= z \text{ face}
\end{align*} \]

The mass in such a cube is \( T^{00} V^3 \). The
moment of inertia of this cube is \( T^{00} L^5 \).

Newton's equation of motion applied to the
cube's rotation around the \( z \)-axis is

\[ \Sigma (T^{00} L^3) = (\text{torque})^3 = (\text{\( \tau \times \Delta P \)})^3 \]

\[ \begin{align*}
\Sigma L^3 &= \left[ \frac{1}{2} \left( -\tau^{x} L^2 \right) + \left( \frac{1}{2} \tau^{x} L^2 \right) \right] \\
&= \left[ \frac{1}{2} \left( -T^{x} L^2 \right) + \left( \frac{1}{2} T^{x} L^2 \right) \right]
\end{align*} \]

\( \text{Lower arm: y-force on lower arm x-force on} \)  
\( \text{to x face x-face to x-face} \)

\( \text{Lower arm: x-force on lower arm x-force on} \)  
\( \text{to y face y-face to y-face} \)

Comments:

1) Note that at \( x = \frac{L}{2} \)

\( \Delta F^y = T^{x} \Delta A_x = T^{x} L^2 \)

is a \( y \)-force exerted by the \( x \)-face on the outside
medium \( (\frac{1}{2} < x) \). This is because this \( y \)-force
expresses a flow of \( y \)-momentum into the
\( +x \) direction at the \( +x \) face.

2) Equivalently:

\( (-) T^{y} L^2 \)

is a \( y \)-force exerted on the \( x \)-face by the outside
medium. This expresses a flow of \( y \)-momentum
into the \( -x \) direction at the \( +x \) face.

In other words, it is the direction of the flow
of momentum that gets reversed when
one reverses the origin and the destination
of the application of a force.
c) At the -x face \( T_{x} \) also expresses a flow of y momentum into the x direction, but here it represents a y force exerted on the -x face by the outside medium (\( x = -\frac{L}{2} \)). In other words, the momentum flow \( T_{x} \) from the outside to the inside of the cube across the -x face.

As an aside, we note that the y force on the +x face can be represented in terms of the vector valued 3-form \( *T \) by

\[ *T = T_{A'B'C} \] where

\[ A' : (1, 0, 0) \]
\[ B' : (0, 0, 1) \]
\[ C' : (0, 0, 0) \]

as follows:

\[ \int T_{x} \Delta A_{x} = \int T_{y} \Delta L_{y} = \int T_{y} \Delta A_{y} \cdot n \Delta s \]

\[ = T_{y} \Delta A_{y} \cdot n \Delta s \Delta A_{x} (A'B'C) \]

Flow of momentum across \( \Delta A_{x} \) into negative direction (into the cube in the picture on page 10).

e) In fact, more generally, we note that the spatial components of the force \( \Delta F \) together with the energy rate ("power")

\[ \sum T_{y} \Delta A_{y} = T_{x} \Delta A_{x} + T_{y} \Delta A_{y} \]

make up the components of the 4-momentum

\[ e^{\mu} T^{\mu}_{x} \Delta A_{x} = \int \rho \Delta V \Delta \omega \xi \Delta \eta \] 1 \( B \Delta C \)

crossing the element of area \( \Delta A = B \times C \) during the time of the vector \( A' : (1, 0, 0) \)

By contrast the 4-momentum in the future

directed 3-volume \( V = A \times B \times C \)

\[ T_{y} \Delta L_{y} = \int \rho \Delta V \Delta \omega \xi \Delta \eta \]

where \( A : (0, 0, 1) \)
\( B : (1, 0, 0) \)
\( C : (0, 1, 0) \)

are spacelike.
Back to Newton's eq'n on p.38.

After simplifying Newton's equations for cube's rotation around the z-axis (p.10), one obtains

\( (S^2) = \frac{T^x y - T^x x}{T_{00}} \frac{\frac{1}{L^5}} \)

We see that if it were true that 
\( T^x y - T^x x \) then

\( (S^2)^2 \to \infty \) as \( L \to 0 \).

In other words, the cube of matter would be spinning with arbitrarily large angular velocities if we would consider it to be of sufficiently small volume \( L^3 \), where 
\( T^{xy} = T^{yx} \).

The fact that matter does not behave this way demands that 
\( T^{xy} = T^{yx} \)
or more generally 
\( T^{ij} = T^{ji} \).

Summary
The stress energy tensor \( T \) has the following symmetric array of components:

\[
T = \begin{bmatrix}
T^{00} & T^{01} & T^{02} & T^{03} \\
T^{10} & T^{11} & T^{12} & T^{13} \\
T^{20} & T^{21} & T^{22} & T^{23} \\
T^{30} & T^{31} & T^{32} & T^{33}
\end{bmatrix}
\]

\( T^{00} \) is the energy density, 
\( T^{ij} \) the stress, 
\( T^{0j} \) the energy flux, 
\( T^{ij} \) the momentum density.