Lecture 11

Creation of momenergy:

a) Expressed in physical terms
b) Expressed as an integral over the boundary [MTW Box 15.1 B] of a 4-volume
c) Expressed as a 4-volume integral:
   Gauss's theorem for a vector valued integral
   [MTW § 5.3, Box 5.3, Box 5.4]
d) Expressed as the exterior derivative of
   vector-valued three form: Generalized Stokes' theorem
   [MTW § 5.3, Box 5.4]

Summary: Conservation of momenergy expressed at 4 levels of mathematical generalization [MTW Box 15.2]
Reminder about "orientation".

The defining property of a vector space is its linearity structure. This structure accommodates the introduction of coordinates, i.e., sets of linearly independent spanning vectors and their corresponding sets of dual basis elements.

\[ \{ e_1 \}, \{ e_2 \}, \ldots, \{ e_n \}, \langle \omega^i | e_j \rangle = \delta_{ij}. \]

The concept "orientation" is an additional structure which, however, is not part of the defining (= essential) properties of a vector space. "Orientation" is a way of classifying ordered sets of vectors. In an n-dimensional space, it is achieved by the orientation form (a.k.a. Levi-Civita tensor)

\[ E = \frac{1}{n!} e_1 \wedge \cdots \wedge e_n, \quad \omega^1 \wedge \cdots \wedge \omega^n = (\omega^1 \wedge \cdots \wedge \omega^n). \]

Thus the order n-tuple \((e_1, e_2, \ldots, e_n)\) has positive orientation because

\[ E(e_1, e_2, \ldots, e_n) > 0, \]

while \((e_2, e_1, \ldots, e_n, e_n)\) and \((e_n, e_3, \ldots, e_{n-1}, e_1)\) have negative orientation because

\[ E(e_2, e_1, \ldots, e_{n-1}, e_n) = E(e_n, e_3, \ldots, e_{n-1}, e_1) < 0. \]

The process of replacing a positively oriented set with a set which is the same except for orientation produces a mere sign change in the value of the multilinear map \(E\). This linearity allows us to bring that change into the domain of \(E\) and write

\[ (e_2, e_1, \ldots, e_n, e_n) = -(e_1, e_2, \ldots, e_{n-1}, e_n), \]

\[ (e_1, e_2, \ldots, e_{n-1}, e_n) = -(e_2, e_1, \ldots, e_{n-1}, e_n), \]

\[ (e_n, e_1, \ldots, e_{n-1}, e_2) = +(e_1, e_2, \ldots, e_{n-1}, e_n), \]

etc.
4-cube and its bounding 3-volumes

Let us consider a region of spacetime spanned by the four 4-vectors

\[
T = \Delta t \frac{\partial}{\partial t} : (\Delta t, 0, 0, 0) \\
X = \Delta x \frac{\partial}{\partial x} : (0, \Delta x, 0, 0) \\
Y = \Delta y \frac{\partial}{\partial y} : (0, 0, \Delta y, 0) \\
Z = \Delta z \frac{\partial}{\partial z} : (0, 0, 0, \Delta z)
\]

whose interior is spanned by the ordered triplet

\[
\text{future: } t + \Delta t, \Delta x, \Delta y, \Delta z \quad (X, Y, Z)
\]

\[
\text{past: } t, \Delta x, \Delta y, \Delta z \quad (X, Y, Z)
\]

\[
\text{right: } x + \Delta x, t, \Delta y, \Delta z \quad (Y, Z, T)
\]

\[
\text{left: } x, t, \Delta y, \Delta z \quad (X, Y, Z)
\]

\[
\text{front: } y + \Delta y, t, \Delta z, \Delta x \quad (Z, X, T)
\]

\[
\text{back: } y, t, \Delta z, \Delta x \quad (Z, X, T)
\]

\[
\text{top: } z + \Delta z, t, \Delta x, \Delta y \quad (X, Y, Z)
\]

\[
\text{bottom: } z, t, \Delta x, \Delta y \quad (X, Y, Z)
\]

Our interest lies in exhibiting a geometrical expression for the amount of momenergy created in the volume \(dx\,dy\,dz\) during the time interval \(\Delta t\); in other words, we will determine

\[
\text{the (amount of momenergy)}
\]

\[
\text{created in a specified (region (\(dx\,dy\,dz\)) of spacetime))}
\]

The boundary of this 4-dimensional region consists of eight 3-volumes whose locations relative to the center \(t + \frac{\Delta t}{2}, x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2}\) of the region and whose volumes are...
The vectorial amount of created monomenergy is determined by the net amount of monomenergy flowing into and out of the specified region of spacetime.

This flow is the sum total flowing across the eight parts of the boundary of this 4-dimensional region. We have

\[
\alpha = \text{(change in m.e. in } \Omega_{xyz}^2) + \text{(outflow of m.e. through the sides of } \Omega_{xyz}^2 \text{ during } \Delta t)
\]

\[
= \text{(monomenergy in specified region of space } \Omega_{xyz}^2 \text{ at end of specified time interval } \Delta t) - \text{(monomenergy in specified region of space } \Omega_{xyz}^2 \text{ at beginning of specified time interval } \Delta t)
\]

\[
= \text{(flow of monomenergy out of top face of region } \Omega_{xyz}^2 \text{ during specified time interval)} + \text{(flow of monomenergy out of bottom face of region } \Omega_{xyz}^2 \text{ during specified time interval)} + \text{(flow of monomenergy out of front face of region } \Omega_{xyz}^2 \text{ during specified time interval)} + \text{(flow of monomenergy out of back face of region } \Omega_{xyz}^2 \text{ during specified time interval)}
\]

The total monomenergy flow across the 6 faces of the region \( \Omega_{xyz}^2 \) can be positive or negative. Taking note of the fact that in nature \( \alpha = 0 \), we see, for example, that

\[
\text{(change in monomenergy in region } \Omega_{xyz}^2) > 0 \Leftrightarrow \text{outflow} < 0,
\]

increase in monomenergy \( \Rightarrow \) inflow > 0.
The mathematical expression for these four contributing pairs is of 3-D \( \mathbf{a} \cdot \mathbf{b} \) followed by \( \mathbf{a} \times \mathbf{b} \). The A1-A5, A2-A5, A3-A5, A4-A5 pairs and the three planar components of momentum, namely the density, are composed of four contributing pairs.

\[
\mathbf{\Delta} = \mathbf{\Delta}_{1} \mathbf{\Delta}_{2} \mathbf{\Delta}_{3} \mathbf{\Delta}_{4}
\]

This representation, which is a reflection of an element \( \Delta x, \Delta y, \Delta z, \Delta r \), is a basis that spans the 4-D volume having as columns a basis of vectors in the vector having as columns a basis of vectors in the vector.

\[
\mathbf{T} = \begin{pmatrix}
\mathbf{T}_{1} & \mathbf{T}_{2} & \mathbf{T}_{3} & \mathbf{T}_{4}
\end{pmatrix}
\]

However, a geometrical (i.e., basis-independent) representation of this is possible.
Each of the eight contributions in the previous page is the vector (i.e., monenergy) valued density-flux 3-form $\star T^{(A,B,C)}$ evaluated on each of 8 ordered triples of vectors $(A, B, C)$:

$$T^{(A,B,C)} = \epsilon_{\mu} T^{\mu} \epsilon_{\nu} T^{\nu} \epsilon_{\lambda} T^{\lambda} \epsilon_{\sigma} T^{\sigma} \epsilon_{\phi} T^{\phi} \epsilon_{\chi} T^{\chi} \epsilon_{\psi} T^{\psi} (A, B, C)$$

$$= \frac{1}{8!} \epsilon_{\mu} \epsilon_{\nu} \epsilon_{\lambda} \epsilon_{\sigma} \epsilon_{\phi} \epsilon_{\chi} \epsilon_{\psi} T^{\mu} T^{\nu} T^{\lambda} T^{\sigma} T^{\phi} T^{\chi} T^{\psi} (A, B, C)$$

These 8 ordered triple of vectors are:

$(x, y, z)$ and $-(x, y, z)$
$(y, z, t)$ and $-(y, z, t)$
$(z, x, t)$ and $-(z, x, t)$
$(x, y, t)$ and $-(x, y, t)$
The basis independent $\mathbf{A}$ can be done for each of the eight oriented 3-D volumes. In terms of the vectors $\mathbf{X}, \mathbf{Y},$ and $\mathbf{Z}$ one has

$$e_{\mu} T^{\alpha \beta \gamma} \delta_{3} = e_{\mu} T^{\alpha \beta} e_{0123} \omega^\gamma \omega^\lambda \omega^\delta (X, Y, Z)$$

$$= e_{\mu} T^{\alpha \beta} e_{0123} \omega^\gamma \omega^\lambda \omega^\delta (X, Y, Z)$$

$$= e_{\mu} T^{\alpha \beta} e_{0123} \omega^\gamma \omega^\lambda \omega^\delta (X, Y, Z)$$

$$= * T (X, Y, Z)$$

$$e_{\mu} T^{\alpha \beta \gamma} \delta_{3} \delta_{4} = e_{\mu} T^{\alpha \beta} (-) e_{1230} \omega^\gamma \omega^\lambda \omega^\delta (Y, Z, T)$$

$$= e_{\mu} T^{\alpha \beta} (-) e_{1230} \omega^\gamma \omega^\lambda \omega^\delta (Y, Z, T)$$

$$= e_{\mu} T^{\alpha \beta} (-) e_{1230} \omega^\gamma \omega^\lambda \omega^\delta (Y, Z, T)$$

$$= (-) * T (Y, Z, T)$$

$$e_{\mu} T^{\alpha \beta \gamma} \delta_{3} \delta_{4} \delta_{5} = e_{\mu} T^{\alpha \beta} (-) e_{23410} \omega^\gamma \omega^\lambda \omega^\delta (X, Y, T)$$

$$= e_{\mu} T^{\alpha \beta} (-) e_{23410} \omega^\gamma \omega^\lambda \omega^\delta (X, Y, T)$$

$$= e_{\mu} T^{\alpha \beta} (-) e_{23410} \omega^\gamma \omega^\lambda \omega^\delta (X, Y, T)$$

$$= (-) * T (X, Y, T)$$

etc.

Introducing these basis invariants into the expression for $Q$ on PB, one finds that it has the form

$$Q = * T (X, Y, Z) - * T (X, Y, Z)$$

$$+ (-) * T (T, Y, Z) - (-) * T (T, Y, Z)$$

$$+ (-) * T (T, Z, X) - (-) * T (T, Z, X)$$

$$+ (-) * T (T, X, Y) - (-) * T (T, X, Y)$$

$$= \sum_{(\alpha, \beta, \gamma)} * T (\alpha, \beta, \gamma)$$

This sum is over the set of ordered triples of vectors,

$$(X, Y, Z), \quad (T, Y, Z), \quad (T, Z, X), \quad (T, X, Y)$$

and

$$(T, Y, Z), \quad (T, Z, X), \quad (T, X, Y)$$

$$= \sum_{(\alpha, \beta, \gamma)} * T (\alpha, \beta, \gamma)$$
The second set consists of those ordered triples which span those four 3-D faces of the cubic region $S_2$, each one of which contains the point event $(x, y, z)$. The normals to these faces are codirectional with the following vectors:

$$
\begin{cases}
-(x, y, z) : T \\
(T, y, z) : X \\
(T, z, x) : Y \\
(T, x, y) : Z
\end{cases}
$$

We see that each of the spanning triples, together with its appropriately scaled normal, is an even permutation of $(T, x, y, z)$:

$$
E(ABC, D) = E(T, x, y, z).
$$

This fact implies that the spanned by the ordered triples form the boundary...
There are two ways that the eight terms of the integral over the boundary $\partial S$ can be converted into an integral over the interior spacetime region $S$.

One leads to a 4-volume integral of a divergence, Gauss's theorem; the other leads to the integral of a 4-form, an example of the generalised Stokes's theorem.

A. Gauss's theorem.

A pair-wise combination reduces the eight term expression for the momemtum-energy created in $S$ from eight to four:

$$ Q = \nabla \cdot \left( \varepsilon_m T^{\mu\nu} g^{-1} \right) \delta X \delta Y \delta Z \delta T $$

$$ (-) \nabla \cdot \left( \varepsilon_m T^{\mu\nu} g^{-1} \right) \delta Y \delta Z \delta T \delta X $$

$$ (-) \nabla \cdot \left( \varepsilon_m T^{\mu\nu} g^{-1} \right) \delta Z \delta X \delta T \delta Y $$

$$ (-) \nabla \cdot \left( \varepsilon_m T^{\mu\nu} g^{-1} \right) \delta X \delta Y \delta T \delta Z $$

Here $(-)$ we used integers for the coordinate labels:

(2) we introduced the covariant derivatives $\nabla_{\mu}$

$$ \nabla_{\mu} \equiv \nabla_{\nu} $$

and kept only the lowest order terms in combining pairs using Taylor series expansion.

(3) we wrote

$$ \varepsilon_{\mu\nu\rho\sigma} = \varepsilon^{\gamma\delta\epsilon\sigma} \left( \gamma \times \delta \times \epsilon \right) $$

in terms of the constant permutation symbol whose values are $+1, -1, 0$.

The four term expression for $Q$ reduces therefore to the infinitesimal volume integral

$$ Q = \nabla_{\nu} \left( \varepsilon_m T^{\mu\nu} g^{-1} \right) \delta X \delta Y \delta Z \delta T $$

$$ = \int_{S} \nabla_{\nu} \left( \varepsilon_m T^{\mu\nu} g^{-1} \right) \delta X \delta Y \delta Z \delta T $$

This integral over $S$ equals the integral over $\partial S$ on the r.h.s. on r/h.
This is Gauss's divergence theorem.
The amount of monenergy created in the space-time region is zero. Consequently,
\[ \nabla \cdot (\varepsilon_m \mathcal{T}^{\mu \nu} \eta - \eta') = 0 \]
This divergence condition is an expression of the local conservation of monenergy.

### B. Stokes' Theorem

Recall that\[ T = \Delta t \frac{\partial}{\partial s}, \quad \omega^1 = \frac{\partial}{\partial x}, \quad \omega^2 = \frac{\partial}{\partial y}, \quad \omega^3 = \frac{\partial}{\partial z} \]
\[ X = \Delta x \frac{\partial}{\partial x}, \quad Y = \Delta y \frac{\partial}{\partial y}, \quad Z = \Delta z \frac{\partial}{\partial z} \]
so that
\[ \omega^1 \omega^2 \omega^3 (T, X, Y, Z) = \Delta t \Delta x \Delta y \Delta z \]

created monenergy as the value of a 4-form evaluated on the vectors \( T, X, Y, \) and \( Z \). Our task is to find this 4-form by "factoring out" these vectors.

To this end we write, using the expression on page 75:
\[ Q = \nabla_0 (\varepsilon_m \mathcal{T}^{\mu \nu} \mathcal{E}_{2035}) \omega^0 \omega^1 \omega^2 \omega^3 (T, X, Y, Z) \]
\[ - \nabla_2 (\varepsilon_m \mathcal{T}^{\mu \nu} \mathcal{E}_{2350}) \omega^0 \omega^1 \omega^2 \omega^3 (T, X, Y, Z) \]
\[ - \nabla_3 (\varepsilon_m \mathcal{T}^{\mu \nu} \mathcal{E}_{2150}) \omega^0 \omega^1 \omega^2 \omega^3 (T, X, Y, Z) \]
\[ - \nabla_5 (\varepsilon_m \mathcal{T}^{\mu \nu} \mathcal{E}_{2130}) \omega^0 \omega^1 \omega^2 \omega^3 (T, X, Y, Z) \]

We wish to consolidate these terms into the integral of single 2-form.

In the restricted summation convention,\[ \varepsilon_{\mu \nu \rho \sigma} = \begin{cases} 1 & \text{if } \mu \nu \rho \sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu \nu \rho \sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases} \]
has only a single term. For example,
\[ \varepsilon_{2130} \omega^0 \omega^1 \omega^2 \omega^3 = \varepsilon_{0213} \omega^0 \omega^1 \omega^2 \omega^3 = \omega_0 \omega^1 \omega^2 \omega^3 \]
and similarly for the others. Thus the created monenergy is

This fact helps us to identify the amount of
\[ Q = \nabla_\nu (\rho T^{\mu \nu} E_{\mu \nu \rho \sigma}) \epsilon^{\rho \sigma \alpha \beta \gamma} \]

\[ + \nabla_\nu (\rho T^{\mu \nu} E_{\mu \nu \rho \sigma}) \epsilon^{\rho \sigma \nu \alpha \beta} \]

\[ + \nabla_\nu (\rho T^{\mu \nu} E_{\mu \nu \rho \sigma}) \epsilon^{\rho \sigma \nu \alpha \beta} \]

\[ + \nabla_\nu (\rho T^{\mu \nu} E_{\mu \nu \rho \sigma}) \epsilon^{\rho \sigma \nu \alpha \beta} \] \( (T, X, Y, Z) \)

The amount of momenergy created in the 4-D region \( S \) spanned by \( T, X, Y, \) and \( Z \) is therefore

\[ Q = \int S \cdot \tau \quad (T, X, Y, Z) \]

**SUMMARY:**

The mathematical theorem we have proved is

\[ \int S S S S E_{\mu \nu} \epsilon^{\nu \alpha \beta \gamma} \omega^{\alpha \beta \gamma} = \int S S S S (\rho \nabla_{\mu} E_{\mu \nu} \omega^{\nu \alpha \beta \gamma}) \]

or more briefly

\[ \int S S S S = \int S S S S \frac{d}{dt} \tau \]

Compare this with the generalized Stokes' theorem.

**For a scalar valued \( p \)-form \( \omega \) it reads,**

\[ \int \frac{d}{dt} \omega = \int \omega \]

because \( d \omega = 0 \) for a coordinate basis, which we have been assuming all along.
The local conservation of momenergy can thus be stated in two mathematically equivalent ways:

1. \( \left[ d \tau \rightarrow \right] = d \left[ \left( \omega_{a} \right) \left( T^{a}_{\mu \nu} \leftrightarrow \omega_{\mu ^{a}} \omega_{\mu ^{b}} \right) \right] = 0 \)

which is based on the "generalized Stokes' theorem".

2. \( \nabla_{\nu} \left( \varepsilon_{\mu \tau} T^{\mu \nu} \right) = 0 \)

which is based on "Gauss's divergence theorem".

In terms of components one can show that this is the same as

3. \( T^{a}_{\mu \nu} \nabla_{\nu} = 0 \)

4. In Minkowski spacetime this reads

\[ \frac{\partial T^{a}_{\mu \nu}}{\partial t} + \frac{\partial T^{a}_{\mu \nu}}{\partial x} + \frac{\partial T^{a}_{\mu \nu}}{\partial y} + \frac{\partial T^{a}_{\mu \nu}}{\partial z} = 0 \]

5. Conservation of momenergy \( \Delta = 0 \)

6. \( 0 = \frac{1}{r^{3}} \nabla_{\nu} \left( \varepsilon_{\mu \tau} T^{\mu \nu} \right) = \varepsilon_{\mu \tau} T^{\mu \nu} \nabla_{\nu} \)

Relative to Minkowski coordinates,

\[ 0 = \frac{1}{r^{3}} \left( \frac{\partial T^{a}_{\mu \nu}}{\partial t} + \frac{\partial T^{a}_{\mu \nu}}{\partial x} + \frac{\partial T^{a}_{\mu \nu}}{\partial y} + \frac{\partial T^{a}_{\mu \nu}}{\partial z} \right) \]
Amount of monopole energy created in 4-volume \( \alpha \):

\[
T \wedge x \wedge y \wedge z = \partial \partial x \partial y \partial z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

\[
Q = \int_{\alpha} \nabla \cdot \left( T \wedge \left[ \frac{\partial}{\partial x} \right] \right) dx dy dz
\]

**Gauß' Theorem**

\[
\int_{\alpha} \nabla \cdot (T \wedge \left[ \frac{\partial}{\partial x} \right]) = \int_{\partial \alpha} T \wedge \left[ \frac{\partial}{\partial x} \right] dx dy dz
\]

**Stokes' Theorem**

\[
\int_{\partial \alpha} T \wedge \left[ \frac{\partial}{\partial x} \right] dx dy dz = \int_{\alpha} \nabla \wedge (T \wedge \left[ \frac{\partial}{\partial x} \right])
\]