Lecture 11

Creation of momenergy:

a) Expressed in physical terms
b) Expressed as an integral over the boundary [MTW Box 15.1 B] of a 4-volume
c) Expressed as a 4-volume integral: Gauß's theorem for a vector valued integral
d) Expressed as the exterior derivative of vector-valued three form: Generalized Stokes' theorem [MTW § 5.8, Box 5.3, Box 5.4]

Summary: Conservation of momenergy expressed at 4 levels of mathematical generalization [MTW Box 15.2]

Reminder about "orientation".

The defining property of a vector space is its <u>linearity structure</u>. This structure accomodates the introduction of coordinates, i.e. of sets of linearly independent spanning vectors and their corresponding sets of dual basis elements,

$$\{e_i\},\ \{\omega^j\},\ \langle \omega^j | e_i \rangle = \delta^j_i .$$

The concept "orientation" is an additional structure which, however, is not part of the defining (= essential) properties of a vector space. "Orientation" is a way of classifying ordered sets of vectors. In an n-dimensional space it is achieved by the <u>orientation form</u> (a.k.a. Levi-Civita tensor)

$$\epsilon = \frac{1}{n!}\, \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_n}\ \omega^{\alpha_1} \wedge \omega^{\alpha_2} \wedge \cdots \wedge \omega^{\alpha_n} \left(= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\right)$$

Thus the order n-tuple $(e_1, e_2, \ldots, e_n)$ has positive orientation because

$$\epsilon(e_1, e_2, \ldots, e_n) > 0,$$

while $(e_2, e_1, \ldots, e_{n-1}, e_n)$ and $(e_n, e_2, \ldots, e_{n-1}, e_1)$ have negative orientation because

$$\epsilon(e_2, e_1, \ldots, e_{n-1}, e_n) = \epsilon(e_n, e_2, \ldots, e_{n-1}, e_1) < 0$$

The process of replacing a positively oriented set with a set which is the same, except for orientation, produces a mere sign change in the value of the multilinear map ϵ. This linearity allows us to bring that change into the domain of ϵ and write

$$(e_2\, e_1, \ldots, e_{n-1}, e_n) = -(e_1, e_2, \ldots, e_{n-1}, e_n)$$

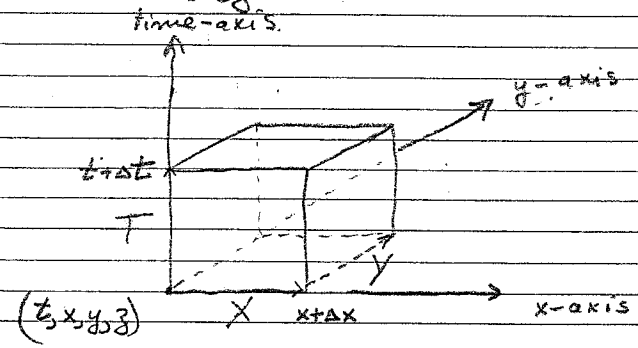
$$(e_n\, e_2, \ldots, e_{n-1}, e_1) = -(e_1, e_2, \ldots, e_{n-1}, e_n)$$

$$(e_n, e_1, \ldots, e_{n-1}, e_2) = +(e_1, e_2, \ldots, e_{n-1}, e_n)$$

etc.

Let us consider a region of spacetime spanned by the four 4-vectors

$$T = \Delta t \frac{\partial}{\partial t} : (\Delta t, 0, 0, 0)$$
$$X = \Delta x \frac{\partial}{\partial x} : (0, \Delta x, 0, 0)$$
$$Y = \Delta y \frac{\partial}{\partial y} : (0, 0, \Delta y, 0)$$
$$Z = \Delta z \frac{\partial}{\partial z} : (0, 0, 0, \Delta z)$$



The boundary of this 4-dimensional region consists of <u>eight</u> 3-volumes whose locations relative to the center $t+\frac{\Delta t}{2}, x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}$ of the region and whose volumes are

	Location of	3-volume	Whose interior is spanned by the ordered triplet
future	$t+\Delta t$	$\Delta x \Delta y \Delta z$	(X, Y, Z)
past	t	$\Delta x \Delta y \Delta z$	$-(X, Y, Z)$
right	$x+\Delta x$	$\Delta t \Delta y \Delta z$	(Y, Z, T)
left	x	$\Delta t \Delta y \Delta z$	$-(Y, Z, T)$
front	$y+\Delta y$	$\Delta t \Delta z \Delta x$	(Z, X, T)
back	y	$\Delta t \Delta z \Delta x$	$-(Z, X, T)$
top	$z+\Delta z$	$\Delta t \Delta x \Delta y$	(X, Y, T)
bottom	z	$\Delta t \Delta x \Delta y$	$-(X, Y, T)$

Our interest lies in exhibiting a geometrical expression for the amount of <u>momenergy</u> <u>created in the volume</u> <u>$\Delta x \Delta y \Delta z$ during the time interval Δt</u>; in other words, we will determine the

$$\left(\begin{array}{l}\text{amount of momenergy} \\ \text{created in a specified} \\ \text{region } (\Delta t \Delta x \Delta y \Delta z) \text{ of spacetime}\end{array}\right) \equiv Q$$

The vectorial amount of created momenergy is determined by the net amount of momenergy flowing into and out of the specified region of spacetime.

This flow is the sum total flowing across the eight parts of the boundary of this 4-dimensional region. We have

$Q =$ created momenergy

$= \left(\begin{array}{l}\text{change in m.e. in } \Delta x\Delta y\Delta z \\ \text{during time } \Delta t\end{array}\right) + \left(\begin{array}{l}\text{out flow of m.e. through} \\ \text{the sides of } \Delta x\Delta y\Delta z \text{ during } \Delta t\end{array}\right)$

$=$

$\left(\begin{array}{l}\text{momenergy in specified} \\ \text{region of space, } \Delta x\Delta y\Delta z, \\ \text{at } \underline{\text{end}}, t+\Delta t, \text{ of the specified} \\ \text{time interval } \Delta t\end{array}\right) - \left(\begin{array}{l}\text{momenergy in specified} \\ \text{region of space, } \Delta x\Delta y\Delta z, \text{ at} \\ \underline{\text{beginning}}, t, \text{ of specified} \\ \text{time interval } \Delta t\end{array}\right)$

$\left(\begin{array}{l}\text{flow of momenergy} \\ \underline{\text{out of right-hand}} \\ \text{face of region } \Delta x\Delta y\Delta z \\ \text{during specified} \\ \text{time interval } \Delta t\end{array}\right) + \left(\begin{array}{l}\text{flow of momenergy} \\ \underline{\text{out of left-hand}} \\ \text{face of region } \Delta x\Delta y\Delta z \\ \text{during specified} \\ \text{time interval } \Delta t\end{array}\right)$

$+ \left(\begin{array}{l}\text{flow of momenergy} \\ \text{out of } \underline{\text{front}} \text{ face} \\ \text{of region } \Delta x\Delta y\Delta z \\ \text{during specified} \\ \text{time interval } \Delta t\end{array}\right) + \left(\begin{array}{l}\text{flow of momenergy} \\ \text{out of } \underline{\text{back}} \text{ face} \\ \text{of region } \Delta x\Delta y\Delta z \\ \text{during specified} \\ \text{time interval } \Delta t\end{array}\right)$

$+ \left(\begin{array}{l}\text{flow of momenergy} \\ \text{out of } \underline{\text{top}} \text{ face} \\ \text{of region } \Delta x\Delta y\Delta z \\ \text{during specified} \\ \text{time interval}\end{array}\right) + \left(\begin{array}{l}\text{flow of momenergy} \\ \text{out of } \underline{\text{bottom}} \text{ face} \\ \text{of region } \Delta x\Delta y\Delta z \\ \text{during specified} \\ \text{time interval}\end{array}\right)$

The total momenergy flow across the 6 faces of the region $\Delta x\Delta y\Delta z$ can be positive or negative. Taking note of the fact that in nature $Q = 0$, we see, for example, that

$\left(\begin{array}{l}\text{change in momenergy} \\ \text{in region } \Delta x\Delta y\Delta z\end{array}\right) > 0 \Leftrightarrow \underbrace{\text{out flow}}_{} < 0.$

increase in momenergy $\Leftrightarrow$ inflow > 0

Q is composed of four constituent pairs, namely the density and the three flux components of momenergy, $\{e_\mu T^{\mu 0}, e_\mu T^{\mu x}, e_\mu T^{\mu y}, e_\mu T^{\mu z}\}$ in their respective four pairs of 3-D volumes $\{\Delta x \Delta y \Delta z, \Delta y \Delta z \Delta t, \Delta z \Delta x \Delta t, \Delta x \Delta y \Delta t\}$ with respective pair separation $\{\Delta t, \Delta x, \Delta y, \Delta z\}$.

The mathematical expression for these four contributing pairs is

$$Q = e_\mu T^{\mu 0}\Big|_{t+\Delta t}^{\Delta x \Delta y \Delta z} - e_\mu T^{\mu 0}\Big|_{t}^{\Delta x \Delta y \Delta z}$$

$$+ e_\mu T^{\mu x}\Big|_{x+\Delta x}^{\Delta y \Delta z \Delta t} - e_\mu T^{\mu x}\Big|_{x}^{\Delta y \Delta z \Delta t}$$

$$+ e_\mu T^{\mu y}\Big|_{y+\Delta y}^{\Delta z \Delta x \Delta t} - e_\mu T^{\mu y}\Big|_{y}^{\Delta z \Delta x \Delta t}$$

$$+ e_\mu T^{\mu z}\Big|_{z+\Delta z}^{\Delta x \Delta y \Delta t} - e_\mu T^{\mu z}\Big|_{z}^{\Delta x \Delta y \Delta t}$$

This representation of Q is a reflection of having chosen a basis of vectors on page 1 which are collinear with the vectors X, Y, Z and T that span the 4-D volume element $\Delta x \Delta y \Delta z \Delta t$.

However, a geometrical, i.e. basis independent representation of Q is possible. Indeed

each of the eight contributions on the previous page is the vector (i.e. momenergy) valued density-flux 3-form $*\Pi(\ ,\ ,\ )$ evaluated on each of 8 ordered triples of vectors (A, B, C):

$$*\Pi(A,B,C) = e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (A, B, C)$$

$$= \sqrt{-g} \begin{vmatrix} e_\mu T^{\mu 0} & e_\mu T^{\mu 1} & e_\mu T^{\mu 2} & e_\mu T^{\mu 3} \\ \omega^0(A) & \omega^1(A) & \omega^2(A) & \omega^3(A) \\ \omega^0(B) & \omega^1(B) & \omega^2(B) & \omega^3(B) \\ \omega^0(C) & \omega^1(C) & \omega^2(C) & \omega^3(C) \end{vmatrix}$$

Each ordered triple (A, B, C) is a particular choice of only three distinct vectors from the ordered set (T, X, Y, Z) with positive orientation, i.e. such that

$$0 < \epsilon(T, X, Y, Z) = \epsilon_{\alpha\beta\gamma\delta}\, T^\alpha X^\beta Y^\gamma Z^\delta$$
$$= \Delta t \Delta x \Delta y \Delta z,$$

as determined by the 4-D orientation (a.k.a. Levi-Civita) tensor

$$\epsilon = \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta$$
$$= \frac{\sqrt{-g}}{4!} [\alpha\beta\gamma\delta]\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta$$

These 8 ordered triple of vectors are

(X, Y, Z) and $-(X, Y, Z)$

(Y, Z, T) and $-(Y, Z, T)$

(Z, X, T) and $-(Z, X, T)$

(X, Y, T) and $-(X, Y, Z)$

The basis independent representation can be done for each of the eight oriented 3-D volumes. In terms of the vectors T, X, Y and Z one has

$$e_\mu T^{\mu 0} \Delta x \Delta y \Delta z = e_\mu T^{\mu 0} \epsilon_{0123} \, \omega^1 \wedge \omega^2 \wedge \omega^3 (X, Y, Z)$$
$$= e_\mu T^{\mu 0} \epsilon_{0|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (X, Y, Z)$$
$$= e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (X, Y, Z)$$
$$= {}^*\mathbb{T}(X, Y, Z)$$

$$e_\mu T^{\mu x} \Delta y \Delta z \Delta t = e_\mu T^{\mu 1} (-) \epsilon_{1230} \, \omega^2 \wedge \omega^3 \wedge \omega^0 (Y, Z, T)$$
$$= e_\mu T^{\mu 1} (-) \epsilon_{1|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (Y, Z, T)$$
$$= e_\mu T^{\mu\nu} (-) \epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (Y, Z, T)$$
$$= (-) {}^*\mathbb{T}(Y, Z, T)$$

$$e_\mu T^{\mu y} \Delta z \, \Delta x \, \Delta t = e_\mu T^{\mu 2} (-) \epsilon_{2310} \, \omega^3 \wedge \omega^1 \wedge \omega^0 (Z, X, T)$$
$$= e_\mu T^{\mu 2} (-) \epsilon_{2|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (Z, X, T)$$
$$= e_\mu T^{\mu\nu} (-) \epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (Z, X, T)$$
$$= (-) {}^*\mathbb{T}(Z, X, T)$$

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etc.

Introducing these basis invariants into the expression for Q on P8, one finds that it has the form

$$Q = {}^*\mathbb{T}(X, Y, Z)\Big|_{t+\Delta t} - {}^*\mathbb{T}(X, Y, Z)\Big|_t$$
$$+ (-) {}^*\mathbb{T}(T, Y, Z)\Big|_{x+\Delta x} - (-) {}^*\mathbb{T}(T, Y, Z)\Big|_x$$
$$+ (-) {}^*\mathbb{T}(T, Z, X)\Big|_{y+\Delta y} - (-) {}^*\mathbb{T}(T, Z, X)\Big|_y$$
$$+ (-) {}^*\mathbb{T}(T, X, Y)\Big|_{z+\Delta z} - (-) {}^*\mathbb{T}(T, X, Y)\Big|_z$$
$$= \sum_{(A,B,C)} {}^*\mathbb{T}(A, B, C)$$

This sum is over the set of 8 ordered triples of vectors,

$$\left.\begin{array}{l} (X, Y, Z) \\ -(T, Y, Z) \\ -(T, Z\ X) \\ -(T, X, Y) \end{array}\right\} \text{ and } \left\{\begin{array}{l} -(X, Y, Z) \\ (T, Y, Z) \\ (T, Z, X) \\ (T, X, Y) \end{array}\right.$$

The second set consists of those ordered triples which span those four 3-D faces, each one of which contains the point event (t, x, y, z).

The normals to these faces are codirectional with the following vectors

$$\mathbb{C}(A,B,C) = \left\{ \begin{array}{ll} -(X,Y,Z) & : T \\ (T,Y,Z) & : X \\ (T,Z,X) & : Y \\ (T,X,Y) & : Z \end{array} \right\} = D$$

We see that each of the spanning triples together with its appropriately scaled normal is an even permutation of (T, X, Y, Z):

$$\epsilon(A,B,C,D) = \epsilon(T\,X\,Y\,Z).$$

This fact implies that the spanned by the ordered triples form the boundary

$\partial\Omega$ of the cubical region Ω.

Consequently,

$$Q = \sum_{ABC} {}^*\mathbb{T}(A,B,C)$$

$$= \iiint_{\partial\Omega} {}^*\mathbb{T} = \iiint_{\partial\Omega} e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \omega^\alpha{}_\lambda \omega^\beta{}_\lambda \omega^\gamma$$

There are two ways that the eight terms of the integral over the boundary $\partial\Omega$ can be converted into an integral over the interior spacetime region Ω. One leads to a 4-volume integral of a divergence, Gauß's theorem; the other leads to the integral of a 4-form, an example of the generalized Stokes' theorem.

A. Gauß's theorem

A pairwise combination reduces the eight term expression for the momenergy created in Ω from eight to four:

$$Q = \nabla_0\left(e_\mu T^{\mu 0}\sqrt{-g}\right)[0123]\,\Delta t\,\Delta x\,\Delta y\,\Delta z$$
$$(-)\,\nabla_1\left(e_\mu T^{\mu 1}\sqrt{-g}\right)[1230]\,\Delta x\,\Delta y\,\Delta z\,\Delta t$$
$$(-)\,\nabla_2\left(e_\mu T^{\mu 2}\sqrt{-g}\right)[2310]\,\Delta y\,\Delta z\,\Delta x\,\Delta t$$
$$(-)\,\nabla_3\left(e_\mu T^{\mu 3}\sqrt{-g}\right)[3120]\,\Delta z\,\Delta x\,\Delta y\,\Delta t$$
$$= \left(\nabla_\nu e_\mu T^{\mu\nu}\sqrt{-g}\right)[0123]\,\Delta t\,\Delta x\,\Delta y\,\Delta z = \iiiint_\Omega \nabla_\nu(\quad)\,d^4x$$

Here (1) we used integers for the coordinate labels:

$$t = x^0 \qquad e_0 = \frac{\partial}{\partial t} \qquad e_2 = \frac{\partial}{\partial y}$$
$$x = x^1$$
$$y = x^2$$
$$z = x^3, \qquad e_1 = \frac{\partial}{\partial x} \qquad e_3 = \frac{\partial}{\partial z}$$

(2) we introduced the covariant derivatives ∇_ν

$$\nabla_\nu \equiv \nabla_{e_\nu}$$

and kept only the lowest order terms in combining pairs using Taylor series expansion,

(3) we wrote

$$\epsilon_{\nu\alpha\beta\gamma} = \sqrt{-g}\,[\nu\alpha\beta\gamma]$$

in terms of the constant permutation symbol whose values are +1, -1, or 0.

The four term expression for Q reduces therefore to the infinitesimal volume integral

$$Q = \nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right)\Delta t\,\Delta x\,\Delta y\,\Delta z$$
$$= \iiiint_\Omega \nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right)d^4x$$

This integral over Ω equals the integral over $\partial\Omega$ on the r.h.s. on p14

$$\boxed{\iiint_{\partial\Omega} e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, dx^\alpha dx^\beta dx^\gamma = \iiiint_\Omega \nabla_\nu (e_\mu T^{\mu\nu}\sqrt{-g})\, d^4x}$$

This is Gauß's divergence theorem.
The amount of momenergy created in the space time region is zero. Consequently

$$\boxed{\nabla_\nu (e_\mu T^{\mu\nu}\sqrt{-g}) = 0}$$

This divergence condition is an expression of the local conservation of momenergy.

B. Stokes' Theorem.

Recall that

$$T = \Delta t \frac{\partial}{\partial x^0} \qquad \langle \omega^i | \frac{\partial}{\partial x^j} \rangle = \delta^i_j$$

$$X = \Delta x \frac{\partial}{\partial x^1}$$

$$Y = \Delta y \frac{\partial}{\partial x^2}$$

$$Z = \Delta z \frac{\partial}{\partial x^2}$$

so that

$$\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 (T, X, Y, Z) = \Delta t\, \Delta x\, \Delta y\, \Delta z \wedge$$

is the volume of region Ω.

This fact helps us to identify the amount of

created momenergy as the value of a 4-form evaluated on the vectors T, X, Y, and Z. Our task is to find this 4-form by "factoring out" these vectors. To this end we write, using the expression on p15

$$\begin{aligned} Q = {} & \nabla_0 (e_\mu T^{\mu 0} \epsilon_{0123})\, \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 (T, X, Y, Z) \\ & - \nabla_1 (e_\mu T^{\mu 1} \epsilon_{1230})(-)\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^0 (T, X, Y, Z) \\ & - \nabla_2 (e_\mu T^{\mu 2} \epsilon_{2310})(-)\omega^2 \wedge \omega^3 \wedge \omega^1 \wedge \omega^0 (T, X, Y, Z) \\ & - \nabla_3 (e_\mu T^{\mu 3} \epsilon_{3120})(-)\omega^3 \wedge \omega^1 \wedge \omega^2 \wedge \omega^0 (T, X, Y, Z) \end{aligned}$$

We wish to consolidate these terms into the integral of single diff'l form.

In the restricted summation convention

$$\epsilon_{\nu|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \qquad \alpha < \beta < \gamma$$

has only a single term. For example

$$\epsilon_{2|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma = \epsilon_{2013}\, \omega^0 \wedge \omega^1 \wedge \omega^3 = (-)\epsilon_{1023}\, \omega^3 \wedge \omega^1 \wedge \omega^0$$

and similarly for the others.

Thus the created momenergy is

$$Q = \Big[\nabla_0 \big(e_\mu T^{\mu 0} \epsilon_{0|\alpha\beta\gamma|} \big) \omega^0 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_1 \big(e_\mu T^{\mu 1} \epsilon_{1|\alpha\beta\gamma|} \big) \omega^1 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_2 \big(e_\mu T^{\mu 2} \epsilon_{2|\alpha\beta\gamma|} \big) \omega^2 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_3 \big(e_\mu T^{\mu 3} \epsilon_{3|\alpha\beta\gamma|} \big) \omega^3 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \Big] (T, X, Y, Z)$$

Note that

$$\epsilon_{0|\alpha\beta\gamma|} \, \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma = 0 \quad \text{unless } \sigma = 0 .$$

In fact we have

$$\epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma = 0 \quad \text{unless } \sigma = \nu$$

Thus

$$Q = \Big[\nabla_\sigma \big(e_\mu T^{\mu 0} \epsilon_{0|\alpha\beta\gamma|} \big) \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$\nabla_\sigma \big(e_\mu T^{\mu 1} \epsilon_{1|\alpha\beta\gamma|} \big) \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$\nabla_\sigma \big(e_\mu T^{\mu 2} \epsilon_{2|\alpha\beta\gamma|} \big) \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$\nabla_\sigma \big(e_\mu T^{\mu 3} \epsilon_{3|\alpha\beta\gamma|} \big) \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \Big] (T, X, Y, Z)$$

$$= \underbrace{\nabla_\sigma \big(e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \big) \omega^\sigma}_{} \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \, (T, X, Y, Z)$$

$$= d \big(e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \big) \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \quad (T, X, Y, Z)$$

PROBLEM: SHOW THIS

$$= d \big(e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \big) \quad (T, X, Y, Z)$$

because $d\omega^\alpha = 0$ for a coordinate basis, which we have been assuming all along.

The amount of momenergy created in the 4-D region Ω spanned by T, X, Y and Z is therefore

$$Q = d^* \Pi \quad (T, X, Y, Z)$$

SUMMARY

The mathematical theorem we have proved is

$$\iiint_{\partial\Omega} e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma = \iiiint_{\Omega} d \big(e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \big)$$

or more briefly

$$\boxed{\iiint_{\partial\Omega} {}^*\Pi = \iiiint_{\Omega} d\, {}^*\Pi}$$

Compare this with the generalized Stokes' Theorem. For a scalar valued p-form $\underset{\sim}{\omega}$ it reads,

$$\int_{\partial R} \underset{\sim}{\omega} = \int_R \underset{\sim}{d\omega}$$

The local conservation of momenergy can thus be stated in two mathematically equivalent ways:

I. $$\boxed{d\,{}^*\mathbb{T} \equiv d\left(e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma\right) = 0},$$

which is based on the "generalized Stokes' Theorem"

II A) $$\boxed{\nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right) = 0}$$

which is based on "Gauß's divergence theorem"

In terms of components one can show that this is the same as

B) $$\boxed{T^{\mu\nu}{}_{;\nu} = 0}$$

C) In Minkowski spacetime this reads

$$\boxed{\frac{\partial T^{\mu t}}{\partial t} + \frac{\partial T^{\mu x}}{\partial x} + \frac{\partial T^{\mu y}}{\partial y} + \frac{\partial T^{\mu z}}{\partial z} = 0}$$

1st Summary of Lecture 9

1.) Amount of momenergy created in a 4-volume $T \wedge X \wedge Y \wedge Z = \Delta t \Delta x \Delta y \Delta z \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$:

$$Q = \iiint_{\partial\Omega} {}^*\mathbb{T}$$ where (a) $\partial\Omega$ = the boundary of region Ω.

(b) ${}^*\mathbb{T} = e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$

2.) Gauß's theorem

$$\iiint_{\partial\Omega} {}^*\mathbb{T} = \iiiint_{\Omega} \nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right)\sqrt{-g}\, d^4x$$

3.) Stoke's theorem

$$\iiint_{\partial\Omega} {}^*\mathbb{T} = \iiiint_{\Omega} d({}^*\mathbb{T})$$

Conservation of momenergy $Q = 0$

4.) $$0 = \frac{1}{\sqrt{-g}} \nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right) = e_\mu T^{\mu\nu}{}_{;\nu}$$

$$= e_\mu\left(\frac{\partial T^{\mu t}}{\partial t} + \frac{\partial T^{\mu x}}{\partial x} + \frac{\partial T^{\mu y}}{\partial y} + \frac{\partial T^{\mu z}}{\partial z}\right)$$

Relative to Minkowski coordinates.

2nd Summary

$$Q = \iiint_{\substack{\partial\Omega = \\ = 8 \text{ bounding} \\ \text{cubes}}} \overbrace{e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}}^{*\mathbb{T}} \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma =$$

$$= \iiiint_{\substack{\Omega = \\ = 4\text{-D region}}} \underbrace{\nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\,[0,1,2,3]\right)}_{e_\mu T^{\mu\nu}{}_{;\nu}\sqrt{-g}\,[0,1,2,3]} \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \qquad (*)$$

A calculation yields

$$d\left(e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma\right) =$$

$$= e_\mu T^{\mu\sigma}{}_{;\sigma}\, \epsilon_{0123}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$= e_\mu T^{\mu\sigma}{}_{;\sigma}\sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

Thus, after insertion into (*)

$$\boxed{Q = \iiint_{\partial\Omega} *\mathbb{T} = \iiiint_{\Omega} (d * \mathbb{T})}$$

$$Q = 0 \;\forall \Omega \Rightarrow \left.\begin{array}{l} \boxed{\nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right) = 0} \\ \text{OR} \\ \boxed{d(*\mathbb{T}) = 0} \end{array}\right\} \; T^{\mu\sigma}{}_{;\sigma} = 0$$

Amount of momenergy created in 4-volume Ω

$$T \wedge X \wedge Y \wedge Z = \Delta t \Delta x \Delta y \Delta z \; \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

$$Q = \iiint_{\partial\Omega} *\mathbb{T} \equiv \iiint e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

Gauß' theorem

$$= \iiiint_{\Omega} \underbrace{\nabla_\nu\left(e_\mu T^{\mu\nu}\sqrt{-g}\right)}_{\text{" definition of ";\nu\text{"}}} \sqrt{-g}\, d^4x$$

$$\equiv \iiiint_{\Omega} \underbrace{e_\mu T^{\mu\nu}{}_{;\nu}}_{\text{relative to Minkowskee coord.}} \sqrt{-g}\, d^4x$$

$$= \iiiint_{\Omega} e_\mu \left(\frac{\partial T^{\mu t}}{\partial t} + \frac{\partial T^{\mu x}}{\partial x} + \frac{\partial T^{\mu y}}{\partial y} + \frac{\partial T^{\mu z}}{\partial z}\right) dt\,dx\,dy\,dz$$

Stokes's theorem

$$= \iiiint_{\Omega} d(*\mathbb{T})$$