LECTURE 12

PART 1: Vectorial Form of Gauss's and Stokes's theorems: Application to momentum conservation.

PART 2: Fluid Kinematics:
Expansion of a moving volume of fluid [MTW § 22.1 - 22.3]
via the scalar version of Gauss's Theorem.
"PART 1": CONTINUED FROM LECTURE II

Figure 2: Cylindrical region $\Sigma$ and its boundary $\partial \Sigma$

(6) $Q =$ net momentum emerging from Vol.$\Sigma = x \cdot E$

$$Q = \sum_{\text{oriented faces } F} \int_{F} \mathbf{T} \cdot d\mathbf{S}$$

$$= \iiint_{\Sigma} \mathbf{e}_{\mu} \mathbf{T}^{\mu} \mathbf{e}_{\nu} \mathbf{E}_{\nu}(x, y, z) \cdot \mathbf{a}_{\mathbf{E}_{\nu}} \mathbf{a}_{\mathbf{E}_{\nu}}$$

$$= \mathbf{T}(xy-z)_{xyz} - \mathbf{T}(xy-z)_{xyz} + \text{cyclic (xyz)}$$

There are two mathematical formulations of expressing the final result:

A) Gauss's theorem and B) Stokes' theorem.

Thus we have proved Gauss's theorem

$$Q = \iiint_{\Sigma} \mathbf{e}_{\mu} \mathbf{T}^{\mu} \mathbf{E}_{\nu}(x, y, z) \cdot \mathbf{a}_{\mathbf{E}_{\nu}} \mathbf{a}_{\mathbf{E}_{\nu}}$$

with the consequence that

$$\mathbf{F} = 0 \iff \mathbf{\nabla} \cdot (\mathbf{e}_{\mu} \mathbf{T}^{\mu} \mathbf{E}_{\nu}(x, y, z)) = 0,$$

which is a differential geometric formulation of the law of momentum conservation.
6 B) Stokes' Theorem

From the top of page 2 we have with the help of the familiar totally antisymmetric Levi-Civita tensor whose components (see page 3, lecture 8) are

\[ \epsilon_{x' y' z'} = \sqrt{\frac{2}{3}} \left[ \nu x' y' z' \right] ; \quad \nu = 0, 1, 2, 3 \]

\[ Q = \nabla_0 (e_\mu T^\mu e_{0123}) \omega_0 \omega^1 \omega^2 \omega^3 (T X Y Z) \]

\[ -\nabla_1 (e_\mu T^\mu e_{1230}) \omega^1 \omega^2 \omega^3 \omega^0 (X Y Z T) \]

\[ -\nabla_2 (e_\mu T^\mu e_{2310}) \omega^2 \omega^3 \omega^0 \omega^1 (Y Z X T) \]

\[ -\nabla_3 (e_\mu T^\mu e_{3120}) \omega^3 \omega^0 \omega^1 \omega^2 (Z X Y T) \]

This expression for Q can be brought into a form which is independent of the chosen basis.

The attentive reader should note the order of the four spanning vectors in each of the terms.

Bringing these vectors into the same order yields (why?)

\[ Q = \nabla_0 (e_\mu T^\mu e_{0123}) \omega_0 \omega^1 \omega^2 \omega^3 (T X Y Z) \]

\[ + \nabla_1 (e_\mu T^\mu e_{1230}) \omega^1 \omega^2 \omega^3 \omega^0 (T X Y Z) \]

\[ + \nabla_2 (e_\mu T^\mu e_{2310}) \omega^2 \omega^3 \omega^0 \omega^1 (T X Y Z) \]

\[ + \nabla_3 (e_\mu T^\mu e_{3120}) \omega^3 \omega^0 \omega^1 \omega^2 (T X Y Z) \]
It follows that the expression for $Q$ on page 4 is

$$Q = \{ V_0 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{0\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta \}
+ V_1 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{1\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta
+ V_2 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{2\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta
+ V_3 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{3\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta (TXYZ)$$

The next to the last step consists of taking advantage of the restrictive sum lemma. It is obtained

$$Q = \{ V_0 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{0\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta \}
+ V_1 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{1\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta
+ V_2 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{2\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta
+ V_3 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{3\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta (TXYZ)$$

and using the restrictive sum lemma, one obtains

$$Q = \{ V_0 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{0\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta \}
+ V_1 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{1\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta
+ V_2 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{2\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta
+ V_3 (e_\mu T^{\alpha\nu} E_{\alpha\nu}^{3\beta\gamma\delta}) \omega^\alpha \omega^\beta \omega^\gamma \omega^\delta (TXYZ)$$

The ambitious reader should verify these.
Introducing a coordinate basis
\[ \omega^\alpha = dx^\alpha \]
so that
\[ \omega_j^\alpha = 0 \]
one finally express \( \mathcal{L} \) in term of an exterior derivative
\[ \mathcal{L} = d \left( e_\mu T^{\mu\nu} e_\nu \wedge dx^\alpha \wedge dx^\beta \right) (1x) \]
\[ \mathcal{L} = \int_{S^2} d(\ast T) \] by the MV theorem
where \( S^2 \) is the cubical region spanned by \( (TXYZ) \).
Summary
Starting with \( \mathcal{L} = \int_{S^2} \ast T \) one finds that
\[ \int_{S^2} \ast T = \int_{S^2} d(\ast T) \]
This is the 3-4 version of the vectorial Stokes' theorem. The frame independent geometrical formulation of moment energy conservation \( \mathcal{L} = 0 \) is therefore expressed by
\[ d(\ast T) = 0 \]
where
\[ \ast T = e_\mu T^{\mu\nu} e_\nu \wedge dx^\alpha \wedge dx^\beta \] is the moment energy density 3-form
The amount of momentum created in the 4-D region $S_2$ spanned by $T, X, Y, Z$ is therefore

$Q = d^4T (T, X, Y, Z)$

**Summary:**

The mathematical theorem we have proved is

$$\int_{S_2} e_{\mu} T^{\mu} e_{\nu}(x) e_{\lambda}(y) e_{\eta}(z) = \int_{S_2} d^4T e_{\nu}(x) e_{\lambda}(y) e_{\eta}(z)$$

or more briefly

$$\int_{S_2} d^4T = \int_{S_2} d^4T$$

Compare with the generalized Stokes' Theorem

For a scalar valued $p$-form $\omega$ it reads:

$$\int_{\partial R} d\omega = \int_{R} \omega$$