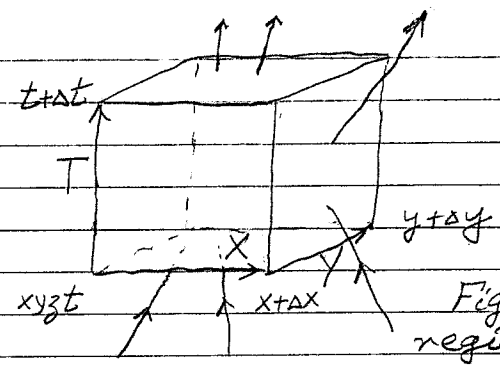
LECTURE 12

PART 1: Vectorial Form of Gauss's and Stokes's theorems: Application to momenergy conservation

PART 2: Fluid Kinematics: Expansion of a moving volume of fluid [MTW §22.1–22.3] via the scalar version of Gauss's theorem.

For Lecture 13

"PART 1": CONTINUED from LECTURE 11



$T = \Delta t \frac{\partial}{\partial t}$

$X = \Delta x \frac{\partial}{\partial x}$

$Y = \Delta y \frac{\partial}{\partial y}$

$Z = \Delta z \frac{\partial}{\partial z}$

Figure 2: Cubical region Ω and its boundary $\partial\Omega$

(6) Q = net momenergy emerging ~~from~~ Vol. $\Omega = \Delta x \Delta y \Delta z$

$$= \sum_{\left\{\begin{matrix}\text{8 oriented}\\ \text{faces of}\\ XYZT\end{matrix}\right\} = \partial\Omega} {}^*T(A,B,C) = \iiint_{\partial\Omega} {}^*\mathbb{T}$$

$$= \iiint_{\partial\Omega} e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$

$$= {}^*\mathbb{T}(XYZ)\big|_{t+\Delta t} - {}^*\mathbb{T}(XYZ)\big|_t$$

$$\leftrightarrow {}^*\mathbb{T}(YZT)_{x+\Delta x} - \overset{*}{\leftrightarrow}\mathbb{T}(\quad)_x + \text{cyclic } (xyz)$$

There are two mathematical of of expressing the final result: A) Gauss's theorem and B) Stokes' thm

(6) A) Gauss's thm (with $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$):

From page 1 and page 11 of Lecture 11 we have

$$Q = \nabla_0 (e_\mu T^{\mu 0}\sqrt{-g})[0123]\Delta t \Delta x \Delta y \Delta z$$
$$- \nabla_1 (e_\mu T^{\mu 1}\sqrt{-g})[1230]\Delta x \Delta y \Delta z \Delta t$$
$$- \nabla_2 (e_\mu T^{\mu 2}\sqrt{-g})[2310]\Delta y \Delta z \Delta x \Delta t$$
$$- \nabla_3 (e_\mu T^{\mu 3}\sqrt{-g})[3120]\Delta z \Delta x \Delta y \Delta t$$
$$= \nabla_\nu (e_\mu T^{\mu\nu}\sqrt{-g})\, \Delta t \Delta x \Delta y \Delta z$$

(mean value theorem for integrals)

$$= \iiiint_\Omega \nabla_\nu (e_\mu T^{\mu\nu}\sqrt{-g})\, d^4x$$

Thus we have proved Gauss's thm

$$Q = \boxed{\iiint_{\partial\Omega} {}^*\mathbb{T} = \iiiint_\Omega \nabla_\nu (e_\mu T^{\mu\nu}\sqrt{-g})\, d^4x}$$

~~with the consequence that~~

$$\boxed{Q = 0 \iff \nabla_\nu (e_\mu T^{\mu\nu}\sqrt{-g}) = 0},$$

which is a differential geometric formulation of the law of momenergy conservation

⑥ B) Stokes' Thm

From the top of page 2 we have with the help of the familiar totally antisymmetric Levi-Civita tensor whose components (see page 3, Lecture 8) are

$$\epsilon_{\nu\alpha\beta\gamma} = \sqrt{-g}\,[\nu\alpha\beta\gamma]; \quad \left.\begin{matrix}\nu\\ \alpha\\ \beta\\ \gamma\end{matrix}\right\} = 0,1,2,3,$$

$$\begin{aligned} Q = {} & \nabla_0\left(e_\mu T^{\mu 0}\epsilon_{0123}\right)\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3(TXYZ) \\ & -\nabla_1\left(e_\mu T^{\mu 1}\epsilon_{1230}\right)\omega^1\wedge\omega^2\wedge\omega^3\wedge\omega^0(XYZT) \\ & -\nabla_2\left(e_\mu T^{\mu 2}\epsilon_{2310}\right)\omega^2\wedge\omega^3\wedge\omega^1\wedge\omega^0(YZXT) \\ & -\nabla_3\left(e_\mu T^{\mu 3}\epsilon_{3120}\right)\omega^3\wedge\omega^1\wedge\omega^2\wedge\omega^0(ZXYT) \end{aligned}$$

The attentive reader should note the order of the four spanning vectors in each of the terms.

Bringing these vectors into the same order yields (Why?)

$$\begin{aligned} Q = {} & \nabla_0\left(e_\mu T^{\mu 0}\epsilon_{0123}\right)\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3(TXYZ) \\ & +\nabla_1\left(e_\mu T^{\mu 1}\epsilon_{1230}\right)\omega^1\wedge\omega^2\wedge\omega^3\wedge\omega^0(TXYZ) \\ & +\nabla_2\left(e_\mu T^{\mu 2}\epsilon_{2310}\right)\omega^2\wedge\omega^3\wedge\omega^1\wedge\omega^0(TXYZ) \\ & +\nabla_3\left(e_\mu T^{\mu 3}\epsilon_{3120}\right)\omega^3\wedge\omega^1\wedge\omega^2\wedge\omega^0(TXYZ) \end{aligned}$$

This expression for Q can be brought into a form which is <u>independent of the chosen basis</u>.

It follows that the expression for Q on page 4 is

$$Q = \{\nabla_0 (e_\mu T^{\mu 0} \epsilon_{0|\alpha\beta\gamma|})\, \omega^0 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_1 (e_\mu T^{\mu 1} \epsilon_{1|\alpha\beta\gamma|})\, \omega^1 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_2 (e_\mu T^{\mu 2} \epsilon_{2|\alpha\beta\gamma|})\, \omega^2 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_3 (e_\mu T^{\mu 3} \epsilon_{3|\alpha\beta\gamma|})\, \omega^3 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma\}(TXYZ)$$

The next to the last step consists of taking advantage of the restrictive sum lemma

$$\boxed{\begin{aligned}
\epsilon_{\nu|\alpha\beta\gamma|}\, \omega^0 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma &= \delta_\nu^0\, \epsilon_{0|\alpha\beta\gamma|}\, \omega^0 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \\
\epsilon_{\nu|\alpha\beta\gamma|}\, \omega^1 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma &= \delta_\nu^1\, \epsilon_{1|\alpha\beta\gamma|}\, \omega^1 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \\
\epsilon_{\nu|\alpha\beta\gamma|}\, \omega^2 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma &= \delta_\nu^2\, \epsilon_{2|\alpha\beta\gamma|}\, \omega^2 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \\
\epsilon_{\nu|\alpha\beta\gamma|}\, \omega^3 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma &= \delta_\nu^3\, \epsilon_{3|\alpha\beta\gamma|}\, \omega^3 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma
\end{aligned}}$$

The ambitious reader should verify these.

Rewriting the expression for Q on page 5 in the form

$$Q = \{\nabla_0 (e_\mu T^{\mu\nu} \delta_\nu^0 \epsilon_{0|\alpha\beta\gamma|})\, \omega^0 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_1 (e_\mu T^{\mu\nu} \delta_\nu^1 \epsilon_{1|\alpha\beta\gamma|})\, \omega^1 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_2 (e_\mu T^{\mu\nu} \delta_\nu^2 \epsilon_{2|\alpha\beta\gamma|})\, \omega^2 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_3 (e_\mu T^{\mu\nu} \delta_\nu^3 \epsilon_{3|\alpha\beta\gamma|})\, \omega^3 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma\}(TXYZ)$$

and using the restrictive sum lemma one obtains

$$Q = \nabla_0 (e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|})\, \omega^0 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_1 (e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|})\, \omega^1 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_2 (e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|})\, \omega^2 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$+ \nabla_3 (e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|})\, \omega^3 \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$
$$= \nabla_\sigma (e_\mu T^{\mu\nu}{}_{\nu|\alpha\beta\gamma|})\, \omega^\sigma \wedge \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma.$$

Introducing a coordinate basis

$$\omega^\alpha = dx^\alpha$$

so that

$$d\omega^\alpha = 0$$

one finally express Q in term of an exterior derivative

$$Q = \underset{\Omega}{d}\Big(\underbrace{e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma}_{*T}\Big)(TXYZ)$$

$$= \iiiint_\Omega d(*T) \quad \text{by the M.V. theorem}$$

where Ω is the cubical region spanned by $(TXYZ)$

Summary

Starting with $Q = \iiint_{\partial\Omega} *T$ one finds that

$$\boxed{\iiint_{\partial\Omega} *T = \iiiint_\Omega d(*T)}$$

This is the 3-4 version of the vectorial Stokes' Theorem. The frame independent geometrical formulation of momenergy conservation, $Q = 0$, is therefore expressed by

$$\boxed{d(*T) = 0}$$

where

$$\boxed{*T = e_\mu T^{\mu\nu} \epsilon_{\nu|\alpha\beta\gamma|}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma}$$

is the momenergy density 3-form

The amount of momenergy created in the 4-D region Ω spanned by T, X, Y and Z is therefore

$$Q = d^*\mathbb{T}\ (T, X, Y, Z)$$

SUMMARY:

The mathematical theorem we have proved is

$$\iiint_{\partial\Omega} e_\mu T^{\mu\nu} \epsilon_{\nu 1\alpha\beta\gamma} \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma = \iiiint_{\Omega} d\left(e_\mu T^{\mu\nu} \epsilon_{\nu 1\alpha\beta\gamma} \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma\right)$$

or more briefly

$$\boxed{\iiint_{\partial\Omega} {}^*\mathbb{T} = \iiiint_{\Omega} d\,{}^*\mathbb{T}}$$

Compare this with the generalized Stokes' Theorem. For a scalar valued p-form $\underset{\sim}{\omega}$ it reads,

$$\int_{\partial R} \underset{\sim}{\omega} = \int_{R} \underset{\sim}{d}\,\underset{\sim}{\omega}$$