LECTURE 21

Curvature induced rotation on a cube add to zero.

Bianchi Identities

Moment of Rotation for a 3-cube.

Its independence of the location of its fulcrum.

Einstein Field Eqs

\[
\text{(Sum of moments of rotation for the faces of a little 3-cube)} = 8\pi \text{ (amount of within that 3-cube)}
\]
1) The curvature induced rotation of a given element of area spanned by vectors $e_2 \Delta y$ and $e_3 \Delta z$ is

$$e_1 \wedge e_2 \wedge e_3 R^{i j k} \omega_{i k} \alpha \wedge R(e_2 \Delta y, e_3 \Delta z) = e_1 \wedge e_2 \wedge e_3 R^{i j k} y_3 \Delta y \Delta z = \left( \text{lin. comb' of 6 bivectors which express the 6 types of spacetime rotations} \right)$$

2) Given a point $P$ and considering the point $P_x$ centered on the area element $\Delta y \Delta z$, one has the lever arm $P - P_x$, an infinitesimal vector and hence one has the corresponding moment of rotation relative to the fulcrum point $P_x$.

3) Relative to the same reference point $P$, apply this construction to the six faces of the 3 opposing pairs of faces which bound a cube of volume $\Delta x \Delta y \Delta z$.
Moment of Rotation

The key ingredient for the Einstein field equation is the moment of rotation for a given 3-D region. Consider the amount of rotation associated with the $xy$ face of a cube spanned by three orthogonal vectors $u, v, w$ of extent $dx, dy, dz$.

\[
\begin{align*}
\text{(rotation associated with face)} &= e_\lambda \wedge e_\mu R^{\lambda\mu\nu}_{\gamma\delta} dy \wedge dz \\
&= e_\lambda \wedge e_\mu R^{\lambda\mu\nu}_{\gamma\delta} (e_\delta dy \wedge e_\gamma dz)
\end{align*}
\]

Consider an arbitrary point $P$ on or near the cube $dx \times dy \times dz$.

\[P = \text{fulcrum}\]

(a) Let $P_x^+$ be the point at the center of $dy \wedge dz$ which is located at $x+dx$. Then

Moment of rotation associated with front face $xyg^+$

\[
= (P_x^+ - P) \wedge e_\lambda \wedge e_\mu R^{\lambda\mu\nu}_{\gamma\delta} dy \wedge dz
\]

(b) Similarly, let $P_x^-$ be the point at the center of the back face of $xyg^-$ at $x$. The amount of moment of rotation associated with that face is the same except that one obtains

Moment of rotation associated with back face $xyg^-$

\[
= (P_x^- - P) \wedge e_\lambda \wedge e_\mu R^{\lambda\mu\nu}_{\gamma\delta} (dy \wedge dz)
\]

back face has opposite orientation.
4. Similar expression holds for the other pair of oppositely directed faces. Therefore:

\[ \text{Total moment of rotation for the faces of a little 3-cube} = \sum_{\text{faces}} \left( \frac{1}{2} \left( \sum x \right) \right) \mathrm{le} \times \mathrm{le} \times R_{xy} \Delta y \Delta z \]

express the six term sum of these moments of rotation in terms of the connecting lever arms:

\[
\begin{align*}
    P_x^+ - P_x^- &= \varepsilon_x \Delta x \\
    P_y^- - P_y^+ &= \varepsilon_y \Delta y \\
    P_z^+ - P_z^- &= \varepsilon_z \Delta z
\end{align*}
\]

The result is the net moment of rotation induced by curvatures on all the faces of the cube with volume \( \Delta x \Delta y \Delta z \):
Mmm. of Rotation is independent of lever point.

4) The moment of rotation is a geometrical property of each of the 6 individual faces, and it depends on the location of the chosen fulcrum point P. However, the sum of these six quantities forms a total which is independent of that choice. This is because of the "equilibrium condition" namely the fact that the sum of the rotations on all faces is zero,

\[ \sum_{i=1}^{6} e_i \cdot \hat{e}_i = 0 \]

(By virtue of the 2-3 version of Stokes' Theorem, this fact is expressed by the Bianchi identity. For a validation of the moment of rotation's fulcrum independence got to the Appendix.)

5) All together there are four 3-D cubes with volumes axayz, ayazx, axzay, and yzaax which are spanned by the corresponding triad of basis vectors \((e_0, e_2, e_3), (e_3, e_0, e_1), (e_1, e_3, e_2), \) and \((e_2, e_1, e_0)\).

By not specifying these triads one arrives at

\[ \begin{align*}
(\text{net amount of moment of rotation in an as-yet-unspecified 3-D cube}) &= e_0 \cdot e_2 \cdot e_3 \cdot \omega \cdot R \\
\text{equiv.} &= d\Phi \wedge R \\
&= \frac{\text{moment of rotation}}{\text{volume}}
\end{align*} \]
6) The isomorphism between tri-vectors and vectors.

The moment of rotation per unit volume is a tri-vector valued three-form, a tensor of rank \( \binom{3}{3} \), in 4-dimensional spacetime.

In 4-D spacetime there is a correspondence between tri-vectors and the vector perpendicular:

\[
\begin{align*}
\vec{e}_1 \times \vec{e}_2 \times \vec{e}_3 &= \vec{e}_0 \\
\text{(in general,)} &
\end{align*}
\]

\[
\begin{align*}
\vec{e}_1 \times \vec{e}_2 \times \vec{e}_3 &= \vec{e}_0 \\
\end{align*}
\]

The map \( \ast \) is an isomorphism.

One has therefore the vector valued 3-form:

\[
\ast \, d\mathcal{P} \, \mathcal{R} = \mathcal{E}_0 \, \mathcal{E}_{\alpha \beta \gamma} \, \mathcal{R} \times \mathcal{R} \times \mathcal{R}
\]

\[
\begin{align*}
&= \mathcal{E}_0 \mathcal{E}_{\alpha \beta \gamma} \left( \mathcal{R}^\alpha \mathcal{R}^\beta \mathcal{R}^\gamma \right) \\
&= \left( \text{vector-valued moment of rotation} \right) \times \left( \text{3-D volume} \right)
\end{align*}
\]

7) Einstein's Field Equation

On the other hand we recall that

\[
\frac{\text{(momentum)}}{\text{(3-D volume)}} = \frac{8\pi G}{c^2} \left( \mathcal{T}^{\alpha \beta} \right)
\]

The Einstein Field Equations are

\[
\ast \, d\mathcal{P} \, \mathcal{R} = \frac{8\pi G}{c^2} \left( \mathcal{T}^{\alpha \beta} \right)
\]

CARTAN-WHEELER statement:

\[
\frac{\text{(momentum)}}{\text{(3-D volume)}} = \frac{8\pi G}{c^2} \left( \mathcal{T}^{\alpha \beta} \right)
\]

In component form one has

\[
\mathcal{E}_{\alpha \beta \gamma} \mathcal{R}^{\alpha} \mathcal{R}^{\beta} \mathcal{R}^{\gamma} = \frac{8\pi G}{c^2} \mathcal{T}^{\alpha \beta}
\]

\[
\left( \mathcal{R}^{\alpha \beta} \right)^{\gamma \delta} = \frac{8\pi G}{c^2} \mathcal{T}^{\alpha \beta}
\]

"Contracted double dual of Riemann."
The restricted double sum on the L.H.S. of Eq. (4) introduces the factor $\frac{1}{2! 2!} = \frac{1}{4}$.

That left hand side becomes

$$\frac{1}{4} \left[ \delta^x_\lambda \delta^y_\mu \delta^z_\nu + \delta^y_\lambda \delta^z_\mu \delta^x_\nu + \delta^z_\lambda \delta^x_\mu \delta^y_\nu - \delta^x_\lambda \delta^y_\mu \delta^z_\nu - \delta^y_\lambda \delta^z_\mu \delta^x_\nu - \delta^z_\lambda \delta^x_\mu \delta^y_\nu \right] R^{\lambda \mu \nu \beta}$$

Introducing the components of the Ricci tensor and the curvature invariant

$$R^{\lambda \mu \nu \beta} = R^{\mu \beta \nu \lambda} \quad \text{and} \quad R^{\lambda \mu \nu \beta} = R^{\mu \beta \nu \lambda} = R$$

one finds

$$\left( R^{\mu \beta} - \frac{1}{2} \delta^{\beta \mu} R \right) = \frac{8\pi G}{c^4} T^{\mu \beta}$$

which are the Einstein field equations in their original 1916 form.
Note that, like the total moment of force on page 20,10, the total moment of rotation, \( d\mathbf{P} \mathbf{\wedge} \mathbf{R} \), is independent of the chosen lever point \( \mathbf{P} \). Let \( \mathbf{P}' \) be another point. One obtains

\[
\sum_{\text{6 faces}} (\mathbf{P}' - \mathbf{P}) \mathbf{e}_i \mathbf{e}_m \mathbf{R}^{i,m} y^z \Delta y^z = \sum_{\text{6 faces}} (\mathbf{P}' - \mathbf{P}) \mathbf{e}_i \mathbf{e}_m \mathbf{R}^{i,m} y^z \Delta y^z
\]

\[
= \sum_{\text{6 faces}} (\mathbf{P}' - \mathbf{P}) \mathbf{e}_i \mathbf{e}_m \mathbf{R}^{i,m} y^z \Delta y^z
\]

\[
= d\mathbf{P}' \mathbf{\wedge} \mathbf{R} + \text{zero.}
\]

\[
\sum_{\text{rotation}} = 0
\]

Thus \( d\mathbf{P} \mathbf{\wedge} \mathbf{R} = d\mathbf{P}' \mathbf{\wedge} \mathbf{R} \), i.e. the total amount of rotation for a 3-cube is independent of the chosen lever point.