LECTURE 21

Curvature induced rotation on a cube add to zero.

Bianchi Identities

Moment of Rotation for a 3-cube

Its independence of the location of its fulcrum.

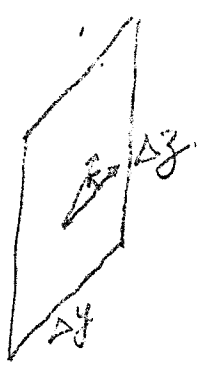
Einstein Field Eq'ns

$$\boxed{\begin{pmatrix}\text{Sum of moments of} \\ \text{rotation for the faces} \\ \text{of a little 3-cube}\end{pmatrix} = 8\pi \begin{pmatrix}\text{amount of} \\ \text{within} \\ \text{that 3-cube}\end{pmatrix}}$$

1.) The curvature induced rotation of a given element of area spanned by vectors $e_y \Delta y$ and $e_z \Delta z$ is

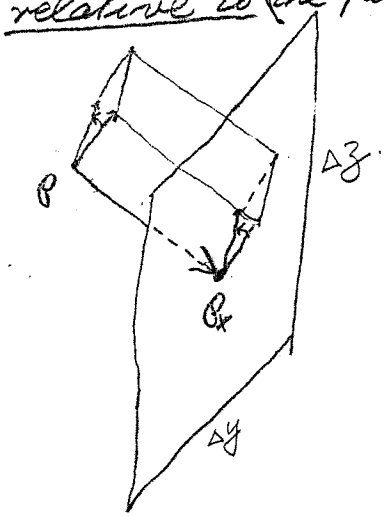
$$e_\lambda \wedge e_\mu \overbrace{R^{|\lambda\mu|}{}_{|\alpha\beta|}\, \omega^\alpha_\lambda \omega^\beta}^{\underline{\underline{R}}^{|\lambda\mu|}} (e_y \Delta y, e_z \Delta z) =$$

$$= e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz} \Delta y \Delta z = \left(\begin{array}{l}\text{lin. comb'n of 6} \\ \text{bivectors which express} \\ \text{the 6 types of} \\ \text{spacetime rotations}\end{array}\right)$$



at $x+\Delta x$

2.) Given a point P and considering the point P_x centered on the area element $\Delta y \Delta z$, one has the lever arm $P_x - P$, an infinitesimal vector, and hence one has the corresponding moment of rotation <u>relative to</u> (the fulcrum point) P,



$$(P - P_x) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz} \Delta y \Delta z$$

3.) Relative to the same reference point P, apply this construction to the six faces of the 3 opposing pairs of faces which bound a cube of volume $\Delta x \Delta y \Delta z$:

Moment of Rotation

The key ingredient for the Einstein field equations is the moment of rotation for a given 3-D region.

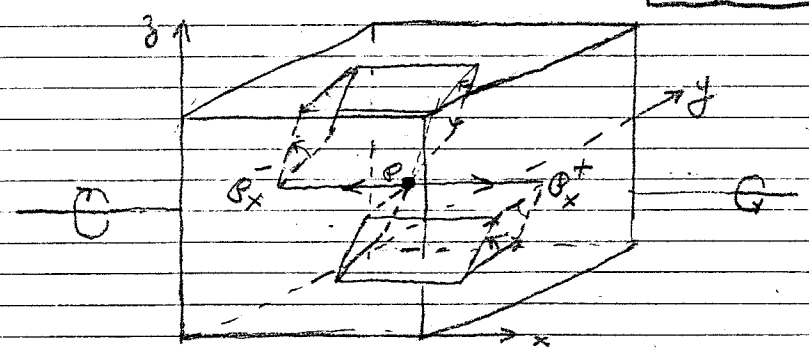
Consider the amount of rotation associated with the $\Delta y, \Delta z$ face of a cube spanned by three orthogonal vectors u, v, t of extent $\Delta x, \Delta y, \Delta z$:

$$\begin{pmatrix}\text{rotation}\\ \text{associated}\\ \text{with face}\\ \Delta y \Delta z\end{pmatrix} = e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz} \Delta y \Delta z$$

$$= e_\lambda \wedge e_\mu \underset{\sim}{R}^{|\lambda\mu|}(e_y \Delta y, e_z \Delta z)$$

Consider an arbitrary point P in or near the cube $\Delta x \Delta y \Delta z$.

$\boxed{P = \text{"Fulcrum"}}$



Moment of rotation associated with a pair of opposite faces.

(a) Let $P_x^{(+)}$ be the point at the center of $\Delta y \Delta z$ which is located at $x + \Delta x$. Then

$$\begin{pmatrix}\text{Moment}\\ \text{[relative to } P\text{]}\\ \text{of rotation}\\ \text{associated}\\ \text{with front}\\ \text{face } \Delta y \Delta z\end{pmatrix} = (P_x^+ - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz} \Delta y \Delta z \quad (*)$$

(b) Similarly, let P_x^- be the point at the center of the back face of $\Delta y \Delta z$ at x. The amount of moment of rotation associated with that face is the same except that one obtains

$$\begin{pmatrix}\text{Moment}\\ \text{[relative to } P\text{]}\\ \text{of rotation}\\ \text{associated}\\ \text{with back}\\ \text{face } \Delta y \Delta z\end{pmatrix} = (P_x^- - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz} (-) \Delta y \Delta z \quad (**)$$

back face has opposite orientation

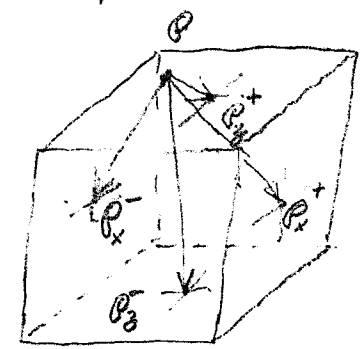
4.) Similar expression hold for the other pairs of oppositely directed faces.

Therefore,

$$\left(\begin{array}{l}\text{Total amount}\\ \text{of moment of}\\ \text{rotation for}\\ \text{the faces of}\\ \text{a little 3-cube}\end{array}\right) = \sum_{\substack{\text{cyclic}\\ xyz}} \left[\text{r.h.s of Eq}(*) + \text{r.h.s of Eq}(**)\right]$$

$$= \sum_{\substack{\text{cyclic}\\ xyz}} \left[(\mathcal{P}_x^+ - \mathcal{P}) - (\mathcal{P}_x^- - \mathcal{P})\right] \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz} \Delta y \Delta z.$$

Total Mom. of Rotation on a Cube



express the six term sum of these moments of rotation in terms of the connecting lever arms

$$\mathcal{P}_x^+ - \mathcal{P}_x^- = e_x \Delta x$$

$$\mathcal{P}_y^+ - \mathcal{P}_y^- = e_y \Delta y$$

$$\mathcal{P}_z^+ - \mathcal{P}_y^- = e_z \Delta z.$$

The result is the

$$\left(\begin{array}{l}\text{net moment of}\\ \text{rotation induced}\\ \text{by curvature}\\ \text{on all the faces}\\ \text{of the cube}\\ \text{with volume}\\ \Delta x \Delta y \Delta z\end{array}\right) = \underbrace{e_\nu \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}}_{\underset{\sim}{R}} \underbrace{\check{\omega}^\alpha \wedge \omega^\beta}_{d\mathcal{P}} (e_x, e_y, e_z) \Delta x \Delta y \Delta z$$

$$\equiv d\mathcal{P} \wedge \underset{\sim}{R}\,(e_x, e_y, e_z)\, \Delta x \Delta z$$

4.) The moment of rotation is a geometrical property of each of the 6 individual faces, and it depends on the location of the chosen fulcrum point P. However, the sum of these six quantities forms a total which is independent of that choice. This is because of the "equilibrium condition," namely the fact that the sum of the rotations on all faces is zero,

$$\sum_{i=1}^{6} e_\lambda \wedge e_\mu \underset{\sim}{R}^{|\lambda\mu|} = 0$$

(By virtue of the 2-3 version of Stokes' Thm this fact is expressed by the Bianchi identities. For a validation of the moment of rotation's fulcrum independence got to the Appendix to LECTURE 21)

5.) All together there are four 3-D cubes with volumes $\Delta y \Delta z \Delta t$, $\Delta z \Delta x \Delta t$, $\Delta x \Delta y \Delta t$, and $\Delta x \Delta y \Delta t$ which are spanned by the corresponding triad of basis vectors (e_0, e_2, e_3), (e_0, e_3, e_1), (e_0, e_1, e_2) and (e_1, e_2, e_3). By not specifying these triads one arrives at

$$\begin{pmatrix}\text{net amount of}\\ \text{mom. of rotation}\\ \text{in an}\\ \text{as-yet-unspecified}\\ \text{3-D cube}\end{pmatrix} = \overbrace{\underset{\sim}{e}_\nu \wedge \underbrace{e_\lambda \wedge e_\mu}_{\underset{\sim}{R}}}^{dP} \; \underset{\sim}{\omega} \wedge \underset{\sim}{R}^{|\lambda\mu|}$$

$$\equiv dP \wedge \underset{\sim}{R}$$

$$= \frac{\text{mom. of rotation}}{\text{volume}}$$

The Isomorphism between Tri-vectors and Vectors.

6.) The moment of rotation per unit volume is a <u>tri-vector-valued</u> three form, a tensor of rank $\binom{3}{3}$, in 4-dimensional spacetime.

In 4-D spacetime there is a correspondence between a tri-vector and the vector <u>perpendicular</u>:

$$e_1 \wedge e_2 \wedge e_3 \rightsquigarrow \star(e_1 \wedge e_2 \wedge e_3) = e_0$$

In general,

$$e_\nu \wedge e_\lambda \wedge e_\mu \rightsquigarrow \star(e_\nu \wedge e_\lambda \wedge e_\mu) = \epsilon_{\nu\lambda\mu}{}^{\sigma} e_\sigma$$

$\boxed{\text{The map } \star \text{ is a isomorphism}}$

One has therefore the <u>vector-valued</u> 3-form

$$\star\, d\mathcal{P} \wedge \underset{\sim}{R} = e_\sigma\, \epsilon_{\nu\lambda\mu}{}^{\sigma}\, \omega^\nu \wedge R^{|\lambda\mu|}$$

$$= e_\sigma\, \epsilon_{\nu\lambda\mu}{}^{\sigma} R^{|\lambda\mu|}{}_{|\alpha\beta|}\, \omega^\nu \wedge \omega^\alpha \wedge \omega^\beta$$

$$= \quad '' \quad '' \quad \epsilon^{\tau\nu\alpha\beta} \underbrace{\epsilon_{\tau|\alpha'\beta'\gamma'|}\, \omega^{\alpha'} \wedge \omega^{\beta'} \wedge \omega^{\gamma'}}_{\underset{\sim}{d}^3\Sigma_\tau}$$

$$= \frac{\left(\begin{array}{c}\text{vector-valued}\\ \text{moment of rotation}\end{array}\right)}{(\text{3-D volume})}$$

7.) Einstein's Field Equation

On the other hand we recall that

$$\frac{(\text{momenergy})}{(\text{3-D volume})} = \star\mathbb{T} = e_\sigma T^{\sigma\tau} \underbrace{\epsilon_{\tau|\alpha\beta\gamma|}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma}_{\underset{\sim}{d}^3\Sigma_\tau}$$

The Einstein Field Equations are

$$\boxed{\star\, d\mathcal{P} \wedge \underset{\sim}{R} = \frac{8\pi G}{c^2} \star \mathbb{T}}$$

Cartan-Wheeler statement.

$$\frac{\left(\begin{array}{c}\text{vector-valued}\\ \text{moment of rotation}\end{array}\right)}{(\text{3-D volume})} = \frac{8\pi G}{c^2} \frac{(\text{momenergy})}{(\text{3-D volume})}$$

$$\underbrace{e_\sigma \underbrace{\epsilon_{\nu\lambda\mu}{}^{\sigma} R^{|\lambda\mu|}{}_{|\alpha\beta|} \epsilon^{\tau\nu\alpha\beta}}_{G^{\sigma\tau}}}_{\star G = \text{"Einstein Tensor"}} \underset{\sim}{d}^3\Sigma_\tau = \frac{8\pi G}{c^2} \underbrace{e_\sigma T^{\sigma\tau} \underset{\sim}{d}^3\Sigma_\tau}_{\star\mathbb{T} = \text{"momenergy tensor"}}$$

In component form one has

$$\epsilon_{\nu\lambda\mu\sigma} R^{|\lambda\mu|}{}_{|\alpha\beta|} \epsilon^{\tau\nu\alpha\beta} = \frac{8\pi G}{c^2} T_\sigma{}^\tau \qquad (\star)$$

$$(\star R \star)_{\nu\sigma}{}^{\nu\tau} = \frac{8\pi G}{c^2} T_\sigma{}^\tau$$

"Contracted double dual of Riemann"

The restricted doublesum on the l.h.s. of Eq. (*) introduces the factor $\frac{1}{2!}\frac{1}{2!}=\frac{1}{4}$

That left hand side becomes

$$lhs = \frac{-1}{4}\Big[\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\tau}_{\sigma} + \delta^{\beta}_{\lambda}\delta^{\tau}_{\mu}\delta^{\alpha}_{\sigma} + \delta^{\tau}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\tau}_{\sigma} - \delta^{\alpha}_{\lambda}\delta^{\tau}_{\mu}\delta^{\beta}_{\sigma} - \delta^{\tau}_{\lambda}\delta^{\beta}_{\mu}\delta^{\alpha}_{\sigma}\Big] R^{\lambda\mu}{}_{\alpha\beta}$$

Introducing the components of the Ricci tensor and the curvature invariant

$$R^{\lambda\mu}{}_{\lambda\beta} \equiv R^{\mu}{}_{\beta} \quad \text{and} \quad R^{\lambda\mu}{}_{\lambda\mu} = R^{\mu}{}_{\mu} \equiv R,$$

one finds

$$\boxed{R^{\mu}{}_{\beta} - \frac{1}{2}\delta^{\mu}{}_{\beta} R = \frac{8\pi G}{c^4} T^{\mu}{}_{\beta}}$$

which are the Einstein field equations in their original 1916 form.

APPENDIX: $d\mathcal{P}\wedge\mathcal{R}$ is independent of the chosen lever point $\mathcal{P}$.

(continuing from p15,12) Similar expressions hold for the other pairs of oppositely directed faces. Therefore the total moment is

$$\begin{pmatrix}\text{Total amount of}\\ \text{moment [relative to } \mathcal{P}\text{]}\\ \text{of rotation from}\\ \text{all 3 pairs}\\ \text{of faces}\end{pmatrix} = \left\{ \begin{array}{l} \sum\limits_{\substack{\text{cyclic}\\ xyz \text{ on}\\ +\text{faces}}} (\mathcal{P}_x^+ - \mathcal{P})\wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz}\,\Delta y\Delta z \\ - \sum\limits_{\substack{\text{cyclic}\\ xyz \text{ on}\\ (-)\text{faces}}} (\mathcal{P}_x^- - \mathcal{P})\wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz}\,\Delta y\Delta z. \end{array}\right.$$

$$= \sum_{\substack{\text{cyclic}\\ xyz \text{ on all}\\ 6 \text{ oriented}\\ \text{faces}}} (\mathcal{P}_x^+ - \mathcal{P}_x^-)\wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz}\,\Delta y\Delta z$$

$$\equiv d\mathcal{P}\wedge\mathcal{R}$$

Note that, like the total moment of force on page 20,10, the total moment of rotation, $d\mathcal{P}\wedge\mathcal{R}$, is independent of the chosen lever point $\mathcal{P}$.

Let $\mathcal{P}'$ be another point. One obtains

$$\sum_{6\text{ faces}} (\mathcal{P}^x - \mathcal{P}' + \mathcal{P}' - \mathcal{P})\wedge \underbrace{e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz}\,\Delta y\Delta z}$$

$$= \sum_{6\text{ faces}} (\mathcal{P}^x - \mathcal{P}')\wedge \quad + (\mathcal{P}' - \mathcal{P})\wedge \underbrace{\sum_{6\text{ faces}} e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{yz}\,\Delta y\Delta z}_{\substack{d\mathcal{R} = 0 \\ \sum \text{rotations} = 0 \\ \text{"Bianchi identities"}}}$$

$$= d\mathcal{P}'\wedge\mathcal{R} + \text{zero}$$

$$= \begin{pmatrix}\text{Moment [relative}\\ \text{to } \mathcal{P}'\text{] of rotation}\\ \text{from all 3 pairs}\\ \text{of faces}\end{pmatrix}$$

Thus $d\mathcal{P}\wedge\mathcal{R} = d\mathcal{P}'\wedge\mathcal{R}$, i.e the total amount of rotation for a 3-cube is independent of the chosen lever point