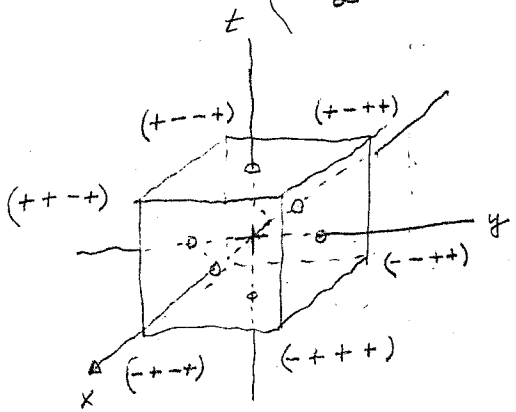
Lecture 24 Appendix

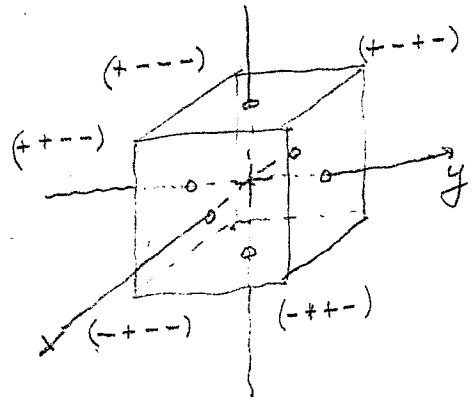
1. The eight bounding cubes of a 4-D domain
2. The trivector-valued moment of rotation on their 48 faces.

Consider a 4-D hypercube Ω having volume $\Delta t \Delta x \Delta y \Delta z$, centered at the origin of the coordinate system and having its 16 vertices located at

$$\left(\pm\frac{\Delta t}{2}, \pm\frac{\Delta x}{2}, \pm\frac{\Delta y}{2}, \pm\frac{\Delta z}{2}\right)_t$$



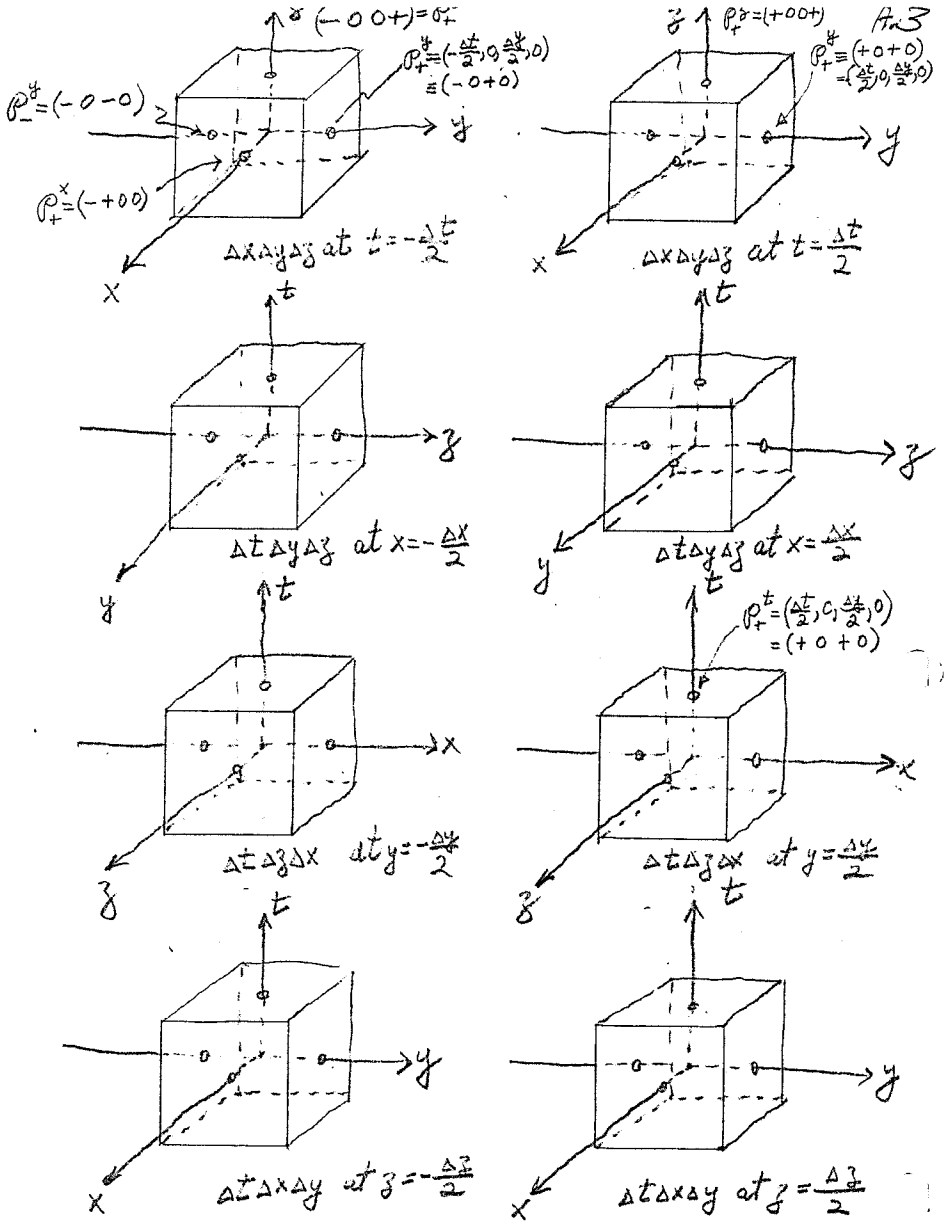
at $z = +\frac{\Delta z}{2}$
The vertices are
$(\pm, \pm, \pm, +)$



at $z = -\frac{\Delta z}{2}$
The vertices are
$(\pm, \pm, \pm, -)$

The hypercube Ω has 4 pairs of 3-D cubes, which form the boundary, $\partial\Omega$, of Ω!
The vertices of these eight 3-D cubes are located at
$(+\pm\pm\pm)$, $(\pm+\pm\pm)$, $(\pm\pm+\pm)$, $(\pm\pm\pm+)$
$(-\pm\pm\pm)$, $(\pm-\pm\pm)$, $(\pm\pm-\pm)$, and $(\pm\pm\pm+)$

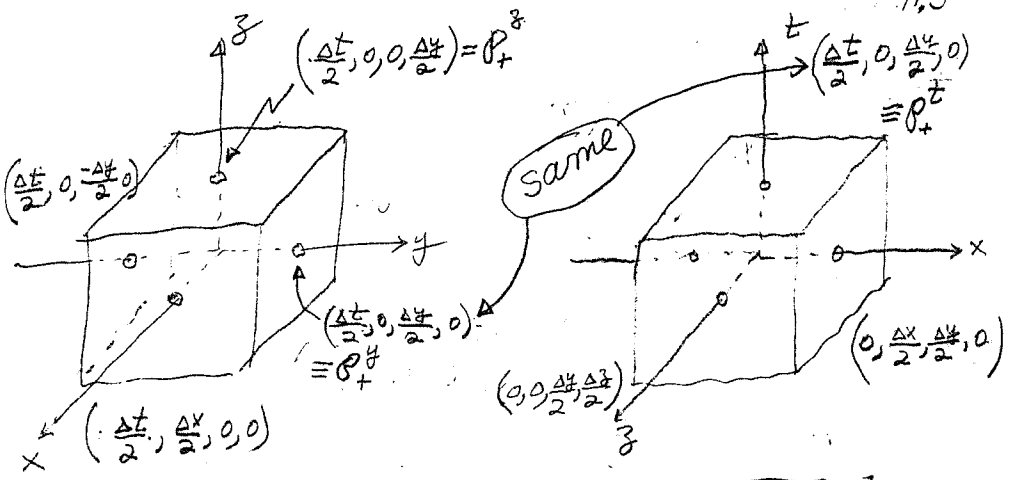
Each 3-D cube has 6 faces. They form the boundary, $\partial\partial\Omega$, of $\partial\Omega$. There are 8×6 such faces. They are depicted on the next page.
Each of them has exactly one point at its center. However, these points are not distinct. In fact, for the space time coordinates (t, x, y, z) of a center on one face there exists the center of another face with the same coordinates.



For example, the point P_+^y at $t=\frac{\Delta t}{2}$ (first row, second column on page A.3) has the same coordinates, $(\frac{\Delta t}{2},0,\frac{\Delta y}{2}0)$, as the point P_+^t at $y=\frac{\Delta y}{2}$ (third row, second column). These points are therefore one and the same

$$\left\{P_+^y \text{ at } t=\frac{\Delta t}{2}\right\}=\left\{P_+^t \text{ at } y=\frac{\Delta y}{2}\right\}=\left(\frac{\Delta t}{2},0,\frac{\Delta y}{2},0\right)$$

Consequently, the corresponding faces are, (except for their opposite orientations) one and the same face, the 2-D domain where two different 3-D cubes meet in the 4-D spacetime.



cube $\Delta x \Delta y \Delta z$ at $t = \frac{\Delta t}{2}$

cube $\Delta t \Delta x \Delta z$ at $y = \frac{\Delta y}{2}$

Consequently, contributions to the trivector valued moment of rotation such as

$$(P_+^y - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{xz}\, \Delta x \Delta z \quad \text{at } t = \frac{\Delta t}{2}$$

and

$$(P_+^t - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{xz}\, (-) \Delta x \Delta z \quad \text{at } y = \frac{\Delta y}{2}$$

cancel because $\Delta x \Delta y \Delta z$ and $\Delta t \Delta x \Delta z$ have

$$P_+^y = \left(\frac{\Delta t}{2}, 0, \frac{\Delta y}{2}, 0\right) = P_+^t,$$

in common, and also share the same (but oppositely oriented) area $\Delta x \Delta z$.

It follows that the corresponding vector-valued moment of rotation

$$\star (P_+^y - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{xz}\, \Delta x \Delta z$$

and

$$\star (P_+^t - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{xz}\, (-) \Delta x \Delta z$$

also cancel.

Applying this reasoning to the remaining 23 pairs of faces one concludes that

$$0 = \{ \star (P_+^y - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{xz} \Delta x \Delta z + \star (P_+^t - P) \wedge e_\lambda \wedge e_\mu R^{|\lambda\mu|}{}_{xz} (-) \Delta x \Delta z$$

+ 46 additional terms which cancel pairwise $\}$

Note that each pair-member does depend on P.

Upon rearranging these 48 surface integrals into 8 groups, with 6 terms for the 6 bounding faces of each of the

eight cubes on page A.3, one obtain

$0 = \frac{1}{8}$ sum with 48 ρ-dependent, but pairwise canceling terms?

$$= \star \sum_{\substack{\text{cyclic} \\ \text{permutation} \\ \text{of } xyz}} \underbrace{\{(\rho_1^x - \rho) - (\rho^x - \rho)\} \wedge e_\lambda \wedge e_\mu \underset{\sim}{R}{}^{\lambda\mu}(e_y, e_z) \Delta y \Delta z \Big|_{t = \frac{\Delta t}{2}}}_{\text{indep. of } \rho}$$

$$\underbrace{- \text{same} \Big|_{t = -\frac{\Delta t}{2}}}_{\text{indep of } \rho}$$

$$\underbrace{+\sum_{t_{yz}} \Big|_{\frac{\Delta x}{2}} - \sum_{t_{yz}} \Big|_{-\frac{\Delta x}{2}} + \sum_{t_{zx}} \Big|_{\frac{\Delta y}{2}} - \sum_{t_{zx}} \Big|_{-\frac{\Delta y}{2}} + \sum_{t_{xy}} \Big|_{\frac{\Delta z}{2}} - \sum_{t_{xy}} \Big|_{-\frac{\Delta z}{2}}}_{\text{each indep of } \rho,}$$