LECTURE 26

1.) Momenergy Conservation without and with gravitation:

2.) Geometrical and matter degrees of freedom.

I. Reminder:

1.) The Einstein field equations on M^4 are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi t_{\mu\nu}$$

For spherically symmetric systems and their processes they reduce to a tensor and a scalar equation on M^2

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} = 0, 1$$

$$G_{AB} = \frac{1}{r^2}\left\{-2r\, r_{;AB} + g_{AB}\left(2r\, r_{;c}{}^{;c} + r_{;c}\, r_{;D}\, g^{cD} - 1\right)\right\} = 8\pi t_{AB}$$

$$G_\theta{}^\theta = G_\varphi{}^\varphi = \frac{1}{2} G_a{}^a \equiv \frac{r_{;c}{}^{;c}}{r} - R = 8\pi t_\theta{}^\theta$$
$$= 8\pi t_\varphi{}^\varphi$$
$$= 8\pi \frac{1}{2} t_a{}^a$$

2.) The relation implied by the E. field eq'ns, namely,

$$\partial\partial v = 0 \Leftrightarrow \text{conservation of momenergy}$$

$$G_\mu{}^\nu{}_{;\nu} = 0 \Leftrightarrow t_\mu{}^\nu{}_{;\nu} = 0$$

on M^4;

for spherically symmetric systems, becomes a single vector eq'n on M^2

$$\left(r^2 G_A{}^B\right)_{;B} - \left(r^2\right)_{;A} \frac{G_a{}^a}{2} = 0 \Leftrightarrow \left(r^2 t_A{}^B\right)_{;B} - \left(r^2\right)_{;A} \frac{t_a{}^a}{2} = 0$$

$$A = 0, 1$$

3.) The Law of Momenergy Conservation applies everywhere at all times, from the smallest nook on earth to most distant corner of the (cosmos) universe. It applies (to) gravitating systems, electromagnetic systems, matter systems. It is mathematized by means of

$$\boxed{t_\mu{}^\nu{}_{;\nu} = 0}$$

Here, for example,

a) for a (perfect) fluid on has

$$t^{\mu\nu}_{matter} = (p+\rho)u^\mu u^\nu + p g^{\mu\nu};$$

b) for electromagnetism

$$t^{\mu\nu}_{e.m} = \frac{1}{4\pi}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{2}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

where $F^{\mu\nu}$ is related to the charge density-flux four vector

$$J^\mu = qNu^\mu$$

by

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu \quad ;$$

c) for Klein-Gordon mesons of rest mass m,

$$t^{K.G.}_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} + m^2\phi^2\right)$$

where the meson field ϕ obeys

$$\phi_{,\alpha}{}^{;\alpha} + \left(\frac{mc}{\hbar}\right)^2\phi = 0$$

The universality of the law of momenergy conservation is mathematized by the statement

$$\underbrace{\left(t^{\mu\nu}_{matter} + t^{\mu\nu}_{e.m.} + t^{\mu\nu}_{K.G} + \cdots\right)}_{t^{\mu\nu}}{}_{;\nu} = 0$$

Comment:

If matter is ionized (or charged) then $q \neq 0$, and then, by virtue of

$$\boxed{F^{\mu\nu}{}_{;\nu} = 4\pi qNu^\mu}, \quad \text{("Maxwell's Equations")}$$

one has

$$\boxed{t^{\mu\nu}_{matter;\nu} = -t^{\mu\nu}_{e.m;\nu}} \quad \text{("Relativistic Hydro eq'ns for a charged fluid")}$$

II. Geometrical vs matter degrees of freedom.

The Einstein field equations are a set of quasi-linear second order partial differential equation for the spacetime metric. "Quasi-linear" means non-linear, in the first and zeroeth derivative, but linear in the second derivative.

The objective is to determine from the field equations not only the spacetime dependence of the metric coefficients

$$g_{\mu\nu}(x^\alpha),$$

which express the geometrical degrees of freedom, but also that of the matter degrees of freedom which go into the construction of the momenergy tensor T. The matter degrees of freedom are typically expressed by the velocity, density, pressure and other attributes that go into the

construction of the momenergy tensor of the medium. For example, its components for a perfect fluid are SKIP this ↓ ALREADY DONE

$$t^{\mu\nu}_{matter} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu},$$

its components for a the electromagnetic field $F_{\mu\nu}$ are

$$t^{\mu\nu}_{e.m.} = \frac{1}{4\pi}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}\right]$$

while its components for the real scalar Klein-Gordon field ψ, which satisfies $\psi_{,\mu}{}^{,\mu} - m^2\psi = 0$ are

$$t_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}(\phi^{,\alpha}\phi_{,\alpha} + m^2\phi^2)$$

In the presence of gravitation
The geometrical and the matter degrees are coupled

to one another by the Einstein field equation

geometry responds to matter $\quad G^{\mu\nu}[g_{\alpha\beta}] = 8\pi\, t^{\mu\nu}[\rho, p, u^{\mu}, g_{\alpha\beta}, F_{\mu\nu}, \phi, \text{etc.}]$, $\quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$

These equations can be viewed as governing the <u>response</u> of the <u>geometrical degrees of freedom</u> <u>to the motion of matter</u>.

By contrast the conservation of momenergy equations, the Euler hydrodynamical equations,

matter responds to geometry $\quad t^{\mu\nu}{}_{;\nu} = 0, \rightarrow \begin{cases} \text{Hydro eq'n} & \text{for fluid} \\ \text{wave eq'n} & \text{for e.m. field} \\ \text{" "} & \text{for K-G field} \end{cases}$

can be viewed as governing the <u>response</u> of the <u>matter degrees of freedom</u> <u>to</u>

<u>the curving of geometry</u>.

A solution to the Einstein field equations consists of specifying the spacetime dependence of all degree of freedom, geometrical and matter.

Let us bring into sharper focus the geometrical degrees of freedom as they are determined by the Einstein field equations. Their adaptation to spacetime with spatial spherical symmetry illustrates the basic idea. Upon being solved, the equations yield the following geometrical degrees of freedom

1+3=4 degrees of freedom: $\quad r(x^0, x^1)$ and $g_{AB}(x^0, x^1)$. $\quad$ (1+3=4 as-yet-unknowns)

These are 1+3=4 functions of space and time (x^1 and x^0). However note that the arbitrariness in the choice of a

coordinate system

$$x^0 = x^0(\bar{x}^0, \bar{x}^1)$$
$$x^1 = x^1(\bar{x}^0, \bar{x}^1)$$

reduces the number of degrees of freedom by 2. For example, starting with

$$g_{00}(dx^0)^2 + 2g_{01}\,dx^0 dx^1 + g_{11}(dx^1)^2 =$$

$$= g_{00}\left(dx^0 + \frac{g_{01}}{g_{00}}dx^1\right)^2 + \left(g_{11} - g_{00}\left(\frac{g_{01}}{g_{00}}\right)^2\right)(dx^1)^2$$

$$= \frac{g_{00}}{\lambda^2}\left[\lambda\left(dx^0 + \frac{g_{01}}{g_{00}}dx^1\right)\right]^2 + \left(\frac{g_{11}g_{00} - g_{01}g_{01}}{g_{00}}\right)(dx^1)^2$$

and by choosing the integration factor λ so as to make [ ] an exact differential, one obtains

$$\lambda\left(dx^0 + \frac{g_{01}}{g_{00}}dx^1\right) = d\bar{x}^0 \Rightarrow \bar{x}^0 = \int^{(x^0, x^1)} \lambda\, dx^0 + \frac{\lambda g_{01}}{g_{00}}dx^1 \quad \left(\text{"path indep. line integral"}\right)$$

i.e. $\boxed{\bar{x}^0 = \bar{x}^0(x^0, x^1)}$. Consequently,

$$g_{AB}\,dx^A dx^B = \bar{g}_{00}(d\bar{x}^0)^2 + \bar{g}_{11}(d\bar{x}^1)^2$$

so that

$$\bar{g}_{01} = 0$$

Thus one has used the coordinate function $\boxed{\bar{x}^0 = \bar{x}^0(x^0, x^1)}$ bring the metric into diagonal form.

Using a similar argument one introduces $r(x^0, x^1) = \bar{x}^1 \equiv r$ as the spatial argument.

Thus

$$g_{\mu\nu}\,dx^\mu x^\nu = g_{AB}\,dx^A dx^B + r^2(x^0)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

$$= \underbrace{-e^{2\Phi(r,t)}}_{\bar{g}_{00}(\bar{x}^0, \bar{x}^1)}dt^2 + \underbrace{e^{2\lambda(r,t)}}_{\bar{g}_{11}(\bar{x}^0, \bar{x}^1)}dr^2 + r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

Thus an appropriate choice of coordinates has reduced the gravitational (degrees of freedom $4 \to 2$)

degrees of freedom from four to two:

4-2 = 2 geometrical degrees of freedom

$$\left.\begin{aligned} \bar{g}_{00} &= -e^{2\Phi(r,t)} \\ \bar{g}_{11} &= e^{2\Lambda(r,t)} \end{aligned}\right\} \text{two independent metric degrees of freedom.}$$

These are indeed two degrees of freedom which express the geometrical properties of spacetime, and not the arbitrary manner in which to label its events. This is as it should be because the 4 Einstein field equations are not independent. There exist the two identities

2 identities $\left(r^2 G_A{}^B\right)_{|B} - r^2 v_A G_0{}^0 = 0 \qquad A = 0, 1$

between them.

In summary, one has

\# of field equations = 4 (equations)

\# of identities = 2 (identities)

\# of independent equations = 4 − 2 = 2 (indep. eq'ns)

\# of to-be-determined functions = 4 (fns)

\# of coordinate conditions = 2 (arbitrary choices for coord)

\# of gravitational degrees of freedom = 4 − 2 = 2 (geom. fns)

Thus we have

4 − 2 = 4 − 2