LECTURE 26

1) Momentum Conservation without and with gravitation.

2) Geometrical and matter degrees of freedom.
For spherically symmetric systems, the Einstein field equations on $M^2$


g_{\mu\nu} R_{\mu\nu} - \frac{1}{2} g_{\lambda\sigma} R = 8\pi G T_{\mu\nu}

becomes a single vector eqn on $M^2$

\( \gamma_{\mu\nu} T_{\mu\nu} = 0 \Leftrightarrow \left( \gamma^2 G_{\mu\nu} \right) - \left( \gamma^4 \right) T_{\mu\nu} \frac{\gamma}{\gamma^2} = 0 \)

\( \gamma = 0.1 \)

For spherically symmetric systems, they reduce to a tensor and a scalar equation on $M^2$

2) The relation implied by the E. field eqns, namely

\( \Box \gamma = 0 \Rightarrow \text{conservation of} \)

\( G_{\mu\nu} T_{\mu\nu} = 0 \Rightarrow T_{\mu\nu} \gamma_{\mu\nu} = 0 \)

3) The law of momentum conservation applies everywhere at all times, from the smallest rock on earth to most distant corner of the universe.

It applies to gravitating systems, electromagnetic systems, matter systems.

It is mathematized by means of

\( T_{\mu\nu} \gamma_{\mu\nu} = 0 \)
Here, for example,

g for a (perfect) fluid one has
\[ t_{\mu\nu} = (p + \rho) u_{\mu} u_{\nu} + \rho g_{\mu\nu} \]

b) for electromagnetism,
\[ t_{\mu\nu} = \frac{1}{4 \pi} F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \]
where \( F_{\mu\nu} \) is related to the charge density-flux four vector
\[ J^\mu = q N u^\mu \]
by
\[ F_{\mu\nu} = 4 \pi J_{\nu} \]

c) for Klein-Gordon mesons of rest

mass \( m \),
\[ t_{\mu\nu} = \phi_{\mu} \phi_{\nu} - \frac{1}{2} g_{\mu\nu} (\phi^2 + m^2 \phi^2) \]
where the meson field \( \phi \) obeys
\[ \phi_{\mu \nu} \phi^\mu + \left( \frac{m^2}{2} \right) \phi = 0 \]

The universality of the law of momentum conservation is mathematized by the statement
\[ (t_{\mu\nu} + t_{e.m.\nu} + t_{m.e.\nu} + \ldots)_{\nu} = 0 \]

Comment:
If matter is ionized (charged) then
\[ q \phi \]
and then by virtue of Maxwell's Equations
\[ F_{\mu\nu} = 4 \pi q N u_{\nu} \] (relativistic Hydro. eq. for a charged fluid)
II. Geometrical vs matter degrees of freedom.

The Einstein field equations are a set of quasi-linear second order partial differential equations for the spacetime metric. "Quasi-linear" means non-linear in the first and second derivatives, but linear in the second derivatives.

The objective is to determine from the field equations only the spacetime dependence of the metric coefficients \( g_{\mu \nu}(x^a) \), which express the geometrical degree of freedom, but also that of the matter degrees of freedom which go into the construction of the momenergy tensor \( T \). The matter degrees of freedom are typically expressed by the velocity, density, pressure and other attributes that go into the...

\[ T_{\mu \nu} = (\rho + p) u^\mu u^\nu + pg^\mu \nu \]

For a perfect fluid, its components for the electromagnetic field \( F_{\mu \nu} \) are

\[ T_{\mu \nu} = \frac{1}{4\pi} [F_{\mu \alpha} F^\alpha_\nu - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta}] \]

while its components for the real scalar Klein-Gordon field \( \psi \), which satisfies \( \psi_{\mu \nu} \rightleftharpoons m^2 \psi \), are

\[ T_{\mu \nu} = F_{\mu \nu} - \frac{1}{2} g_{\mu \nu} (\phi^2 + m^2 \phi) \]
In the presence of gravitation, the curving of geometry.

The geometrical and the matter degrees of freedom are coupled.

A solution to the Einstein field equations consists of specifying the spacetime dependence of all degree of freedom, geometrical and matter.

Let us bring into sharper focus the geometrical degrees of freedom as they are determined by the Einstein field equations. Their adaptation in spacetime with isotropic spherical symmetry illustrates the basic idea. Upon being solved, the equations yield the following geometrical degrees of freedom:

By contrast, the conservation of momenergy equations the Euler hydrodynamical equations, the wave equation for fluid and electromagnetic field.

These are $1+3=4$ functions of space and time ($x^0 \text{ and } x^0$). However note that the arbitrariness in the choice of a
coordinate system

\[ x'' = x^0(x^0, x') \]
\[ x' = x'(x^0, x') \]

reduces the number of degrees of freedom by 2. For example, starting with

\[ g_{00}(dx^0)^2 + 2g_{00}dx^0dx^1 + g_{11}(dx^1)^2 = \]
\[ = g_{00}(dx^0 + \frac{\partial \ln \lambda}{\partial x^0})^2 + \left( \frac{\partial \ln \lambda}{\partial x^0} \right)^2(dx^1)^2 \]
\[ = \frac{\partial \ln \lambda}{\partial x^0} \left[ \lambda (dx^0 + \frac{\partial \ln \lambda}{\partial x^0} dx^1) \right]^2 + \left( \frac{\partial \ln \lambda}{\partial x^0} \right)^2(dx^1)^2 \]

and by choosing the integration factor \( \lambda \) so as to make \( \lambda dx^0 \) an exact differential, one obtains

\[ \lambda (dx^0 + \frac{\partial \ln \lambda}{\partial x^0} dx^1) = dx^0 + \int \frac{\partial \ln \lambda}{\partial x^0} dx^1 \]

Consequently

\[ \int g_{00} dx^0 dx^1 = g_{00} (dx^0)^2 + \frac{\partial \ln \lambda}{\partial x^0} (dx^1)^2 \]

so that

\[ \frac{\partial \ln \lambda}{\partial x^0} = 0 \]

Thus one has used the coordinate function \( x^0 = x^0(x^0, x') \) to bring the metric into diagonal form.

Using a similar argument one introduces \( \tau(x^0, x') = \overline{x}^1 = \tau \) as the spatial argument.

Thus

\[ g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = g_{\mu\nu} dx^\mu dx^\nu + \tau^2(x^0) \left[ dt^2 + \frac{\partial \tau}{\partial x^0} \right] \]

\[ = \frac{\partial \tau}{\partial x^0} dx^0 dx^0 + \frac{\partial \tau}{\partial x^0} dx^0 dx^0 + \tau^2(x^0) \left[ dt^2 + \frac{\partial \tau}{\partial x^0} \right] \]

Thus an appropriate choice of coordinates has reduced the gravitational (degrees of freedom 4 \( \rightarrow 2 \))
degrees of freedom from four to

\[ \tilde{g}_{00} = -\frac{2}{3}(5t) \]
\[ \tilde{g}_{11} = e^{2\Lambda(5t)} \]

\# of to-be-determined functions = 4 (fns)
\# of coordinate conditions = 2 (chosen coord)
\# of gravitational degrees of freedom = 4 - 2 = 2 (geom fns)

Thus we have

These are indeed two degrees of freedom which express the geometrical properties of spacetime, and not the arbitrary manner in which to label its events.

This is as it should be because the 4 Einstein field equations are not independent. There exist the two identities

\[ 2 \text{identities} (\nabla^2 G_{\mu\nu} = 0, \nabla^2 G = 0) \quad A = 0, 1 \]

between them.

In summary, one has

\# of field equations = 4 (equations)
\# of identities = 2 (identities)
\# of independent equations = 4 - 2 = 2 (indep eqns)