Lecture 27

Helmholtz's theorem

Helmholtz's Theorem

Vector fields in two dimensions a particularly simple property which makes it useful in unravelling the properties of spherically symmetric space times. The property is this (although technically incorrect, we shall refer to a tensor field by its components):

For any covector field J_A on M^2 there exist two scalars ψ and ϕ such that

$$\boxed{J_A = \psi_{,A} + \phi_{,C}\,\epsilon^C{}_A}$$

$$J_A dx^A = \psi_{,A} dx^A + \phi_{,C}\,\epsilon^C{}_A dx^A$$

$$\underset{\sim}{J} = d\psi + *d*\phi\, dx^A dx^B$$

$$d\psi + *d\phi$$

or figuratively

"covector" = "gradient" + "curl"

The two scalars are solutions to the 2-D inhomogeneous wave equations in M^2:

$$\psi_{,A}{}^{|A} = J_A{}^{|A}$$

$$\phi_{,C}{}^{|C} = J_{A|C}\,\epsilon^{AC} \left(= (J_{0,1} - J_{1,0}) \frac{-1}{|g|^{1/2}} \right)$$

where

$$\epsilon_{AC} = |g|^{1/2}\,[AC] = \begin{bmatrix} \epsilon_{00} & \epsilon_{01} \\ \epsilon_{10} & \epsilon_{00} \end{bmatrix} = |g|^{1/2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\epsilon^{AC} = g^{AB} g^{CD} \epsilon_{BC} = g^{AB} g^{CD} (-g)^{1/2} [BC] = \frac{1}{g}[AC](-g)^{1/2} = -(-g)^{-1/2}[AC]$$

Minus sign because g is negative for Lorentzian spacetime.

is the Levi Civita tensor field the Lorentzian M^2.

This equation is the 2-d version of Helmholtz's theorem in 3-d according to which a 3-d vector field is the sum of the gradient of a scalar field plus the curl of a vector field,

$$\vec{F} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}.$$

The 2-d Helmholtz theorem is based on the following Theorem ("zero divergence ppty")

Given: A vector field J^A

Conclusion:

$$\boxed{\exists \text{ a function } \phi \text{ such that } J_A = \phi_{,C}\,\epsilon^C{}_A} \iff \boxed{J^A{}_{,A} = 0}$$

Here $\epsilon^C{}_A = g^{CB}\epsilon_{BA}$; $[\epsilon_{BA}] = |g|^{1/2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\epsilon^{CD} = g^{CB} g^{DA} |g|^{1/2} [BA] = [CD] \frac{1}{g} (-g)^{1/2}$$

$$= [CD] \frac{-1}{|g|} |g|^{1/2} = \frac{-1}{|g|^{1/2}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

proof:

$$\Rightarrow \quad J^A{}_{|A} = \left(\Phi_{,c}\,\epsilon^{cA}\right)_{|A}$$

$$= \Phi_{,c|A}\,\epsilon^{cA} + \phi_{,c}\,\underbrace{\epsilon^{cA}{}_{|A}}_{\text{zero}}$$

$= \sum$ sym × antisym

$= 0$ (3,11 a in MTW)

$\Leftarrow$ 1. Note that

$$0 = J^A{}_{|A} = \frac{1}{\sqrt{-g}}\left[\frac{\partial\left(J^0\sqrt{-g}\right)}{\partial x^0} + \frac{\partial\left(J^1\sqrt{-g}\right)}{\partial x^1}\right]$$

2. Consequently, we consider

$$\phi(x^0, x^1) = \int^{x^1} J^0\sqrt{-g}\,dx^1$$

$$\frac{\partial\phi}{\partial x^0} = \int^{x^1} \frac{\partial\left(J^0\sqrt{-g}\right)}{\partial x^0}\,dx^1$$

$$= -\int^{x^1} \frac{\partial\left(J^1\sqrt{-g}\right)}{\partial x^1}\,dx^1$$

$$= -\sqrt{-g}\,J^1(x^0, x^1)$$

3. $J^0 = \frac{1}{\sqrt{-g}}\frac{\partial\phi}{\partial x^1}$, $J^1 = -\frac{1}{\sqrt{-g}}\frac{\partial\phi}{\partial x^0}$

Using $\left[\epsilon^{CB}\right] = \begin{bmatrix} \epsilon^{00} & \epsilon^{01} \\ \epsilon^{10} & \epsilon^{11} \end{bmatrix} = \frac{-1}{\sqrt{-g}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ one obtains

$$J^0 = \frac{\partial\phi}{\partial x^c}\,\epsilon^{c0}, \quad J^1 = \frac{\partial\phi}{\partial x^c}\,\epsilon^{c1} \Rightarrow \boxed{J_A = \phi_{,c}\,\epsilon^c{}_A}$$

Q.E.D.

Theorem ("Zero curl property")

Given: A covector field J_A

Conclusion

$$\boxed{\exists \text{ a function } \psi \text{ such that } J_A = \psi_{,A}} \iff \boxed{0 = J_{A|B} - J_{B|A}} \left(= -J_{C|D}\,\epsilon^{CD}\epsilon_{AB}\right)$$

proof

$$\Rightarrow \quad J_{A|B} - J_{B|A} = J_{A,B} - J_{B,A} \quad (\text{"the } \Gamma\text{'s cancel"})$$

$$= \psi_{,A,B} - \psi_{,B,A} = 0$$

$\Leftarrow$ 1.) Note that

$$0 = J_{A|B} - J_{B|A} = J_{A,B} - J_{B,A}$$

2.) Consequently, we consider

$$\psi = \int^{x^0} J_0\,dx^0$$

$$\frac{\partial\psi}{\partial x^1} = \int^{x^0} J_{0,1}\,dx^0 = \int^{x^0} J_{1,0}\,dx^0$$

$$= J_1$$

3.) $J_0 = \psi_{,0}$ $\quad J_1 = \psi_{,1}$

or $\boxed{J_A = \psi_{,A}}$ Q.E.D.

Corollary (Helmholtz's theorem)

Given: J_A a covector field

Conclusion: $\exists\ \psi$ and ϕ such that

$$J_A = \psi_{,A} + \phi_{,c}\epsilon^c{}_A$$

proof:

1. solve $\psi_{,A}{}^{|A} = J_A{}^{|A}$ for ψ

2. Note that

$$J_A - \psi_{,A} = J_A^{\text{transverse}}$$

has the ppty that

$$J_A^{\text{trans}\,|A} = 0$$

The zero divergence ppty implies $\exists\ \phi$ such that

$$J_A - \psi_{,A} = \phi_{,c}\epsilon^c{}_A$$

$$\boxed{J_A = \psi_{,A} + \phi_{,c}\epsilon^c{}_A} \quad \text{Q.E.D.}$$

We shall apply Helmholtz's theorem to integrate, at least partially, the Einstein field equations governing any spherically symmetric configuration.