Lecture 27

Helmholtz's theorem
Helmholtz's Theorem

Vector fields in two dimensions have a particularly simple property which makes it useful in unravelling the properties of spherically symmetric space times. The property is this (although technically incorrect, we shall refer to a tensor field by its components):

For any covector field \( J^a \) on \( \mathbb{R}^2 \), there exist two scalars \( \psi \) and \( \phi \) such that

\[
J^a = \psi \delta^a + \phi \epsilon^a
\]

or, figuratively,

"covector" = "gradient" + "curl"

The two scalars are solutions to the 2-D inhomogeneities wave equations in \( \mathbb{R}^2 \):

\[
\begin{align*}
\nabla^2 \psi &= 0 \\
\nabla^2 \phi &= 0
\end{align*}
\]

The 2-D Helmholtz theorem is based on the following ("zero divergence property")

**Theorem**

Given: A vector field \( J^a \)

**Conclusion:**

\[ J^a \] exists if and only if

\[ J^a_{;a} = 0 \]

\[ J^a = \Phi_a \epsilon^a \]

Here

\[
\epsilon^{a} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\]

\[
\epsilon^{cd} g_{cd} = g_{cd} \delta^{bc} = \begin{bmatrix} 1 \end{bmatrix} [g_{cd} \delta^{bc}] = \begin{bmatrix} 1 \end{bmatrix} [g_{cd}] = \begin{bmatrix} 1 \end{bmatrix} \quad \text{(1)}
\]

\[
\epsilon^{cd} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

\[
\epsilon^{cd} = \delta^{cd} \quad \text{(2)}
\]

\[
\epsilon^{cd} g_{cd} = \begin{bmatrix} 1 \end{bmatrix} [g_{cd} \delta^{cd}] = \begin{bmatrix} 1 \end{bmatrix} [g_{cd}] = \begin{bmatrix} 1 \end{bmatrix}
\]

\[
\epsilon^{cd} = \delta^{cd} \quad \text{(3)}
\]

\[
\epsilon^{cd} g_{cd} = \begin{bmatrix} 1 \end{bmatrix} [g_{cd} \delta^{cd}] = \begin{bmatrix} 1 \end{bmatrix} [g_{cd}] = \begin{bmatrix} 1 \end{bmatrix}
\]

\[
\epsilon^{cd} = \delta^{cd} \quad \text{(4)}
\]
proof:

\[ J^A_{IA} = (\Phi_j c)^A_{IA} \]
\[ = \Phi_j c_{IA} E^A + \Phi_j c_{IA} E^A \]
\[ = \sum_{\text{sym, antisym}} (\Phi_j c_{IA}) \]
\[ = 0 \quad (3.11 \text{ a in MTW}), \]

\[ \Leftrightarrow \text{ Note that} \]
\[ 0 = J^A_{IA} = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial x^0} (J^0)_{IA} + \frac{\partial}{\partial x^1} (J^1)_{IA} \right] \]

2. Consequently, we consider

\[ \phi(x^0, x^1) = \int_{x^0}^{x^1'} J^0_{IB} d x^1 \]
\[ \frac{\partial \phi}{\partial x^0} = \int_{x^0}^{x^1'} \frac{\partial (J^0_{IB})}{\partial x^0} d x^1 \]
\[ = \int_{x^0}^{x^1'} \frac{\partial (J^0_{IB})}{\partial x^0} d x^1 \]
\[ = - \int_{x^0}^{x^1'} \frac{\partial (J^0_{IB})}{\partial x^0} d x^1 \]
\[ = - \frac{\partial \phi}{\partial x^0} J(x^0, x^1) \]

3. \( J^0 = \frac{1}{\sqrt{g}}, \quad J^1 = -\frac{1}{\sqrt{g}} \frac{\partial \phi}{\partial x^0} \)

Using:

\[ \left[ E^C_B \right] = \left[ \begin{array}{cc} E^{00} & E^{01} \\ E^{10} & E^{11} \end{array} \right] = \frac{1}{\sqrt{g}} \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \]

\[ J^0 = \frac{\partial \phi}{\partial x^0} E^{00}, \quad J^1 = \frac{\partial \phi}{\partial x^0} E^{01} \Rightarrow J_A = \phi_j c E^A \]

Q.E.D.

Theorem ("Zero curl property")

Given: A covector field \( J_A \)

Conclusion

\[ \exists \text{ a function } \psi \]
\[ \text{such that } \]
\[ J_A = \psi_A \]

\[ \Leftrightarrow 0 = J_{A1B} - J_{B1A} \]

\[ = J_{c1B} e^C e_{AB} \]

proof:

\[ J_{A1B} - J_{B1A} = J_{[A}B - J_{B,A} \]

("the R's cancel")

\[ = \psi_{[A}B - \psi_{B,A} = 0 \]

\[ \Leftrightarrow \text{ Note that} \]
\[ 0 = J_{A1B} - J_{B1A} = J_{[A}B - J_{B,A} \]

2. Consequently, we consider

\[ \psi = \int_{x^0}^{x^1} J^0_{IA} d x^0 \]
\[ \frac{\partial \psi}{\partial x^1} = \int_{x^0}^{x^1} J^0_{IB} d x^0 = \int_{x^0}^{x^1} J_{00} d x^0 \]
\[ = J_0 \]

3. \( J_0 = \psi_0, \quad J_{0} = \psi_0, \quad \psi_0 \]
\[ \Rightarrow J_A = \psi_A \quad \text{Q.E.D.} \]
Corollary (Helmholtz's theorem)

Given: $J^a$ a vector field

Conclusion: $\exists \psi$ and $\phi$ such that

$$ J^a = \psi^a + \phi_c e^c_a $$

Proof:

1. Solve $\psi^a_{,a} = J^a_{,a}$ for $\psi$

2. Note that $J^a - \psi^a_{,a} = J^a$ has the property that $J^a_{,a} = 0$

The zero divergence property implies $\exists \phi$ such that

$$ J^a - \psi^a_{,a} = \phi_c e^c_a $$

$$ [J^a = \psi^a_{,a} + \phi_c e^c_a] \quad \Box$$

We shall apply Helmholtz's theorem to integrate, at least partially, the Einstein field equations governing any spherically symmetric configuration.