Lecture 28

Integration of the Einstein field equations using a conservation law. Mass distribution determines spatial geometry.

The spacetime behaviour of <u>any</u> spherically symmetric system is governed by the Einstein field equations

(i) $$-2r\, r_{,AB} + g_{AB}\left(2r\, r_{,C}{}^{|C} + r_{,C}\, r_{,D}\, g^{CD} - 1\right) \equiv r^2 G_{AB} = 8\pi r^2 t_{AB}$$

(ii) $$\frac{r_{,C}{}^{|C}}{r} - R \equiv \frac{1}{2} G_a{}^a = \frac{1}{2} 8\pi t_a{}^a$$

and the implied Euler hydrodynamical eq'ns:

$$(i)+(ii) \Rightarrow \left(r^2 t_{AB}\right)^{|B} - r\, r_{,A}\, t_a{}^a = 0 \qquad (iii)$$

These equations make <u>geometrical</u> statements about M^2, and hence about M^4. For example, the Gaußian curvature of M^2 is

$$\boxed{R = \frac{r_{,C}{}^{|C}}{r} - 4\pi t_a{}^a}$$

("What does it mean?")

(In vacuum the answer should be very simple)

The significance of this statement is presently unclear.

The tensor equations, although not as terse, make a statement which is discernable because they can be partially integrated with the help of Helmholtz's theorem.

Multiply the <u>tensor</u> equation by the vector $-\frac{1}{2} r_{,C}\, \epsilon^{CA}$ and obtain the covector field:

$$J_B = -\frac{1}{2} r_{,C}\, \epsilon^{CA}\, r^2 G_{AB}$$

With the help of $(r^2 G_{AB})^{|B} = r\, r_{,A}\, G_a{}^a$ we find that its divergence vanishes:

$$J^B{}_{|B} = 0.$$

Thus, using Helmholtz's theorem, we know that there exists a scalar function from J_B can be derived. In order to find this scalar function let us convert the divergence condition into a "curl" condition

$$\epsilon_{BC} = |g|^{1/2}\, [BC]$$
$$\epsilon^{CA} = \frac{|g|^{1/2}}{g}\, [CA] = \frac{1}{|g|^{1/2}}\, [CA]$$
$$\epsilon_{0a}\, \epsilon^{c0} = 1 = \epsilon_{0a}\, \epsilon^{c0}$$

$$0 = J^B{}_{|B}$$
$$= \left(J^B \delta_B{}^A\right)_{|A}$$
$$= \left(J^B \epsilon_{BC}\, \epsilon^{CA}\right)_{|A}$$
$$= \left(*J_{C|A}\right) \epsilon^{CA}$$

$$\boxed{*J_C \equiv J^B \epsilon_{BC}}$$

$$= \left(*J_{0|1} - *J_{1|0}\right) \epsilon^{01} = \left(*J_{1|0} - *J_{0|1}\right) \epsilon^{10}$$

$$\Rightarrow *J_{A|B} - *J_{B|A} = 0 \qquad (\text{"}\underline{\text{zero curl}}\text{"})$$

by Helmholtz:

$$\Rightarrow \exists \psi \ni \psi_{,E} = *J_E \left(\equiv J^B \epsilon_{BE}\right)$$

Thus the covector field, the double dual of the Einstein tensor contracted into the covector $r^2 r_{,C}$,

$$*J_E = -\tfrac{1}{2} r^2 r_{,C} \epsilon^{CA} G_A{}^B \epsilon_{BE}$$

must be a pure gradient. With redoubled efforts we try to write this quantity as a gradient and we find that indeed (from Eq (i))

$$\boxed{*J_E = -\tfrac{1}{2} r_{,C} \epsilon^{CA} r^2 G_A{}^B \epsilon_{BE} = \tfrac{1}{2}\Big[\underbrace{r\,(1 - r_{,D}\, r^{,D})}_{\equiv\, \psi}\Big]_{,E}} = \frac{\partial}{\partial x^E} m(x^0, x') \qquad (1)$$

$\boxed{*J_E \text{ is a conservative covector field!}}$

This is a remarkable statement about the tensor part,

$$G_A{}^B = 8\pi\, t_A{}^B, \qquad (2)$$

of the Einstein field equations on M^2. The statement asserts that a certain part of this tensor field, namely the covector field $*J_E$, a geometrical and non-linear expression in the to-be-found solutions $r(x^C)$ and $g_{AB}(x^C)$, is a conservative vector field. This means it can be integrated

along any path in M^2 to yield a scalar function on M^2,

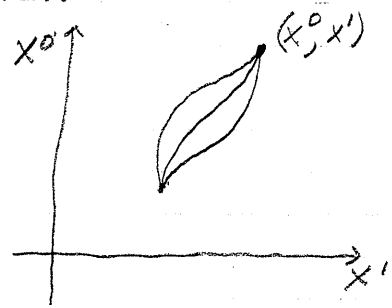
$$m(x^0, x') = \int^{(x^0, x')} *J_E \, dx^E$$
$$= \int^{(x^0, x')} \tfrac{1}{2}\big[r(1 - r_{,D}\, r^{,D})\big]_{,E}\, dx^E \qquad (*)$$
$$= \int^{(x^0, x')} -\tfrac{1}{2} r_{,C} \epsilon^{CA} r^2 G_A{}^B \epsilon_{BE}\, dx^E \qquad (**)$$

This function provides the causal relation between (i) the geometrical mathematization of gravitation on M^2 and (ii) the distribution of matter as expressed by its momenergy tensor on M^2.

Indeed, the integrand of Eq (*) is the gradient of scalar constructed solely from geometrical objects such as $r(x^0, x')$ and $r_{,C}\, r_{,D}\, g^{CD}$.

Furthermore, the value of the integral depends only on the integration endpoints,

not on the path itself if M^2 is simply connected.



1.) The geometrical mathematization of gravitation on M^2 gets into sharper focus using a new coordinate system with respect to which

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

so that

$$\left.\begin{aligned} x^1 &= r \\ x^0 &= t \\ r(x^0, x^1) &= r \end{aligned}\right\} \quad (***)$$

In terms of these coordinates on M^2

the scalar

$$m(x^0, x^1) = \frac{1}{2}\, r\left(1 - r_{,C}\, r_{,D}\, g^{CD}\right)$$

becomes

$$m(t, r) = \frac{1}{2}\, r\left(1 - g^{rr}\right)$$

or in terms of the spatial metric g_{rr},

$$g_{rr} = \frac{1}{g^{rr}} = \frac{1}{1 - \frac{2m(t,r)}{r}}.$$

2.) The relation of this to the distribution of matter is expressed by applying the field Eqs. (2) on page 3 to Eq. (**) on page 4. The result is the integral

$$m(x^0, x^1) = -4\pi \int^{(x^0, x^1)} r_{,C}\, \epsilon^{CA}\, r^2\, t_A{}^B\, \epsilon_{BE}\, dx^E \quad (*)$$

Using the coordinates on P5, evaluate this line integral

along t = constant. The result is

$$m(t,r) = -4\pi \int^{r} r_{,C}\, \epsilon^{CA} r^2 t_A{}^B \epsilon_{Br}\, dr$$

$$= -4\pi \int^{r} \epsilon^{r0} r^2 t_0{}^0 \epsilon_{0r}\, dr$$

In light of

$$\epsilon_{0r} = |g|^{1/2} 1$$

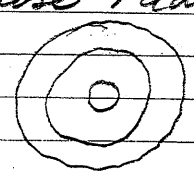
$$\epsilon^{r0} = \frac{-1}{|g|^{1/2}}(-1)$$

$$-t_0{}^0 = \rho(t,r) \quad (= \text{mass density})$$

one obtains

$$\boxed{m(t,r) = \int^{r} \underbrace{4\pi r^2}_{\text{area of concentric spheres}} \rho(t,r)\, dr = \text{mass-energy}}$$

If one has a spherical mass distribution, say a star, whose center is at $r=0$ and whose radial coordinate is r



then

$$m(t,r) = \int_0^{r} 4\pi r^2 \rho(t,r)\, dr$$

and

$$g_{rr} = \left(1 - \frac{2m(r,t)}{r}\right)^{-1}$$

3.) As noted on page 4, the value of the integral depends only on the integration end points

It follows that the integral over a closed path vanishes

$$0 = \oint * J_E\, dx^E = \oint m_{,E}\, dx^E \quad (3)$$

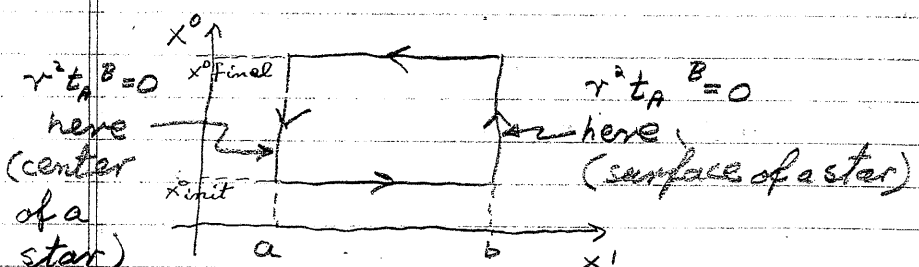


Figure : Closed path on M^2

Consider the integral over a closed path. If the matter tensor $t_A{}^B$ or

its product with r^2 vanishes along two opposite time-like pathsegments, then one obtains the following conservation law from Eq.(3).

$$-4\pi \underbrace{\int_a^b r_{,c}\, \epsilon^{CA} r^2 t_A{}^B \epsilon_{B1}\, dx^1}_{m(x^0_{init},\, x^1=b)} \Big|_{x^0_{init}} =$$

$$= \underbrace{-4\pi \int_a^b r_{,c}\, \epsilon^{CA} r^2 t_A{}^B \epsilon_{B1}\, dx^1}_{m(x^0_{final},\, x^1=b)} \Big|_{x^0_{final}}$$

In terms of the (t,r)-coordinates with $b=R$ one has

$$m(t_{final}, R) = m(t_{init}, R)$$

Consequently, one concludes that in spite of any time dependence $\left(\frac{\partial \rho}{\partial t} \neq 0\right)$ of the matter distribution inside a spherical configation

$$\boxed{m(R) = \int^R 4\pi r^2 \rho(r,t)\, dr} \qquad \rho(r,t)=0 \text{ for } r>R.$$

$m^{conv}(t,r)$ = "mass" enclosed by sphere of area $4\pi r^2$ at t = constant.

This mass function is related to the metric by

$$\frac{G}{c^2} m(t,r) = \frac{1}{2} r \left(1 - r_{,D}\, r_{,C}\, g^{DC}\right) = \frac{1}{2}\left(1 - \frac{\partial r}{\partial r}\frac{\partial r}{\partial r} g^{rr}\right)$$

$$= \frac{1}{2} r \left(1 - g^{rr}\right)$$

or

$$\boxed{g_{rr} = \frac{1}{1 - \frac{2(G/c^2)\, m^{conv}(t,r)}{r}} \equiv \frac{1}{1 - \frac{2m(t,r)}{r}}}$$

Here we introduce the $m = \frac{G}{c^2} m^{conv}$ $\left(= \left[\frac{\text{Length}}{\text{mass}}\right] \times [\text{mass}] = [\text{Length}]\right)$ the the geometrical mass.

For example, solutions to the Einstein corresponding to empty space and a homogeneous static star yield

$$t_A{}^B = 0 \implies m(t,r) = \text{constant} \equiv M \quad (\text{vacuum})$$

$$(-)t_0{}^0 = \rho \implies m(t,r) = \frac{4\pi}{3} r^3 \rho \quad (\text{star})$$

where ρ is a constant, independent of t and ρ.

Consequently

$$g_{rr} = \begin{cases} \left(1 - \frac{2M}{r}\right)^{-1} & (\text{vacuum}) \\ \left(1 - \frac{8\pi}{3}\rho r^2\right)^{-1} & (\text{INSIDE of a homogeneous star}) \end{cases}$$

Appendix

Comments about units in Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \begin{cases} \frac{8\pi G}{c^4} T_{\mu\nu}^{phys} \\ \frac{8\pi G}{c^2} T_{\mu\nu}^{astro} \\ 8\pi \, T_{\mu\nu}^{geom} \end{cases}$$

Being expressed in terms of curvature, the Einstein tensor $G_{\mu\nu}$ has units of $\left(\frac{1}{\text{length}}\right)^2$;

$\frac{G}{c^2}$ has units $\left[\frac{\text{length}}{\text{mass}}\right]$

$\frac{G}{c^4}$ has units $\left[\frac{\text{length}}{\text{energy}}\right]$

$T_{\mu\nu}^{phys}$ has units $\left[\frac{\text{energy}}{(\text{length})^3}\right]$ ("physics" units)

$T_{\mu\nu}^{astr}$ has units $\left[\frac{\text{mass}}{(\text{length})^3}\right]$ "astrophysics units")

$\therefore T_{\mu\nu}^{astr} = \frac{1}{c^2} T_{\mu\nu}^{phys}$

$T_{\mu\nu}^{geom}$ has units $\left[\frac{1}{(\text{length})^2}\right]$ ("geometrical units")

$$\therefore T_{\mu\nu}^{geom} = \frac{G}{c^2} T_{\mu\nu}^{astro} = \frac{G}{c^4} T_{\mu\nu}^{phys}$$

Example (perfect fluid)

(i) $T_{phys}^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$

Pressure; $p = \left[\frac{\text{force}}{\text{area}}\right] = \left[\frac{\text{energy}}{\text{volume}}\right]$

Energ density: $\rho = \left[\frac{\text{energy}}{\text{volume}}\right]$

$\boxed{\text{physics units}}$

(ii) $T_{astro}^{\mu\nu} = \left(\rho^{astr} + \frac{p}{c^2}\right) u^\mu u^\nu + \frac{p}{c^2} g^{\mu\nu}$

pressure; $\frac{p}{c^2} = \left[\frac{\text{force}}{(\text{length})^2}\right] \frac{1}{c^2} = \left[\frac{\text{mass}}{\text{volume}}\right]$

mass density; $\rho^{astr} = \left[\frac{\text{mass}}{\text{volume}}\right]$

$\boxed{\text{astro-physics units}}$

(iii) $T_{geom}^{\mu\nu} = (\rho^* + p^*) u^\mu u^\nu + p^* g^{\mu\nu}$

$\boxed{\text{geometrical units}}$

pressure; $p^* = \frac{G}{c^4} p = \left[\frac{\text{length}}{\text{energy}} \times \frac{\text{energy}}{(\text{length})^3}\right] = \left[\frac{1}{(\text{length})^2}\right]$

mass density $\rho^* = \frac{G}{c^4} \rho = \frac{\text{length}}{\text{energy}} \times \frac{\text{energy}}{(\text{length})^3} = \frac{1}{(\text{length})^2}$