Lecture 28

Integration of the Einstein field equations using a conservation law. Mass distribution determines spatial geometry.
The spacetime behaviour of any spherically symmetric system is governed by the Einstein field equations:

(i) \[ 2\Pi F_{AB} + j_{AB} = \left( \frac{2}{\rho} \gamma_{AC} + \gamma_{BD} \right) \gamma^{CD} - 1 \equiv \gamma^B_{AB} = \frac{\gamma}{r^2} \gamma_{AB} \]

(ii) \[ \frac{\gamma}{\gamma} R = \delta^a \gamma^a = \frac{1}{2} \Pi B a^2 \]

and the implied Euler hydrodynamical eqns:

(i)\(\times\) (ii) \[ (\gamma^a \gamma^B) \gamma_{AB} + \gamma^a \gamma^B \gamma^c - \gamma^a \gamma^B \gamma^c = 0 \] (ii)

These equations make geometrical statements about \(M^2\) and hence about \(M^4\). For example, the Gaussian curvature of \(M^2\) is

\[ R = \frac{\gamma}{\gamma} \gamma_{AB} \]

:\(\text{What does it mean?\?}\)

(:\(\text{Just vacuum the answer}\))

Thus, using Helmholtz's theorem, we know that there exists a scalar function from \(J_B\) that can be derived. In order to find this scalar function let us convert the divergence condition into a "curl" condition:

\[ 0 = J_B \]

\[ \Rightarrow \nabla \times J_B = 0 \] (\text{"zero curl\")

\[ \Rightarrow \nabla \times J_B = 0 \]

\[ \Rightarrow \nabla \times \psi = \psi \nabla \times \left( J_B E_B \right) \]
Thus the covector field, the double dual of the Einstein tensor contracted into the covector \( \tau^c \),

\[
x J^c = -\frac{1}{2} \tau^c \varepsilon^{ca} G_a^B e_B E
\]

must be a pure gradient. With redoubled efforts we try to write this quantity as a gradient and we find that indeed from Eq. (1)

\[
*J^c = -\frac{1}{2} \varepsilon^{ca} \tau^c \varepsilon^{BE} G_a^B e_B E = \frac{1}{2} \varepsilon^{ca} \left[ \tau^c (1 - \tau^a \tau^a) \right]_E dx^E = \frac{\partial}{\partial x^E} m(\mathbf{x})
\]

This is a remarkable statement about the tensor part

\[
G_a^B = \partial \tau^a \tau^B
\]

of the Einstein field equations on \( \mathbb{M}^2 \). The statement asserts that a certain part of this tensor field, namely the covector field \(*J^c\), a geometrical and non-linear expression in the to-be-found solutions \( \tau(\mathbf{x}) \) and \( G_{ab}(\mathbf{x}) \), is a conservative vector field. This means it can be integrated along any path in \( \mathbb{M}^2 \) to yield a scalar function on \( \mathbb{M}^2 \)

\[
m(\mathbf{x}, \mathbf{x}') = \int_{\mathbf{x}}^{\mathbf{x}'} *J^c \; dx^E
\]

\[
= \int_{\mathbf{x}}^{\mathbf{x}'} \frac{1}{2} \varepsilon^{ca} \left[ \tau^c (1 - \tau^a \tau^a) \right]_E dx^E
\]

\[
= \int_{\mathbf{x}}^{\mathbf{x}'} \frac{1}{2} \varepsilon^{ca} \tau^c \varepsilon^{BE} G_a^B e_B E dx^E
\]

This function provides the causal relation between (1) the geometrical mathematization of gravitation on \( \mathbb{M}^2 \) and (2) the distribution of matter as expressed by its monenergetic tensor on \( \mathbb{M}^2 \).

Indeed, the integrand of Eq. (1) is the gradient of scalar constructed solely from geometrical objects such as \( \tau(\mathbf{x}, \mathbf{x}') \) and \( \tau^c \tau^d \varepsilon^{cd} \).

Furthermore, the value of the integral depends only on the integration endpoints.
not on the path itself if \( M^2 \) is simply connected.

\[
\begin{array}{c}
\text{(x, y)} \\
\text{(x, y')}
\end{array}
\]

1. The geometrical mathematicalization of gravitation on \( M^2 \) gets into sharper focus using a new coordinate system with respect to which

\[
ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

so that

\[
\begin{cases}
  x^1 = \tau \\
x^0 = t \\
r(x^0, x^1) = \tau
\end{cases}
\]

\( m(x^0, x^1) = \frac{1}{2} \gamma (1 - \gamma \gamma^D g^{CD}) \)

becomes

\( m(\tau, r) = \frac{1}{2} \gamma (1 - g^{rr}) \)

or in terms of the spatial metric \( g_{\mu\nu} \)

\[
g^{rr} = \frac{1}{g^{xx}} = \frac{1}{1 - 2m(\tau, r)}
\]

2. The relation of this to the distribution of matter is expressed by applying the field Eqs. (2) on page 3 to Eq. (4) on page 4. The result is the integral

\[
m(x^0, x') = -4\pi \int \frac{E^C}{\gamma^2 \gamma^D} T_{\gamma^C \gamma^D} B_{E} E d x^E (x)
\]

\[
\text{Using the coordinates on PS, evaluate}
\]

\text{in terms of these coordinates on } M^2 \text{ this line integral}
along \( t = \text{constant} \), the result is
\[
m(t, r) = -4\pi \int_0^r \rho(r') r'^2 \, dr'
\]
\[
= -4\pi \int_0^r r'^2 \, dE_0 \, dr'
\]
In light of
\[
E_0 = \frac{1}{2} \rho
\]
\[
t_0 = \rho(t, r) \quad (\text{mass density})
\]

One obtains
\[
m(t, r) = \frac{1}{2} \int_0^r r'^2 \rho(t, r) \, dr = \text{mass-energy}
\]
If one has a spherical mass distribution, say a star, whose center is at \( r = 0 \)
and whose radial coordinate is \( r \)

then
\[
m(t, r) = \int_0^r 4\pi r'^2 \rho(t, r) \, dr
\]
and
\[
g_{rr} = \left( 1 - \frac{2m(t, r)}{r} \right)^{-1}
\]

3) As noted on page 4, the value of the integral depends only on the integration endpoints.

It follows that the integral over a closed path vanishes
\[
o = \oint \xi J_r \, dx^B = \oint m_E \, dx^B \quad (3)
\]

Consider the integral over a closed path. If the matter tensor \( \tau^B_{\alpha} \)
its product with \( r^2 \) vanishes along two opposite time-like path segments, then one obtains the following conservation law from Eq. (3):

\[
-4\pi \int_{a}^{b} \int_{E_{\text{init}}}^{E_{\text{final}}} x^2 \, dE \, dx = m(x^0_{\text{init}}, x^1 = b) - m(x^0_{\text{final}}, x^1 = b)
\]

In terms of the \((t, r)\)-coordinates with \( b = R \) one has

\[
m(t_{\text{final}}, R) = m(t_{\text{init}}, R)
\]

Consequently, one concludes that in spite of any time dependence \((\frac{\partial}{\partial t} \neq 0)\) of the matter distribution inside a spherical configuration

\[
m(R) = \int_{4\pi r^2}^{\infty} \sigma(r, t) \, dr
\]

for \( r > R \).

\[m^{\text{conw}}(t, r) = "\text{mass" enclosed by sphere of area } 4\pi r^2 \text{ at } t = \text{const.}\]

The mass function is related to the metric by

\[
\frac{\partial}{\partial \rho} m(t, r) = \frac{1}{2} x^0 \left( -g^{00} g^{00} \right) = \frac{1}{2} \left( -\frac{2M}{r^2} g^{00} \right) = \frac{1}{2} \rho \left( 1 - \frac{R}{r} \right)^2
\]

or

\[
g^{00} = \left[ \frac{1}{1 - \frac{R}{r}} \right] \left[ \frac{1}{1 - \frac{R}{r}} \right] = \frac{1}{2} \rho \left( 1 - \frac{R}{r} \right)^2
\]

Here we introduce the mass

\[
m = \frac{\partial}{\partial \rho} m^{\text{conw}} = \left[ \frac{\partial}{\partial \rho} m \right] \left[ \frac{\partial}{\partial \rho} \rho \right] = \left[ \frac{\partial}{\partial \rho} \rho \right] \left[ \frac{\partial}{\partial \rho} \rho \right] = \rho \left( 1 - \frac{R}{r} \right)^2
\]

the geometrical mass. For example, solutions to the Einstein equations corresponding to empty space and a homogeneous static star yield

\[t_{\text{vac}} = 0 \Rightarrow m(t, r) = \text{constant} = M \] (vacuum)

\[-t_{\text{vac}} = \rho \Rightarrow m(t, r) = 4\pi r^2 \rho \] (star)

where \( \rho \) is a constant, independent of \( r \) and \( t \).

Consequently

\[
\left[ 1 - \frac{2M}{r} \right]^{-1} \quad \text{(vacuum)}
\]

\[
\left[ 1 - \frac{R}{r} \right]^{-1} \quad \text{(inside of a homogeneous star)}
\]
Appendix

Comments about units in Einstein Field equations

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8 \pi G}{c^4} T_{\mu \nu}^{\text{phys}} \]

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8 \pi G}{c^4} T_{\mu \nu}^{\text{astro}} \]

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8 \pi G}{c^4} T_{\mu \nu}^{\text{geom}} \]

Being expressed in terms of curvature, the Einstein tensor \( G_{\mu \nu} \) has units of

\[ \frac{1}{\text{length}^2} \]

\[ \frac{1}{c^2} \text{ has units } \text{length} \text{ mass}^{-1} \]

\[ \frac{1}{c^4} \text{ has units } \text{length} \text{ energy}^{-1} \]

\[ T_{\mu \nu}^{\text{phys}} \text{ has units } \text{energy} \left( \text{length}^{-3} \right)^2 \text{ ("physics units") } \]

\[ T_{\mu \nu}^{\text{astr}} \text{ has units } \text{mass} \left( \text{length}^{-3} \right)^{1} \text{ "astrophysical units" } \]

\[ T_{\mu \nu} \text{ has units } \text{mass} \left( \text{length}^{-3} \right)^{1} \text{ "astro units" } \]

\[ T_{\mu \nu} = \frac{1}{c^2} T_{\mu \nu}^{\text{phys}} \]

\[ T_{\mu \nu} \text{ has units } \left( \text{length}^{-2} \right)^{1} \text{ ("geometrical units") } \]

\[ T_{\mu \nu} \text{ has units } \frac{1}{c^4} T_{\mu \nu}^{\text{astro}} \]

\[ T_{\mu \nu} \text{ has units } \frac{1}{c^2} T_{\mu \nu}^{\text{geom}} \]

Example (perfect fluid)

(i) \[ T_{\mu \nu}^{\text{phys}} = (p + \rho) u^\mu u_\nu + \rho g_{\mu \nu} \]

Pressure: \[ p = \text{[force]} \left( \text{area} \right) = \text{[energy]} \left( \text{volume} \right) \]

Energy density: \[ \rho = \text{[energy]} \left( \text{volume} \right) \]

(ii) \[ T_{\mu \nu}^{\text{astr}} = (\rho + \rho c^2) u^\mu u_\nu + \rho c^2 g_{\mu \nu} \]

Pressure: \[ \rho c^2 = \text{[force]} \left( \text{length} \right)^2 \text{[mass]} \left( \text{volume} \right) \]

Mass density: \[ \rho = \text{[mass]} \left( \text{volume} \right) \]

(iii) \[ T_{\mu \nu}^{\text{geom}} = (\rho + \rho c^2) u^\mu u_\nu + \rho c^2 g_{\mu \nu} \]

Pressure: \[ \rho c^2 = \frac{\text{[force]} \times \text{[energy]} \left( \text{volume} \right)}{\text{[length]} \left( \text{length} \right)^3} \]

Geometrical units: \[ \rho c^2 = \frac{\text{[length]} \times \text{[energy]} \left( \text{length} \right)^3}{\text{[length]}^2} \]

Mass density: \[ \rho = \frac{\rho c^2}{c^4} = \frac{\text{[length]} \times \text{[energy]} \left( \text{length} \right)^3}{\text{[length]}^2} \]