Supplement to Lecture 21:

Constructive interference $\Rightarrow$ worldline have infinite width determined by Planck's constant. Derivation of Heisenberg's indeterminacy relation.
Wave Packet $\Rightarrow$ Constructive
Destructive interference.

(iv) The principle of constructive and destructive interference emerges directly from the quantum principle (see comment 3.) below.

How to construct a finite wave packet

$$\Psi(x,t) = \Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots = \text{sum of interfer} \atop \text{alternatives}$$

$$= \int_{-\infty}^{\infty} f(E) e^{i E / \hbar} S_E(x,t) dE.$$ 

In the asymptotic limit $S_E \gg \hbar$, a wave packet is characterized by a finite frequency window, e.g.,

$$f(E) = A e^{-\frac{(E-E_0)^2}{2\epsilon^2}}$$

The width of this window is $2\epsilon$.

In this window the phase has the Taylor series expansion

$$S_E(x,t) = S_E(x,t) + \frac{\partial S_E}{\partial E} \bigg|_{E_0} (E-E_0) + \frac{1}{2} \frac{\partial^2 S_E}{\partial E^2} \bigg|_{E_0} (E-E_0)^2$$

+ neglected higher order terms.
Applying the result

\[ \int_{-\infty}^{\infty} \exp(\alpha z^2 + \beta z) \, dz = \sqrt{\frac{\pi}{-\alpha}} \, e^{-\frac{\beta^2}{4\alpha}} \quad \text{for} \ Re(\alpha) < 0 \]

to this wave function \( \psi(x, t) \) with

\[ \alpha = -\frac{1}{\epsilon^2} + \frac{1}{2\hbar} \frac{\partial^2 S(x,t)}{\partial E^2} \bigg|_{E_0} = -\frac{1}{\epsilon^2} (1 - i\sigma) \]

\[ -\frac{1}{\alpha} = \frac{\epsilon^2}{1 - i\sigma} = \epsilon^2 \frac{1 + i\sigma}{1 + \sigma^2} \]

\[ \beta = \frac{\hbar}{\hbar} \frac{\partial S(x,t)}{\partial E} \bigg|_{E_0} \]

yields the wave packet:

\[ \psi(x, t) = \mathcal{A}(x, t) \cdot \exp \left\{ i \frac{S(x,t)}{\hbar} \right\} \]

or

\[ \psi(x, t) = A \sqrt{\frac{1}{\pi \epsilon^2}} \, e^{-\frac{1}{4} \left( \frac{\partial S(x)}{\partial E} \right)^2 \frac{1 + i\sigma}{1 + \sigma^2}} \, e^{ i \frac{2S(x,t)}{\hbar} \frac{1 + i\sigma}{1 + \sigma^2}} \]

\[ \text{location of maximum in space-time} \]

\[ x = \text{fixed} \]

\[ t = \text{fixed} \]
This wave packet expresses very precisely the phenomena of destructive and constructive interference. We see that 
\[ \frac{\partial S}{\partial E} = 0 \] implies \[ |\psi(x,t)| \] has its maximum value, i.e., one has constructive interference and this occurs precisely along the Newtonian worldline which is given by \[ \frac{\partial S}{\partial E} = 0 \]

By contrast, \[ \frac{\partial S}{\partial E} \neq 0 \] implies destructive interference because \[ |\psi(x,t)| < \text{its maximum value} \].

Even though we started with
\[ \psi(x) = e^{iS/E} \]

to arrive at the extremal path in space-time, the constant \( t_0 \), the quantum of action, never appeared in the final result.

The reason is that, in the limit
\[ \delta \to 0 \]

the location of the wave packet reduces to the location of the wave crest.

The location of the wave crest is what is governed by \( S(x,t) \) and the condition of constructive interference \( \frac{\partial S}{\partial E} = 0 \) gives without approximation the location of the sharply defined Newtonian worldline \( x = x(t) \).
Thus the sum of interfering alternative

\[ \psi(x,t) \quad (-\infty < E < \infty) \]

weighted by the Gaussian window

\[ f(E) = A e^{-\frac{(E - E_0)^2}{\sigma^2}} \]
yields

\[ \psi(x,t) = \int_{-\infty}^{\infty} f(E) e^{\frac{i}{\hbar} S_0(x,t) / \hbar} \, dE \]

where

\[ A = A \sqrt{\frac{1+i\sigma^2}{1+i\sigma^2}} \exp\left\{ -\frac{1}{4} \sigma^2 \frac{(1+i\sigma^2)}{1+i\sigma^2} \left( \frac{\partial S(x,t)}{\partial E} \right)^2 \right\} \]

\[ 0 = \frac{\varepsilon^2 - \sigma^2 \varepsilon^2}{2} \frac{1}{\varepsilon \sigma^2} \]

is a slowly varying amplitude which modulates the rapidly oscillating function

\[ i \frac{S_0(x,t)}{\hbar} \]

**Comments:**

1) Such a product represents a wave packet whose maximum amplitude traces in spacetime a worldline given implicitly by

\[ \frac{\partial S_0(x,t)}{\partial E} = 0 \]

Locates where in \( x-t \) space constructive interference occurs, i.e., where the wave packet's maximum is located.
2) Such a wave packet has a finite extent in space and in time. This extent can be inferred from its squared modulus

\[ |\psi(x,t)|^2 = |\varphi|^2 = A^2 \frac{2}{\sqrt{1 + \frac{\hbar^2}{2mE}}} \exp\left\{ - \frac{\hbar^2}{2m} \left( \frac{\partial \frac{\partial S}{\partial E}}{\partial x} \right)^2 \right\} \]

We see that this squared amplitude has a non-zero value even if the condition for constructive interference is violated:

\[ \frac{\partial^2 S(x,t)}{\partial E} = 0 \]

Thus the worldline of the particle is not a sharp one, but instead is spread out in space and time.
The magnitude of this quantum mechanical (non-classical) spread in the worldline is given implicitly by the condition that

\[ \Delta S = \varepsilon \frac{\partial S(x, t)}{\partial \varepsilon} \bigg|_{E=E_0} \approx h = 1.01 \times 10^{-27} \text{ erg s} \]

(Planck's constant)

Expanding the left hand side to first order in \( \Delta x \) and \( \Delta t \) around \((t, x)\), a point on the classical worldline determined by

\[ \frac{\partial S(x, t)}{\partial \varepsilon} \bigg|_{E=E_0} = 0 \] ("constructive interference")

one obtains

\[ \Delta S = \varepsilon \frac{\partial^2 S}{\partial \varepsilon^2} \Delta x - \varepsilon \frac{\partial^2 S}{\partial \varepsilon \partial t} \Delta t \]

\[ = \varepsilon \frac{\partial p}{\partial \varepsilon} \Delta x + \varepsilon \frac{\partial E}{\partial \varepsilon} \Delta t \]

\[ = \Delta p \Delta x + \Delta E \Delta t \]
This spatial spread in the world line is determined by
\[ \Delta S(x+\Delta x, t) = \Delta p \Delta x \approx \hbar \]
while the temporal spread is determined by
\[ \Delta S(x, t+\Delta t) = \Delta E \Delta t \approx \hbar \]
These two equations are, of course, the familiar Heisenberg indeterminacy relations. 

The virtue of our high frequency wave packet construction is that it focuses both the physical aspects and implications behind these relations.

The key principle is the Quantum Principle, about which an excellent discussion is given by Feynman in volume III of his Physics Lectures. Applied to a single particle the Quantum Principle is a statement about the meaning of the equation
\[ \psi(x, t) = \sum_{E} \psi_{E}(x, t) + \psi_{E+\Delta E}(x, t) + \cdots \]
as follows:

Consider repeating the same experiment over and over.

\[ \psi_E : \text{If a particle has energy } E, \text{ then it has probability amplitude } \psi_E(x,t) \text{ for being found at } (x,t) \]

\[ \psi^*_E : \text{If a particle is found at } (x,t), \text{ then it has probability amplitude } \psi^*_E(x,t) \text{ for having energy } E. \]

\[ |\psi_E|^2 \propto P = \text{probability for the occurrence of the above circumstance.} \]

Suppose, upon being found at \((x,t)\), the particle has alternate possible energies \( E, E + \Delta E, \ldots \). Then its probability amplitude for this circumstance is the sum of interfering alternatives:

\[ \psi(x,t) = \psi_E(x,t) + \psi_{E+\Delta E}(x,t) + \ldots \]

with corresponding probability

\[ P \propto |\psi_E(x,t) + \psi_{E+\Delta E}(x,t) + \ldots|^2 \]

If the alternate possible energies were actually determined to be \( E, E + \Delta E, \ldots \), then

\[ P \propto |\psi_E(x,t)|^2 + |\psi_{E+\Delta E}(x,t)|^2 + \ldots \]