Supplement to Lecture 31:

Constructive interference $\Rightarrow$ worldline have a finite width determined by Planck's constant. Derivation of Heisenberg's indeterminacy relation.

Wave Packet $\Rightarrow$ Constructive & Destructive interference.

(iv) The principle of constructive and destructive interference emerges directly from the <u>quantum principle</u> (See COMMENT 3.) below)

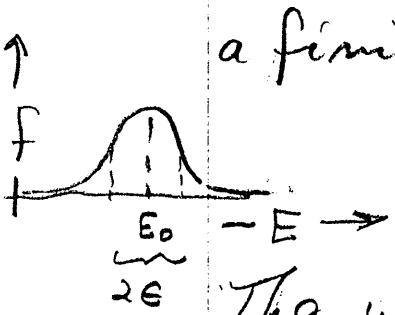
How to construct a finite wave packet

$$\psi(x,t) = \psi_E(x,t) + \psi_{E+\Delta E}(x,t) + \cdots$$ = sum of interfering alternatives

$$= \int_{-\infty}^{\infty} f(E)\, e^{\frac{i}{\hbar} S_E(x,t)}\, dE .$$

in the asymptotic limit $S_E \gg \hbar$.
A wave packet is characterized by

a finite frequency window, e.g.



$$f(E) = A\, e^{-\frac{(E-E_0)^2}{\epsilon^2}}$$

The width of this window is 2ϵ

In this window the phase has the Taylor series expansion

$$S_E(x,t) = S_{E_0}(x,t) + \left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0} (E-E_0) + \frac{1}{2} \left.\frac{\partial^2 S(x,t)}{\partial E^2}\right|_{E_0} (E-E_0)^2$$

+ neglected higher order terms.

Applying the result

$$\int_{-\infty}^{\infty} \exp(\alpha z^2 + \beta z)\, dz = \sqrt{\frac{\pi}{-\alpha}}\; e^{-\frac{\beta^2}{4\alpha}} \quad \text{for } \mathrm{Re}(\alpha) \leq 0$$

to this wave function $\psi(x, t)$ with

$$\alpha = -\frac{1}{\epsilon^2} + \frac{i}{2\hbar} \left.\frac{\partial^2 S(x,t)}{\partial E^2}\right|_{E_0} \equiv -\frac{1}{\epsilon^2}(1 - i\sigma)$$

$$-\frac{1}{\alpha} = \epsilon^2 \frac{1}{1 - i\sigma} = \epsilon^2 \frac{1 + i\sigma}{1 + \sigma^2}$$

$$\beta = \frac{i}{\hbar} \left.\frac{\partial S(x,t)}{\partial E}\right|_{E_0}$$

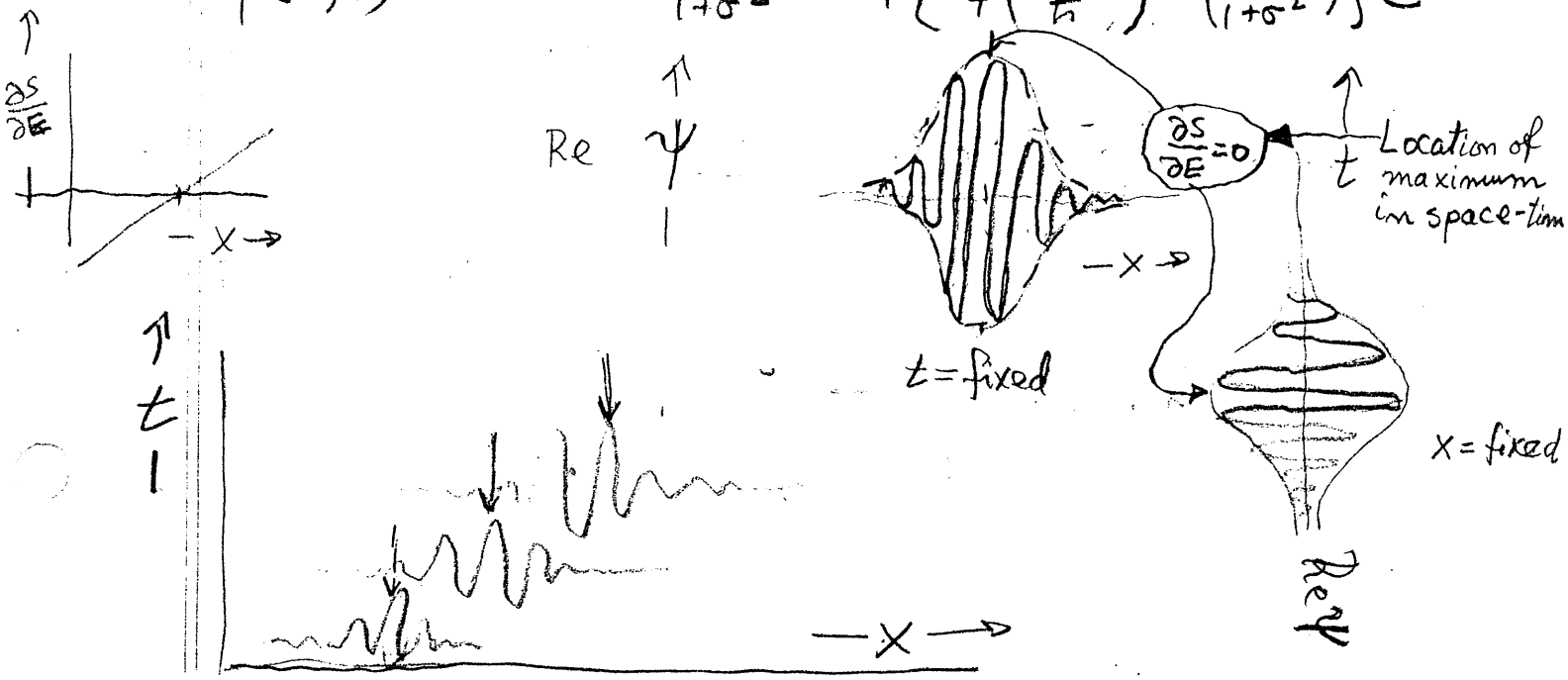
yields the wave packet:

$$\psi(x,t) = \mathcal{A}(x,t) \times e^{i \frac{S_{E_0}(x,t)}{\hbar}}$$

extreme value of action

or

$$\psi(x,t) = \overbrace{A\sqrt{\pi}\,\epsilon \sqrt{\frac{1+i\sigma}{1+\sigma^2}} \exp\left\{-\frac{1}{4}\left(\frac{\frac{\partial S(x,t)}{\partial E_0}}{\hbar}\right)^2 \epsilon^2 \left(\frac{1+i\sigma}{1+\sigma^2}\right)\right\}}^{\mathcal{A}(x,t)} e^{\frac{i}{\hbar} S_{E_0}(x,t)}$$



This wave packet expresses very precisely the phenomena of <u>destructive</u> and <u>constructive</u> interference. We see that

$\frac{\partial S}{\partial E} = 0$ implies $|\psi(x,t)|$ has its maximum value, i.e. one has <u>constructive</u> interference and this occurs precisely along the Newtonian worldline which is given by $\frac{\partial S}{\partial E} = 0$

By contrast $\frac{\partial S}{\partial E} \neq 0$ implies <u>destructive</u> interference because $|\psi(x,t)| <$ its maximum value.

(v) Even though we started with

$$\psi \sim e^{iS/\hbar}$$

to arrive at the extremal path in space time, the constant $\hbar$, the quantum of action, never appeared in the final result.

The reason is that in the limit

$$\boxed{\text{as } \frac{S}{\hbar} \to 0 \quad \text{the location of the } \underline{\text{wave packet}} \text{ reduces to the location of the } \underline{\text{wave crest}}.}$$

The location of the wave crest is what is governed by $S(x,t)$ and the condition of constructive interference $\frac{\partial S}{\partial E} = 0$ gives <u>without approximation</u> the location of the sharply defined Newtonian world line $x = x(t)$.

Thus the sum of interfering alternatives $\psi_E(x,t)$ $(-\infty < E < \infty)$ weighted by the Gaussian window

$$f(E) = Ae^{-\left(\frac{E-E_0}{\epsilon}\right)^2}$$

yields

$$\psi(x,t) = \int_{-\infty}^{\infty} f(E)\, e^{\frac{i}{\hbar} S_E(x,t)}\, dE$$

$$\boxed{\psi(x,t) = \mathcal{A}(x,t)\, e^{i\, S_{E_0}(x,t)/\hbar}}$$

where

$$\mathcal{A} = A\sqrt{\pi}\,\epsilon\sqrt{\frac{1+i\sigma}{1+\sigma^2}}\, \exp\left\{-\frac{1}{4}\epsilon^2\left(\frac{1+i\sigma}{1+\sigma^2}\right)\left(\frac{\left.\frac{\partial S(x,t)}{\partial E}\right|_{E_0}}{\hbar}\right)^2\right\} \qquad \sigma = \frac{\epsilon^2}{2}\frac{\partial^2 S}{\partial E^2}\frac{1}{\hbar}$$

is a slowly varying amplitude which modulates the rapidly oscillating function

$$e^{i\, S_{E_0}(x,t)/\hbar}$$

COMMENTS:

1.) Such a product represents a <u>wave packet</u> whose maximum amplitude traces in spacetime a worldline given implicitly by

$$\boxed{\left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0} = 0}$$

<u>Locates</u> where in x-t space <u>constructive</u> <u>interference</u> occurs, i.e. where the wave packet's maximum is located.

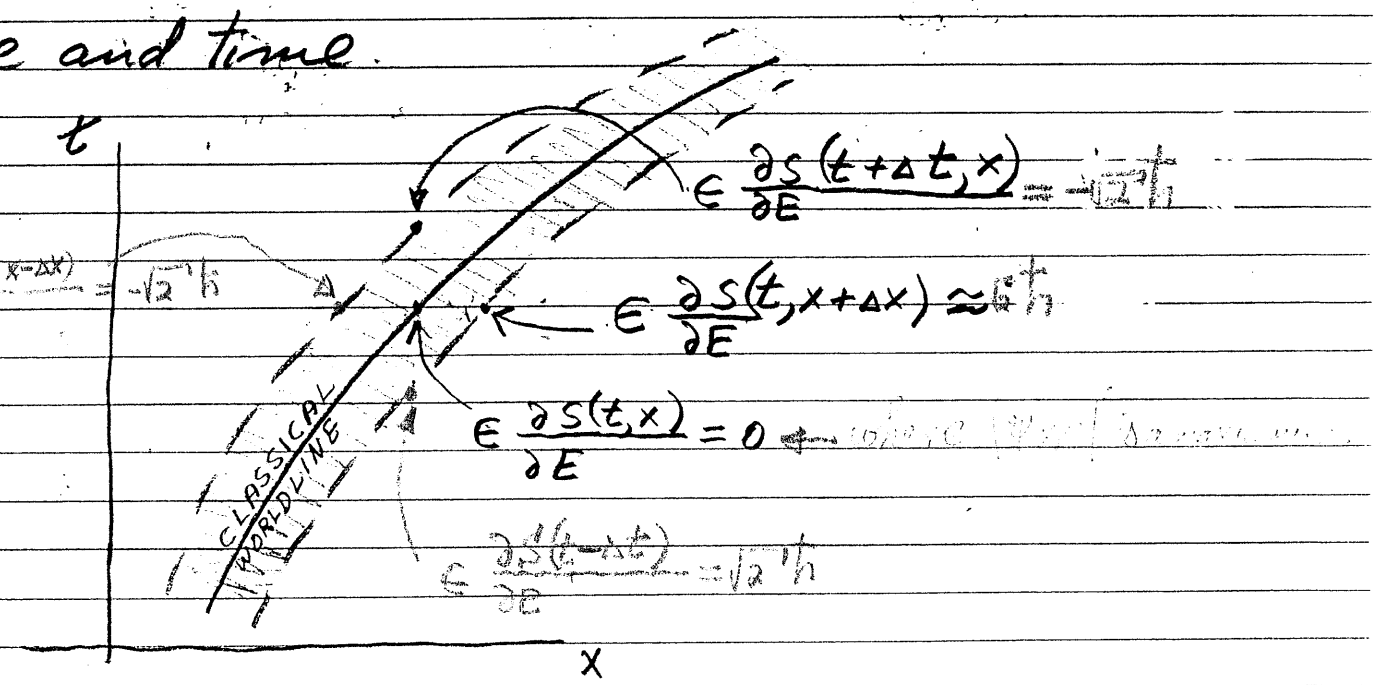
2.) Such a wave packet has a <u>finite</u> <u>extent</u> in space and in time. This extent can be inferred from its squared modulus

$$|\psi(x,t)|^2 = |\mathcal{A}|^2 = A^2 \pi \epsilon^2 \frac{1}{\sqrt{1+\sigma^2}} \exp\left\{ -\frac{\epsilon^2}{2} \frac{1}{1+\sigma^2} \frac{\left(\frac{\partial S(x,t)}{\partial E}\Big|_{E_0} \right)^2}{\hbar^2} \right\}$$

We see that this squared amplitude has a non-zero value even if the condition for constructive interference is violated:

$$\frac{\partial S_E(x,t)}{\partial E}\Big|_E \neq 0$$

Thus the worldline of the particle is not a sharp one, but instead is spread out in space and time.



The magnitude of this quantum mechanical (non-classical) spread in the worldline is given implicitly by the condition that

$$\Delta S \equiv \epsilon \left.\frac{\partial S(x+\Delta x, t-\Delta t)}{\partial E}\right|_{E=E_0} \approx \hbar = 1.01 \times 10^{-27} \text{ ergsec}$$

(Plack's constant)

Expanding the left hand side to first order in Δx and Δt, around (t, x), a point on the classical worldline determined by

$$\left.\frac{\partial S(x,t)}{\partial E}\right|_{E=E_0} = 0 \,,$$

("constructive interference")

one obtains

$$\Delta S = \epsilon \frac{\partial^2 S}{\partial E \partial x} \Delta x - \epsilon \frac{\partial^2 S}{\partial E \partial t} \Delta t$$

$$= \epsilon \frac{\partial p}{\partial E} \Delta x + \epsilon \frac{\partial E}{\partial E} \Delta t$$

$$= \Delta p \, \Delta x + \Delta E \, \Delta t$$

Thus the <u>spatial spread</u> in the world line is determined by

$$\Delta S(x+\Delta x, t) = \Delta p \Delta x \approx \hbar,$$

while the <u>temporal spread</u> is determined by

$$\Delta S(x, t+\Delta t) = \Delta E \Delta t \approx \hbar$$

These two equations are, of course, the familiar Heisenberg <u>indeterminacy</u> relations.

2) The virtue of our high frequency wave packet construction is that it focusses on the physical aspects and implications behind these relations.

The key principle is the <u>Quantum Principle</u>, about which an excellent discussion is given by Feynman in volume III of his Physics Lectures. Applied to a single particle the Quantum Principle is a statement about the meaning of the equation

$$\psi(x,t) = \psi_E(x,t) + \psi_{E+\Delta E}(x,t) + \cdots$$

as follows:
Consider repeating the same experiment over and over.

ψ_E : If a particle has energy E, then it has probability amplitude $\psi_E(x,t)$ for being found at (x,t)

ψ_E^* : If a particle is found at (x,t), then it has probability amplitude $\psi_E^*(x,t)$ for having energy E.

$|\psi_E|^2 \propto P$ = probability for the occurrence of the above circumstance.

Suppose, upon being found at (x,t) the particle has alternate possible energies $E, E+\Delta E, \ldots$, which were not determi in the experiments,

Then its probability amplitude for this circumstance is the <u>sum</u> of <u>interfering alternatives</u>,

$$\psi(x,t) = \psi_E(x,t) + \psi_{E+\Delta E}(x,t) + \cdots$$

with corresponding probability

$$P \propto |\psi_E(x,t) + \psi_{E+\Delta E}(x,t) + \cdots|^2$$

If the alternate possible energies where actually determined to be $E, E+\Delta E, \ldots$ then

$$P \propto |\psi_E(x,t)|^2 + |\psi_{E+\Delta E}(x,t)|^2 + \cdots$$