Lecture 33

Hamilton-Jacobi analysis of the orbits of a particle in the spacetime of a spherically symmetric vacuum configuration

[MTW Box 25.4, §25.5]

The most important way of solving the Hamilton-Jacobi equation is by the method of "separation of variables". In this method we observe that if the equation has the form

$$\mathcal{H}\left(x^i, \frac{\partial S}{\partial x^i}, \phi\left(x^1, \frac{\partial S}{\partial x^1}\right)\right) = 0,$$

where ϕ depends only on x^1, and x^i refer to all the other coordinates, say x^0, x^2, x^3, then we may assume that

$$S = S'(x^0, x^2, x^3) + X^1(x^1).$$

This is because upon substituting this assumed form into the H-J equation and solving for ϕ we find that

$$\phi\left(x^1, \frac{dX^1}{dx^1}\right) = \text{expression in } x^0, x^2, x^3, \frac{\partial S'}{\partial x^0}, \frac{\partial S'}{\partial x^2}, \frac{\partial S'}{\partial x^3}$$

We see that the l.h.s is independent of x^0, x^2, x^3, while the r.h.s. is independent of x^1. Consequently,

$$\phi\left(x^1, \frac{dX^1}{dx^1}\right) = \text{const} = \alpha_1$$

As a consequence we see that if the H-J equation has the form assumed above, then under the separation of variables assumption this equation becomes

$$\mathcal{H}\left(x^i, \frac{\partial S'}{\partial x^i}, \alpha_1\right) = 0$$

If the Hamilton-Jacobi equation permits this application of the separation procedure to all the remaining variables, then the separated solution has the additive form

$$S(x^0, x^1, x^2, x^3, \alpha_1, \alpha_2, \alpha_3) = X^0(x^0) + X^1(x^1) + X^2(x^2) + X^3(x^3)$$

$$S(x^0, x^1, x^2, x^3) = \int^{x^0} p_0\, dx^0 + \int^{x^1} p_1\, dx^1 + \int^{x^2} p_2\, dx^2 + \int^{x^3} p_3\, dx^3 + \text{const} = \sum_{\mu=0}^{3} \int p_\mu\, dx^\mu + \text{const}$$

where the integrand of each term,

$$p_\mu = p_\mu(x^\mu) = \frac{\partial S}{\partial x^\mu} = \frac{dX^\mu}{dx^\mu}, \quad \mu = 0, 1, 2, 3,$$

NO SUM

depends on each respective coordinate x^μ only.

Side comment: In wave mechanics the additive phases become the multiplicative amplitude profiles of a normal mode.

Let us apply this separation procedure to solving the H-J equation and finding the motion of a particle on the spacetime of a black hole.

The metric is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The corresponding H-J equation $g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0$ is

$$-\frac{1}{1-\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)^2 + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + m^2 = 0$$

Note that the coordinates t and φ are "cyclic", i.e. the metric does not depend on these coordinates. Consequently, the separation of variables method can be applied to both of them. We obtain

$$\frac{\partial S}{\partial t} = \text{const} \equiv -E$$

$$\frac{\partial S}{\partial \varphi} = \text{const} \equiv p_\varphi$$

The H-J equation reduces therefore to a p.d.e. of two variables only

$$-\frac{1}{1-\frac{2M}{r}}E^2 + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\underbrace{\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right]}_{\ell^2} + m^2 = 0$$

The separation process can now be applied to the θ-coordinate. It is therefore evident that the solution to the H-J equation separates into

$$S = T(t) + R(r) + \Theta(\theta) + \Phi(\varphi)$$

where

$$\frac{dT}{dt} = -E$$

$$\frac{d\Phi}{d\varphi} = p_\varphi$$

$$\left(\frac{d\Theta}{d\theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta} = \ell^2$$

$$-\frac{1}{1-\frac{2M}{r}}E^2 + \left(1-\frac{2M}{r}\right)\left(\frac{dR}{dr}\right)^2 + \frac{\ell^2}{r^2} + m^2 = 0$$

The general separated solution is therefore

$$S = -Et + \int \frac{1}{1-\frac{2M}{r}}\sqrt{E^2 - \left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+m^2\right)}\,dr$$

$$+ \int \sqrt{\ell^2 - \frac{p_\varphi^2}{\sin^2\theta}}\,d\theta + p_\varphi\varphi + \beta(E, \ell^2, p_\varphi)$$

$$= \int p_t\,dt + \int p_r\,dr + \int p_\theta\,d\theta + \int p_\varphi\,d\varphi$$

$$= \int p_\mu\,dx^\mu$$

I. Constructive interference yields the worldlines
II classically allowed vs classically forbidden regions

Constructive interference applied to the Hamilton-Jacobi ("dynamical") phase"

$$S = \int p_\mu dx^\mu = \int p_t dt + \int p_r dr + \int p_\theta d\theta + \int p_\phi d\phi + \beta(E, \ell^2, p_\phi)$$

$$p_t = -E$$

$$p_r = \frac{1}{1-\frac{2M}{r}}\left[E^2 - \left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+m^2\right)\right]^{1/2}$$

$$p_\theta = \left[\ell^2 - \frac{p_\phi^2}{\sin^2\theta}\right]^{1/2}$$

$$p_\phi = p_\phi$$

yields

$$\frac{\partial S}{\partial E} = 0 \quad \frac{\partial S}{\partial p_\phi} = 0 \quad \frac{\partial S}{\partial \ell^2} = 0$$

which implicitly define the worldline of the particle.

A) Instead of an algebraically exact analysis one can give a qualitative analysis.

It is based on the requirement that classically the particle satisfy

$$\left(\frac{\partial S}{\partial r}\right)^2 \geq 0 \quad \left(\frac{\partial S}{\partial \theta}\right)^2 \geq 0$$

Thus space is divided into regions which are classically allowed and those which are classically forbidden.

There

$$\left(\frac{\partial S}{\partial r}\right)^2 < 0 \quad \left(\frac{\partial S}{\partial \theta}\right)^2 < 0 .$$

The boundary between these regions is located where

$$\boxed{\left(\frac{\partial S}{\partial r}\right)^2 = 0} \quad \boxed{\left(\frac{\partial S}{\partial \theta}\right)^2 = 0}$$

The significance of this boundary we infer from the Hamilton's equations of motion. They imply that

$$\frac{\partial \mathcal{H}}{\partial p_r} = \frac{dr}{d\tau} = 2N(\tau)\, g^{rr}\frac{\partial S}{\partial r} = 2N(\tau)\sqrt{E^2 - \left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+m^2\right)}$$

$$\frac{\partial \mathcal{H}}{\partial p_\theta} \quad \frac{d\theta}{d\tau} = 2N(\tau)\, g^{\theta\theta}\frac{\partial S}{\partial \theta} = 2N(\tau)\frac{1}{r^2}\sqrt{\ell^2 - \frac{p_\phi^2}{\sin^2\theta}}$$

Thus the boundary between what classically is allowed and what is forbidden is the locus of turning points of the radial or polar angle motion of the particle.

From this locus of turning points one can infer major qualitative aspects such as bounded vs unbounded motion, stable vs unstable motion. As an example, consider the radial motion as determined by its locus of turning points:

$$\frac{dr}{d\tau} = 0 \Rightarrow E^2 - V_{eff}^2(r) = 0$$

Upon considering equatorial motion $\theta = \frac{\pi}{2}$ one has $\ell^2 = p_\varphi^2$ so that

$$V_{eff}^2 = m^2 - \frac{2M}{r} m^2 + \frac{p_\varphi^2}{r^2} - \frac{2M}{r}\frac{p_\varphi^2}{r^2}$$

Upon introducing dimensionless quantities

$$\frac{2M}{r} = \frac{1}{\bar{r}}, \quad \frac{p^2}{2Mm} = a$$

we obtain the following contributions to the radial potential:

$$\frac{E^2}{m^2} = \underbrace{1}_{\text{rest mass}} - \underbrace{\frac{1}{\bar{r}}}_{\text{Newtonian attraction}} + \underbrace{\frac{a^2}{\bar{r}^2}}_{\text{centrifugal repulsion}} - \underbrace{\frac{a^2}{\bar{r}^3}}_{\text{Angular kinetic energy has weight.}} = \left(1 - \frac{1}{\bar{r}}\right)\left(1 + \frac{a^2}{\bar{r}^2}\right)$$

which expresses the locus of turning points that seperates a classically allowed from a classically forbidden region.

$\bar{r} = \frac{r}{2M}$ $\qquad M = M_{conv} \cdot \frac{G}{c^2}$

$a = \frac{p_\varphi}{2Mm}$ $\qquad m = m_{conv}\, c$

$\bar{E} = \frac{E_{conv}}{c}$

Let $\frac{2M^*}{r} = \frac{1}{\bar{r}}$

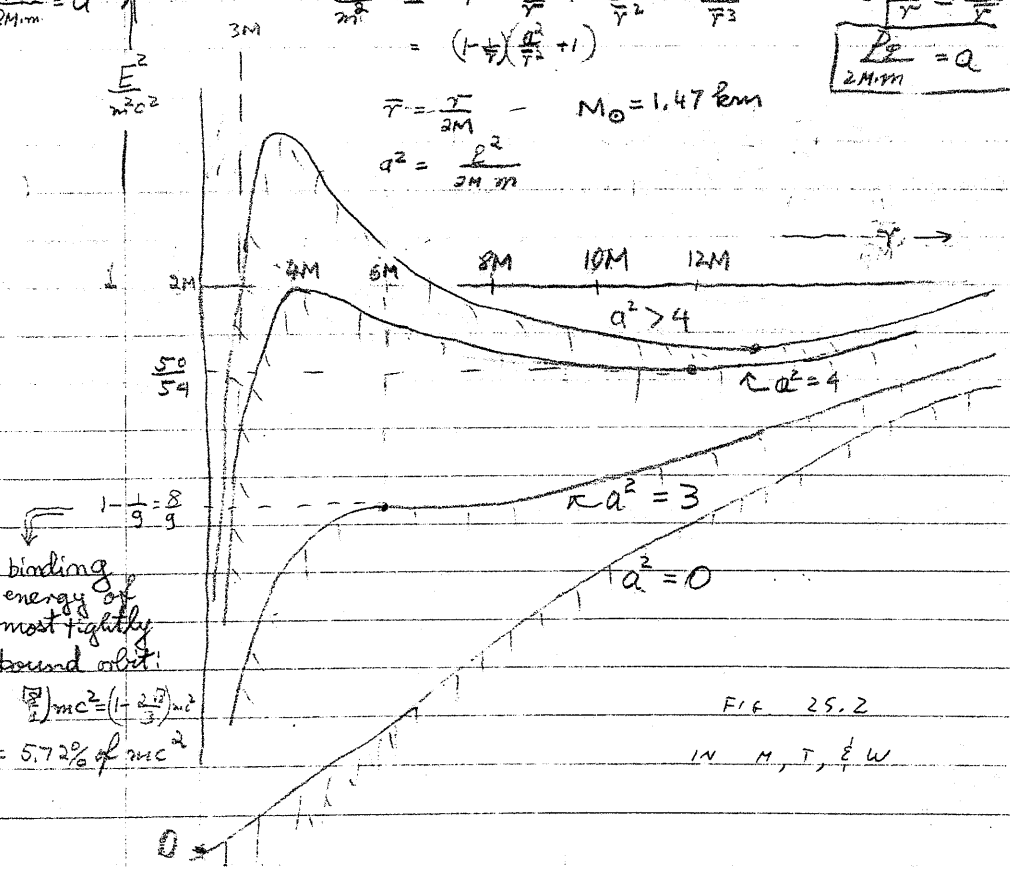
$\frac{p_\varphi}{2M \cdot m} = a$

$$\frac{E^2}{m^2}(\bar{r}) = 1 - \frac{1}{\bar{r}} + \frac{a^2}{\bar{r}^2} - \frac{a^2}{\bar{r}^3} = \left(1 - \frac{1}{\bar{r}}\right)\left(\frac{a^2}{\bar{r}^2} + 1\right)$$

where $\boxed{\frac{2M^*}{r} = \frac{1}{\bar{r}}}$ $\boxed{\frac{p_\varphi}{2M \cdot m} = a}$

$\bar{r} = \frac{r}{2M}$ — $M_\odot = 1.47$ km

$a^2 = \frac{\ell^2}{2M\, m}$



binding energy of most tightly bound orbit:

$\left(1 - \frac{2\sqrt{2}}{3}\right) mc^2 = \left(1 - \frac{2\sqrt{2}}{3}\right) mc^2$

$= 5.72\%$ of mc^2

FIG 25.2

IN M, T, & W

We note that for large enough angular momentum ($p_\varphi > \sqrt{3}\, 2M \cdot m$)

(i) there is bounded motion $E < m$

unbounded motion $E > m$

as well as motion in which the particle disappears into the black hole ($r < 2M$).

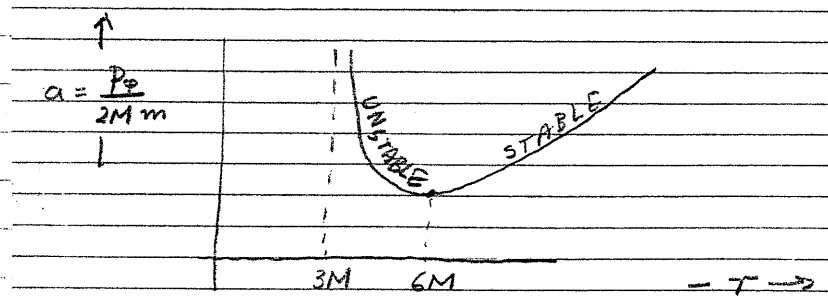
(ii) there exist stable ("Newtonian") as well as unstable ("relativistic") circular orbits.

They are determined by

$\frac{d E^2_{(r)}}{dr} = 0$, which implies

$$\frac{r}{2M} = a^2\left(1 \pm \sqrt{1 - \frac{3}{a^2}}\right) \qquad a = \frac{p_\varphi}{2Mm}$$

From the catalogue of circular orbits



one can see that there exist no circular orbits, stable or unstable, for $r < 3M$.
and that the most tightly bound stable circular orbit has radius

$$r = 6M,$$