Lecture 33

Hamilton-Jacobi analysis of the orbits of a particle in the spacetime of a spherically symmetric vacuum configuration

[MTW Box 25.4, §25.5]
The most important way of solving the Hamilton-Jacobi equation is by the method of "separation of variables." In this method, we observe that if the equation has the form

\[ H(x^i, \frac{\partial \phi}{\partial x^i}, \phi(x^i, \frac{\partial \phi}{\partial x^i})) = 0, \]

where \( \phi \) depends only on \( x^i \) and \( x^j \) refers to all the other coordinates, say \( x^0, x^2, \ldots, x^3 \), then we may assume that

\[ S = S'(x^0, x^2, x^3) + S'(x^i). \]

This is because upon substituting this assumed form into the H-J equation and solving for \( \phi \), we find that

\[ \phi(x^i, \frac{d^2 x^i}{dx^2}) = \text{expression in } x^0, x^2, x^3 \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}. \]

We see that the l.h.s. is independent of \( x^0, x^1 \), while the r.h.s. is independent of \( x^0 \). Consequently, we have

\[ \phi(x^i, \frac{d^2 x^i}{dx^2}) = \text{const} = \alpha_k. \]

As a consequence, we see that if the H-J equation has the form assumed above, then under the separation of variables assumption, this equation becomes

\[ H(x^i, \frac{\partial \phi'}{\partial x^i}, \phi') = 0, \]

where \( \phi' \) is a function of \( x^i \) only.

Side comment: In wave mechanics, the additive phases become the multiplicative amplitude profiles of a normal mode.
H-J equation & its solution.

Let us apply this separation procedure to solving the H-J equation and finding the motion of a particle in the spacetime of a black hole. The metric is

$$ ds^2 = -(1 - \frac{2M}{r}) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) $$

The corresponding H-J equation

$$ g^{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + m^2 = 0 $$

is

$$ \frac{1}{1 - \frac{2M}{r}} \left( \frac{ds}{dr} \right)^2 + (1 - \frac{2M}{r}) \left( \frac{ds}{d\theta} \right)^2 + \frac{1}{(1 - \frac{2M}{r}) \sin^2 \theta} \left( \frac{ds}{d\phi} \right)^2 + m^2 = 0 $$

Note that the coordinates $t$ and $\phi$ are "cyclic", i.e., the metric does not depend on these coordinates. Consequently, the separation of variables method can be applied to both of them. We obtain

$$ \frac{ds}{dt} = \text{const} = -E $$

$$ \frac{ds}{d\phi} = \text{const} = P_\phi $$

The H-J equation reduces therefore to a PDE of two variables only

$$ - \frac{1}{1 - \frac{2M}{r}} E^2 + (1 - \frac{2M}{r}) \left( \frac{dR}{dt} \right)^2 + \frac{E^2}{r^2} + m^2 = 0 $$

The separation process can now be applied to the $\theta$-coordinate. It is therefore evident that the solution to the H-J equation separates into

$$ S = T(t) + R(r) + \Theta(\theta) + \Phi(\phi) $$

where

$$ \frac{dT}{dt} = -E $$

$$ \frac{d\phi}{d\phi} = P_\phi $$

$$ (\frac{d\theta}{d\theta})^2 + \frac{P_\theta^2}{\sin^2 \theta} = L^2 $$

The general separated solution is therefore

$$ S = -Et + \int \sqrt{E^2 - (1 - \frac{2M}{r}) \left( \frac{E^2}{r^2} + m^2 \right)} \, dt $$

$$ + \int \sqrt{P_\phi^2 - \frac{P_\phi^2}{\sin^2 \theta}} \, d\theta + P_\phi \phi + \Phi(E, L, P) $$

$$ = \int S_{\phi dt} + \int S_{\phi dr} + \int S_{\phi d\theta} + \int S_{\phi d\phi} $$

$$ = \int S_{\phi dx}^n $$
I. Constructive interference yields the world lines.
II. Classically allowed vs. classically forbidden regions.

Constructive interference applied to the Hamilton-Jacobi (dynamical) phase:

\[ S = S_p \, dx^m = S_p(t) + S_p(\theta) + S_p(\phi) + S_p dp^2 \]

\[ + p^2 \left( \frac{1}{2} \frac{d^2 \theta}{d\tau^2} \right) \]

\[ p^2 = -E \]

\[ p^\theta = \frac{1}{2m} \left[ E^2 - \left( \frac{1}{2} \frac{d^2 \theta}{d\tau^2} \right) \right] \]

\[ p^\phi = \left[ \frac{p^2 - p^\theta^2}{m^2} \right] \]

\[ p^\phi = p^\phi \]

This space is divided into regions which are classically allowed and those which are classically forbidden.

There \( \left( \frac{\delta S}{\delta \theta} \right)^2 < 0 \) \( \left( \frac{\delta S}{\delta \phi} \right)^2 < 0 \).

The boundary between these regions is located where \( \left( \frac{\delta S}{\delta \theta} \right)^2 = 0 \) \( \left( \frac{\delta S}{\delta \phi} \right)^2 = 0 \).

The significance of this boundary we infer from the Hamilton's equations of motion. They imply that

\[ \frac{\partial}{\partial p^\theta} = \frac{d^2}{d\tau^2} - 2N(r) \]

\[ \frac{\partial}{\partial p^\phi} = 2N(r) - 2N(r) \frac{d^2 \delta S}{d\phi^2} \]

which implicitly define the world line of the particle.

Instead of an algebraically exact analysis, one can give a qualitative analysis. It is based on the requirement that classically the particle satisfy

\( \left( \frac{\delta S}{\delta \theta} \right)^2 = 0 \) \( \left( \frac{\delta S}{\delta \phi} \right)^2 = 0 \).

Thus the boundary between what is classically allowed and what is forbidden is the locus of turning points of the radial or polar angle motion of the particle.
From this locus of turning points one can infer major qualitative aspects such as bounded vs unbounded motion, stable vs unstable motion. As an example, consider the radial motion as determined by its locus of turning points:

\[
\frac{d\mathcal{E}}{dr} = 0 \Rightarrow E^2 - V_{\text{eff}}(r) = 0
\]

Upon considering equatorial motion when \( \vartheta = \frac{\pi}{2} \) one has \( L^2 = p_{\vartheta}^2 \) so that

\[
V_{\text{eff}} = m^2 - \frac{2M}{r} m^2 + \frac{p_{\vartheta}^2}{r^2} - \frac{2M}{r^2} \frac{p_{\vartheta}^2}{r^2}
\]

Upon introducing dimensionless quantities

\[
\frac{2M}{r} = \frac{1}{a} \Rightarrow \frac{p_{\vartheta}^2}{2M m^2} = a
\]

We obtain the following contributions to the radial potential

\[
\frac{E^2}{m^2} = 1 - \frac{1}{a} + \frac{p_{\vartheta}^2}{2a} - \frac{p_{\vartheta}^2}{a^3} = (\gamma^2) (\gamma^2)
\]

mass attraction, repulsion; \( \gamma \) kinetic energy, gravitational potential, weight.
We note that for large enough angular momentum $p \gg 13.2M^2m$:

1. There is bounded motion $E < m$
2. Unbounded motion $E > m$

As well as motion in which the particle disappears into the black hole ($r = 2M$)

(iii) There exist stable (“Newtonian”) as well as unstable (“relativistic”) circular orbits.

They are determined by

$$\frac{dE}{dt} = 0$$

Which implies

$$\frac{\gamma}{2M} = a^2 \left(1 - \frac{3}{a^2} \right)$$

From the catalogue of circular orbits:

$$a = \frac{p_0}{2Mm}$$
one can see that there exist no
circular orbits, stable or unstable,
for \( r < 3M \).
and that the most tightly bound
stable circular orbit has radius
\( r = 6M \).