Lecture 36

Schwarzschild spacetime:

Regular behaviour of proper time, proper distance, and curvature at the Schwarzschild radius $r = 2M$

[MTW §31.2]

Schwarzschild spacetime: Geometry and Dynamics.

I. Geometry.

The most important solution of the Einstein field equations is the one discovered by Karl Schwarzschild,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

The constant

$$2M = \frac{G}{c^2}\, 2M_{conv} = \frac{1}{15\,000\,000}\,\frac{1}{10^{21}}\, 2M_{conv} \quad \left[\begin{matrix}\text{c.g.s.}\\ \text{units}\end{matrix}\right]$$

$$= 1.3\times 10^{-28}\, M_{conv}\ [\text{cm}]$$

is the Schwarzschild radius. Typical examples are

	M_{conv} [gr]	$\times 1.33\cdot 10^{-28}\left[\frac{cm}{gr}\right] =$	$2M$
SUN	2×10^{33}		2.7 [km]
GALAXY	$2\cdot 10^{44}$		2.7×10^{11} [km]
EARTH	6×10^{27}		~1 [cm]
MOON	7.3×10^{25}		10^{-2} [cm]
ELECTRON	9×10^{-28}		$\sim 10^{-56}$ [cm] $= 10^{-43}$ electron radii

In spite of the singularity in the metric coefficient functions, spacetime is non singular at $r = 2M$:

A. The radial free fall proper time to $r = 2M$ is finite
B. The proper distance to $r = 2M$ is finite
C. The tidal forces at $r = 2M$ are finite.

A. The radial free fall time is governed the equations of motion $\left(\frac{dx^\mu}{d\tau} = \frac{1}{2m}\frac{\partial \mathcal{H}}{\partial p_\mu}\right)$ of a particle without any angular momentum

$$\frac{dr}{d\tau} = \sqrt{\frac{E^2}{m^2} - 1 + \frac{2M}{r}}$$

$$\frac{dt}{d\tau} = \sqrt{\frac{E^2}{m^2}}\,\frac{1}{1-\frac{2M}{r}}$$

They imply that

$$\text{proper time} = \tau = \int^{2M} \frac{dr}{\sqrt{\frac{E^2}{m^2} - 1 + \frac{2M}{r}}} = \text{finite} < \infty$$

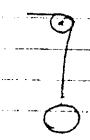
By contrast, the Schwarzschild time,

$$t = \sqrt{\frac{E^2}{m^2}} \int^{2M} \frac{1}{1-\frac{2M}{r}} \frac{dr}{\sqrt{\frac{E^2}{m^2} - 1 + \frac{2M}{r}}} = \infty,$$

yields an infinite result.

This illustrates the fact that proper time for a distant observer (r = very large) proceeds as a rate different from the proper time of a observer falling freely towards $r = 2M$. In particular one sees that the finite amount of proper time (τ) it takes to cross the Schwarzschild radius $r = 2M$ takes an infinite amount of time (t) as seen by a distant observer.

B. The amount of rope it takes to reach the Schwarzschild radius is finite:



$$ds = \int^{2M} \frac{dr}{\sqrt{1-\frac{2M}{r}}} = \text{finite}$$

C. At $r = 2M$ the components of the Riemann tensor, the tidal forces, are finite relative to any orthonormal frame moving radially.

Relative to the static Schwarzschild frame with its orthonormal basis

$$\left\{\omega^t = \left(1-\frac{2M}{r}\right)^{1/2} dt,\ \omega^r = \left(1-\frac{2M}{r}\right)^{-1/2} dr,\ \hat{\omega}^\theta = r\,d\theta,\ \hat{\omega}^\varphi = r\sin\theta\, d\varphi\right\}$$

the component of the Riemann tensor are

$$\boxed{\begin{aligned} &\hat{R}_{rtrt} = \frac{2M}{r^3},\ \hat{R}_{\theta t\theta t} = \hat{R}_{\varphi t\varphi t} = -\frac{M}{r^3} \\ &\hat{R}_{\theta\varphi\theta\varphi} = -\frac{2M}{r^3},\ \hat{R}_{\theta r\theta r} = \hat{R}_{\varphi r\varphi r} = \frac{M}{r^3} \end{aligned}}$$

Relative to an instantaneous frame which is moving into the radial direction with four velocity

$$\{\hat{v}^\mu\} = \{\hat{v}^t, \hat{v}^r, \hat{v}^\theta, \hat{v}^\varphi\} = \{\gamma, \beta\gamma, 0, 0\}$$

and hence is related to the static frame by the Lorentz transformation

$$\Lambda = \begin{array}{c} \begin{array}{cccc} t & r & \theta & \varphi \end{array} \\ \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ t \\ r \\ \theta \\ \varphi \end{array}$$

The Riemann components are given by (suppressing the ;

$$\begin{aligned} R_{\tau\rho\tau\rho} &= R_{\alpha\beta\gamma\delta}\Lambda^\alpha{}_\tau \Lambda^\beta{}_\rho \Lambda^\gamma{}_\tau \Lambda^\delta{}_\rho \\ &= R_{trtr}\Lambda^t{}_\tau \Lambda^r{}_\rho \Lambda^t{}_\tau \Lambda^r{}_\rho \\ &\quad + R_{rttr}\Lambda^r{}_\tau \Lambda^t{}_\rho \Lambda^t{}_\tau \Lambda^r{}_\rho \\ &\quad + R_{trrt}\Lambda^t{}_\tau \Lambda^r{}_\rho \Lambda^r{}_\tau \Lambda^t{}_\rho \\ &\quad + R_{rtrt}\Lambda^r{}_\tau \Lambda^t{}_\rho \Lambda^r{}_\tau \Lambda^t{}_\rho \\ &= R_{trtr}\left(\gamma^2\gamma^2 - \beta^2\gamma^2\gamma^2 - \gamma^2\beta^2\gamma^2 + \beta^2\gamma^2\beta^2\gamma^2\right) \\ &= R_{trtr}\,\gamma^4(1-2\beta^2+\beta^4) \\ &= R_{trtr}\,\gamma^4(1-\beta^2)^2 = R_{trtr}\,1 = \frac{2M}{r^3} \end{aligned}$$

Similarly

$$R_{\theta t\theta t} = R_{\theta t\theta t} \qquad R_{\varphi t\varphi t} = R_{\varphi t\varphi t} = -\frac{M}{r^3}$$

$$R_{\theta\rho\theta\rho} = R_{\theta r\theta r} \qquad R_{\varphi\rho\varphi\rho} = R_{\varphi r\varphi r} = \frac{M}{r^3}$$

Conclusion: Regardless of the radial velocity, of the frame, the components of Riemann stay finite even as the frame approaches the Schwarzschild radius $2M$,

I. Spacelike geometry of Schwarzschild spacetime is best visualized by an Imbedding Diagram.

For

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Focus attention on

$t = \text{const}$

$\theta = \frac{\pi}{2}$, the equatorial plane.

$$ds^2 = \frac{d\bar{r}^2}{1-\frac{2M}{r}} + \bar{r}^2 d\bar{\varphi}^2$$

$$= d\bar{z}^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\varphi}^2$$

For the metric to be on a surface $\bar{z}(\bar{r})$ we must have

$$\left(\frac{d\bar{z}}{d\bar{r}}\right)^2 + 1 = \frac{1}{1-\frac{2M}{r}} \Rightarrow \bar{z} = \pm\sqrt{8M(r-2M)}$$