Lecture 3.6

Schwarzschild spacetime.

Regular behaviour of proper time, proper distance, and curvature at the Schwarzschild radius \( r = 2M \).

[MTW §31.2]
Schwarzschild spacetime: Geometry and Dynamics

I. Geometry

The most important solution of the Einstein field equations is the one discovered by Karl Schwarzschild:

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \]

The constant

\[ 2M = \frac{G}{c^2} 2\text{Me} \approx \frac{1}{15 \, 000 \, 000} \frac{1}{10^{21}} 2\text{Me} \ 	ext{[c.g.s. units]} \]

\[ = 1.3 \times 10^{-28} \text{Me} \ 	ext{[cm]} \]

is the Schwarzschild radius. Typical examples are:

<table>
<thead>
<tr>
<th></th>
<th>Me [g]\r</th>
<th>(1.33 \times 10^{-28}) cm [g]\r</th>
<th>= 2M</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUN</td>
<td>(2 \times 10^{33})</td>
<td>2.7 [km]</td>
<td></td>
</tr>
<tr>
<td>GALAXY</td>
<td>(2 \times 10^{44})</td>
<td>2.7 \times 10^{10} [km]</td>
<td></td>
</tr>
<tr>
<td>EARTH</td>
<td>(6 \times 10^{27})</td>
<td>\approx 1 [cm]</td>
<td></td>
</tr>
<tr>
<td>MOON</td>
<td>(7.3 \times 10^{25})</td>
<td>(10^{-2}) [cm]</td>
<td></td>
</tr>
<tr>
<td>ELECTRON</td>
<td>(9 \times 10^{-28})</td>
<td>(\approx 10^{-56}) [cm] = (10^{-43}) electron radius</td>
<td></td>
</tr>
</tbody>
</table>
In spite of the singularity in the metric coefficient functions, spacetime is non-singular at $r = 2M$.

A. The radial free fall proper time to $r = 2M$ is finite.

B. The proper distance to $r = 2M$ is finite.

C. The tidal forces at $r = 2M$ are finite.

A. The radial free fall time is governed by the equations of motion $(\frac{dx}{dt} = \frac{1}{\sqrt{E^2 - 1 + \frac{2M}{r}}})$ of a particle without any angular momentums.

$$\frac{dr}{dt} = \sqrt{\frac{E^2 - 1 + \frac{2M}{r}}{m^2}}$$

$$\frac{dr}{dt} = \sqrt{\frac{E^2 - 1 + \frac{2M}{r}}{m^2}}$$

They imply that

proper time $\tau = \int \frac{dr}{\sqrt{E^2 - 1 + \frac{2M}{r}}} = \text{finite} < \infty$

By contrast, the Schwarzschild time:

$$T = \sqrt{\int_{m}^{2M} \frac{dM}{\sqrt{E^2 - 1 + \frac{2M}{r}}}} = \infty$$

yields an infinite result.

This illustrates the fact that proper time for a distant observer ($r$ very large) proceeds at a rate different from the proper time of an observer falling freely towards $r = 2M$, in particular one sees that the finite amount of proper time it takes to cross the Schwarzschild radius $r = 2M$ takes an infinite amount of time ($t$) as seen by a distant observer.

B. The amount of rope it takes to reach the Schwarzschild radius is finite:

$$ds = \int \frac{2M}{\sqrt{1 - \frac{2M}{r}}} dr = \text{finite}$$

C. At $r = 2M$ the components of the Riemann tensor, the tidal forces are finite relative to any orthonormal frame moving radially.
Relative to the static Schwarzschild frame with its orthonormal basis
\[ \{ \omega^t = (-\frac{2M}{r^2}) dt, \omega^\tau = (-\frac{2M}{r^2}) d\tau, \omega^\phi = r d\phi, \omega^\theta \text{ r.m.s.} \} \]
the components of the Riemann tensor are
\[ \hat{R}^\tau_{\tau r t} = \frac{2M}{r^3}, \quad \hat{R}^\tau_{\tau t} = \hat{R}^\tau_{t \tau r} = \frac{M}{r^3} \]
\[ \hat{R}^{\phi \phi}_{\theta \theta} = -\frac{2M}{r^3}, \quad \hat{R}^{\phi \phi}_{\phi \phi} = \hat{R}^{\phi \phi}_{\phi \phi} = \frac{M}{r^3} \]

Relative to an instantaneous frame which is moving into the radial direction with four velocity
\[ \{ \hat{v}^\alpha = \{ \hat{v}^t, \hat{v}^\tau, \hat{v}^\phi, \hat{v}^\theta \} = \{ \gamma, 0, \beta \gamma, 0 \} \]
and hence is related to the static frame by the Lorentz transformation
\[ \Lambda = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ \frac{\beta \gamma}{\gamma} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

The Riemann components are given by (expressions:
\[ R_{\tau \tau r t} = R_{\tau \tau t} \Lambda^\tau_{\tau} \Lambda^r_{r} \Lambda^t_{t} \Lambda^\tau_{\tau} \Lambda^\phi_{\phi} \]
\[ = R_{\tau \tau r t} \Lambda^\tau_{\tau} \Lambda^r_{r} \Lambda^t_{t} \Lambda^\tau_{\tau} \Lambda^\phi_{\phi} \]
\[ + R_{\tau \tau r t} \Lambda^\tau_{\tau} \Lambda^r_{r} \Lambda^t_{t} \Lambda^\tau_{\tau} \Lambda^\phi_{\phi} \]
\[ + R_{\tau \tau r t} \Lambda^\tau_{\tau} \Lambda^r_{r} \Lambda^t_{t} \Lambda^\tau_{\tau} \Lambda^\phi_{\phi} \]
\[ + R_{\tau \tau r t} \Lambda^\tau_{\tau} \Lambda^r_{r} \Lambda^t_{t} \Lambda^\tau_{\tau} \Lambda^\phi_{\phi} \]

\[ = R_{\tau \tau r t} \left( \gamma^2 \gamma^2 - \beta^2 \gamma^2 - \gamma^2 \beta^2 \gamma^2 + \beta^2 \gamma^2 \gamma^2 \right) \]
\[ = R_{\tau \tau r t} \gamma^4 (1 - 2\beta^2 + \beta^4) \]
\[ = R_{\tau \tau r t} \gamma^4 (1 - \beta^2)^2 = R_{\tau \tau r t} \tau = \frac{2M}{r^3} \]

Similarly
\[ R_{\tau \tau t} = R_{\tau \tau t} \Lambda^\tau_{\tau} \Lambda^t_{t} \Lambda^\tau_{\tau} \Lambda^\phi_{\phi} \]
\[ R_{\phi \phi \phi} = R_{\phi \phi \phi} \Lambda^\phi_{\phi} \Lambda^\phi_{\phi} \Lambda^\phi_{\phi} \Lambda^\phi_{\phi} \]

Conclusion: Regardless of the radial velocity, of the frame, the components of the Riemann tensor stay finite even as the static frame approaches the Schwarzschild radius 2M,
In spacelike geometry of Schwarzschild spacetime is best visualized by an embedding diagram.

For
\[ ds^2 = -(\frac{-2M}{r}) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
Focus attention on
\[ t = \text{const} \]
\[ \theta = \frac{\pi}{2}, \text{ the equatorial plane} \]
\[ ds^2 = \frac{d\bar{r}^2}{1 - \frac{2M}{r}} + \bar{r}^2 d\bar{\varphi}^2 \]
For the metric to be on a surface \( \bar{z}(\bar{r}) \) we must have
\[ \left( \frac{d\bar{z}}{d\bar{r}} \right)^2 + 1 = \frac{1}{1 - \frac{2M}{r}} \Rightarrow \bar{z} = \pm \sqrt{8M(r-2M)} \]