Lecture 37

Schwarzschild spacetime: Dynamics

Lightcone structure near $r=2M$;

Eddington-Finkelstein coordinates

[MTW §31.4, Box 31.2.]

Kruskal-Szekeres coordinates

[MTW §31.5]

II. Dynamics

The causal structure of the Schwarzschild solution manifests itself through the shape and distribution of light cones through out spacetime.

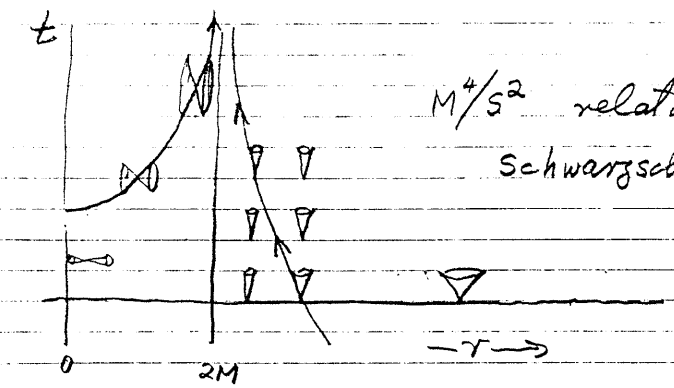
Because of spherical symmetry it is sufficient to set $\theta = \text{const}$, $\varphi = \text{const}$ and consider only the causal structure on $M^2 = M^4/S^2$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}$$

Light cones are generated by tangents to photon world lines.

$(\Delta s)^2 = 0$: $\frac{dr}{dt} = \left(1-\frac{2M}{r}\right)$ (outgoing)

or $\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right)$ (ingoing)



M^4/S^2 relative to Schwarzschild Coordinates.

The problem with the Schwarzschild coordinates is that the time coordinate t is a bad coordinate. It prevents geodesics from being continued across the locus of events $r=2M$. Geodesics ingoing from the outside $(r>2M)$, as well those outgoing from the inside $(r<2M)$ converge towards $(t=\infty, r=2M)$.

Figuratively speaking, the time coordinate pulls geodesics in a finite amount of proper time towards $t=\infty$ off the Schwarzschild (t, r) coordinate chart.

This deficiency can be overcome by straightening out the null geodesics, which generate the null cone structure

(i) Straightening out the ingoing null geodesics implies the introduction of the ingoing Finkelstein-Eddington coordinates. They cover two Schwarzschild neighborhoods.

ii) Straightening out the <u>outgoing</u> null geodesics implies the introduction of the <u>outgoing</u> <u>Finkelstein</u>-<u>Eddington</u> coordinates. They also cover two Schwarzschild neighborhoods.

iii) The introduction of the Kruskal-Szekeres coordinates yields a global coordinate chart which covers the whole inextendible Schwarzschild space time. The Kruskal chart is maximal: it has no nonsingular boundaries across which one can enlarge the spacetime by introducing additional coordinate chart(s).
The K-S coordinates also straighten out ingoing and outgoing radial null geodesics.

(iv) One can also straighten out radial timelike geodesics. This gives rise to the maximal Novikov coordinate chart. See MTW §31.4

We shall only discuss (i) and (iii).

(i) <u>Ingoing</u> <u>Eddington</u>-<u>Finkelstein</u> Coordinates.

In order to straighten out the ingoing radial photon worldlines we integrate their differential equation

$$(\Delta s)^2 = 0 \; : \quad \frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)$$

By separating variables one obtains

$$dt = \pm \frac{r\,dr}{r-2M}$$

whose solution is

$$t = r + 2M \ln\left(\frac{r}{2M} - 1\right) + \tilde{U}$$

$$t = -r - 2M \ln\left(\frac{r}{2M} - 1\right) + \tilde{V}$$

Here $\tilde{V}$ = constant is a coordinate surface containing ingoing photons. This coordinate function is called the "advanced time". The coordinates

$$(\tilde{V}, r)$$

are the <u>ingoing</u> Eddington-Finkelstein coordinates.

The coordinates $(\tilde{U}, r)$ are the outgoing E-F coordinates

a) Metric Relative to Eddington–Finkelstein coordinates

By implementing the coordinate transformation

$$dr = dr'$$

$$dt = -\frac{dr'}{1-\frac{2M}{r}} + d\tilde{V}$$

Relative to these coordinates the metric tensor assumes the Eddington–Finkelstein form

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right] + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -\left(1-\frac{2M}{r}\right)\left(dt + \frac{dr}{1-\frac{2M}{r}}\right)\left(dt - \frac{dr}{1-\frac{2M}{r}}\right) + r^2(\quad)$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}\left(\frac{-2dr}{1-\frac{2M}{r}} + d\tilde{V}\right) + r^2(\quad)$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}^2 + 2d\tilde{V}dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The utility of using these coordinates is that in this straightening out process we have enlarged the spacetime domain of the coordinate chart.

This enlargement becomes evident when one exhibits the null cone structure.

b) Spacetime Picture

In order to make the ingoing ($\tilde{V} = \text{const}$) photon worldlines appear as 45° lines, we introduce the timelike coordinate

$$V = \tilde{V} - r$$

and hence

$$d\tilde{V} = dV + dr.$$

The metric

$$ds^2 = -d\tilde{V}(d\tilde{V} - 2dr) + \frac{2M}{r} d\tilde{V}^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

becomes for these ingoing E-F coordinates

$$ds^2 = -(dV + dr)(dV - dr) + \frac{2M}{r}(dV + dr)^2 + r^2\, d\Omega^2$$

Comment:

Had we introduced the outgoing E-F coordinate $\tilde{U}$,

$$dt = \frac{dr}{1-\frac{2M}{r}} - d\tilde{U}, \quad \text{and then } d\tilde{U} = dU - dr,$$

the metric would have had the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)d\tilde{U}^2 - 2d\tilde{U}dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -$$

$$= -d\tilde{U}(d\tilde{U}+2dr)+\frac{2M}{r}d\tilde{U}^2 + r^2 d\Omega^2$$

$$= -(dU-dr)(dU+dr)+\frac{2M}{r^2}(dU-dr)^2 + r^2 d\Omega^2$$

End of Comment.

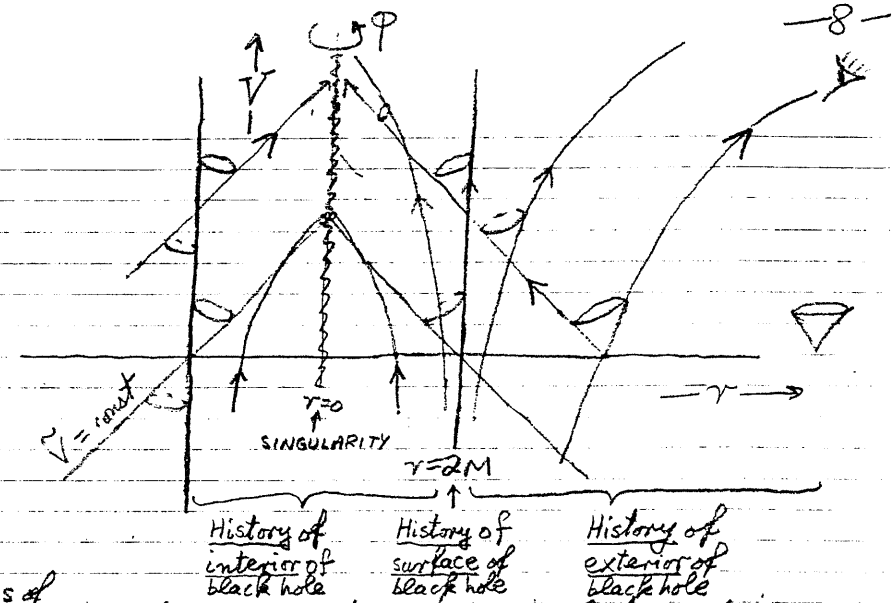
The null cone structure is determined by the condition

$$(\Delta s)^2 = 0$$

Relative to the ingoing E-F coordinates (V, r) one therefore obtain the radially directed tangents to the null cones

$$\boxed{\frac{dV}{dr} = -1} \; ; \; \boxed{\frac{dV}{dr} = \frac{1+\frac{2M}{r}}{1-\frac{2M}{r}} = \frac{r+2M}{r-2M}}$$

The corresponding null cone structure is therefore as follows



Dynamics of a Black Hole: Light Cone Structure Relative to ingoing E-F coordinates

Note that

(i) To obtain picture relative to outgoing E-F, turn the picture upside down.

(ii) Once inside $r = 2M$, all particle (timelike) and photon (null) geodesic terminate at $r = 0$, a natural boundary of spacetime where the curvature becomes singular.