Lecture 38-39-40

Interior vs exterior domain of a blackhole mathematized by means of the Finkelstein Coordinates

Global Rindler coordinate atlas as a guide to the globally defined Kruskal Szekeres coordinate atlas (MTW § 31.5)

Event Horizons via Finkelstein Rindler and Kruskal charts

II. Dynamics

The causal structure of the Schwarzschild solution manifests itself through the shape and distribution of light cones through out spacetime.

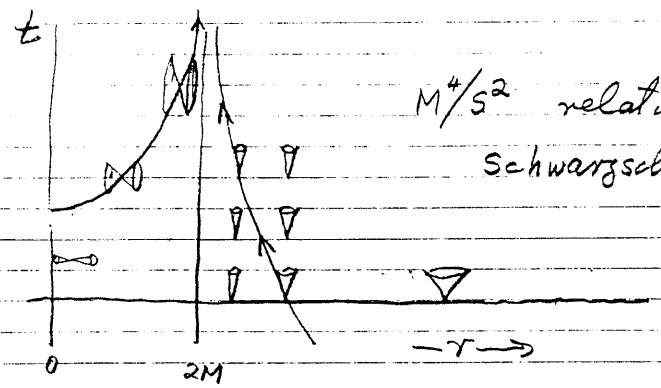
Because of spherical symmetry it is sufficient to set $\theta = \text{const}$, $\varphi = \text{const}$ and consider only the causal structure on $M^2 = M^4/S^2$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}$$

Light cones are generated by tangents to photon world lines.

$(\Delta s)^2 = 0$: $\frac{dr}{dt} = \left(1-\frac{2M}{r}\right)$ (outgoing)

or $\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right)$ (ingoing)



M^4/S^2 relative to Schwarzschild Coordinates.

The problem with the Schwarzschild coordinates is that the time coordinate t is a bad coordinate. It prevents geodesics from being continued across the locus of events $r = 2M$. Geodesics ingoing from the outside $(r > 2M)$, as well those outgoing from the inside $(r < 2M)$ converge towards $(t = \infty, r = 2M)$.

Figuratively speaking, the time coordinate pulls geodesics in a finite amount of proper time towards $t = \infty$ off the Schwarzschild (t, r) coordinate chart.

This deficiency can be overcome by straightening out the null geodesics, which generate the null cone structure

(i) Straightening out the <u>ingoing</u> null geodesics implies the introduction of the <u>ingoing</u> <u>Finkelstein</u>-<u>Eddington</u> coordinates. They cover two Schwarzschild neighborhoods.

(ii) Straightening out the outgoing null geodesics implies the introduction of the outgoing Finkelstein-Eddington coordinates. They also cover two Schwarzschild neighborhoods.

(iii) The introduction of the Kruskal-Szekeres coordinates yields a global coordinate chart which covers the whole inextendible Schwarzschild space time. The Kruskal chart is maximal: it has no nonsingular boundaries across which one can enlarge the spacetime by introducing additional coordinate chart(s).

The K-S coordinates also straighten out ingoing and outgoing radial null geodesics.

(iv) One can also straighten out radial timelike geodesics. This gives rise to the maximal Novikov coordinate chart. See MTW §31.4

We shall only discuss (i) and (iii).

(i) Ingoing Eddington-Finkelstein Coordinates

In order to straighten out the ingoing radial photon worldlines we integrate their differential equation

$$(\Delta s)^2 = 0 \; : \quad \frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)$$

By separating variables one obtains

$$dt = \pm \frac{r\,dr}{r-2M}$$

whose solution is

$$t = r + 2M \ln\left(\frac{r}{2M} - 1\right) + \tilde{U}$$

$$t = -r - 2M \ln\left(\frac{r}{2M} - 1\right) + \tilde{V}$$

Here $\tilde{V}$ = constant is a coordinate surface containing ingoing photons. This coordinate function is called the "advanced time". The coordinates

$$(\tilde{V}, r)$$

are the ingoing Eddington-Finkelstein coordinates.

The coordinates $(\tilde{U}, r)$ are the outgoing E-F coordinates

a) Metric Relative to Eddington–Finkelstein coordinates

By implementing the coordinate transformation

$$dr = dr'$$

$$dt = -\frac{dr'}{1-\frac{2M}{r}} + d\tilde{V}$$

Relative to these coordinates the metric tensor assumes the Eddington–Finkelstein form

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$= -\left(1-\frac{2M}{r}\right)\left(dt + \frac{dr}{1-\frac{2M}{r}}\right)\left(dt - \frac{dr}{1-\frac{2M}{r}}\right) + r^2(\quad)$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}\left(\frac{-2dr}{1-\frac{2M}{r}} + d\tilde{V}\right) + r^2(\quad)$$

$$= -\left(1-\frac{2M}{r}\right) d\tilde{V}^2 + 2d\tilde{V}dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The utility of using these coordinates is that in this straightening out process we have enlarged the spacetime domain of the coordinate chart.

This enlargement becomes evident when one exhibits the null cone structure.

b) Spacetime Picture

In order to make the ingoing ($\tilde{V} = \text{const}$) photon worldlines appear as 45° lines, we introduce the timelike coordinate

$$V = \tilde{V} - r$$

and hence

$$d\tilde{V} = dV + dr.$$

The metric

$$ds^2 = -d\tilde{V}\left(d\tilde{V} - 2dr\right) + \frac{2M}{r} d\tilde{V}^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

becomes for these <u>ingoing</u> E-F coordinates

$$ds^2 = -(dV+dr)(dV-dr) + \frac{2M}{r}(dV+dr)^2 + r^2\, d\Omega^2$$

Comment:

Had we introduced the outgoing E-F coordinate $\tilde{U}$,

$$dt = \frac{dr}{1-\frac{2M}{r}} - d\tilde{U}, \text{ and then } d\tilde{U} = dU - dr,$$

the metric would have had the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)d\tilde{U}^2 - 2d\tilde{U}dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$= -$$

$$= -d\tilde{U}(d\tilde{U}+2dr)+\frac{2M}{r}d\tilde{U}^2 + r^2 d\Omega^2$$

$$= -(dU-dr)(dU+dr)+\frac{2M}{r^2}(dU-dr)^2 + r^2 d\Omega^2$$

End of Comment.

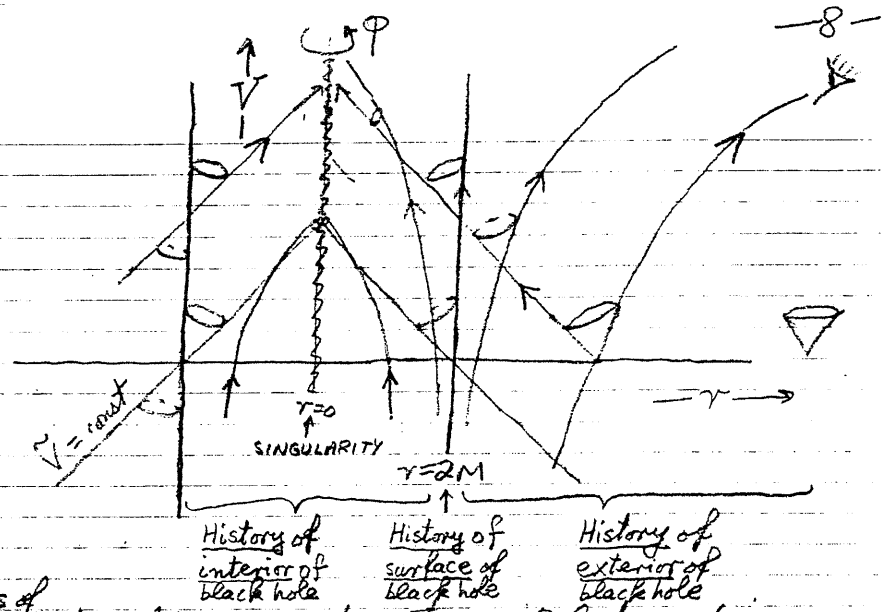
The null cone structure is determined by the condition

$$(\Delta s)^2 = 0$$

Relative to the ingoing E-F coordinates (V, r) one therefore obtain the radially directed tangents to the null cones

$$\boxed{\frac{dV}{dr} = -1} \; ; \; \boxed{\frac{dV}{dr} = \frac{1+\frac{2M}{r}}{1-\frac{2M}{r}} = \frac{r+2M}{r-2M}}$$

The corresponding null cone structure is therefore as follows



Dynamics of a Black Hole: Light Cone Structure Relative to ingoing E-F coordinates

Note that

(i) To obtain picture relative to outgoing E-F, turn the picture upside down.

(ii) Once inside $r=2M$, all particle (timelike) and photon (null) geodesic terminate at $r=0$, a natural boundary of spacetime where the curvature becomes singular.

Kruskal-Szekeres Coordinates: a Maximal Chart.

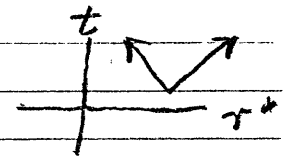
The Schwarzschild space-time has a metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

whose radial null lines can be straightened out easily by introducing the Regge-Wheeler ("tortoise") coordinate r^*:

$$dr^* \equiv \frac{dr}{1-\frac{2M}{r}} \; ; \; r^* = r + 2M\ln\left(\frac{r}{2M}-1\right)$$

In terms of this coordinate, the radial null lines are straight because the metric has the form



$$ds^2 = -\left(1-\frac{2M}{r(r^*)}\right)\left[dt^2 - dr^{*2}\right] + r^2(r^*)\left(d\theta^2 + \sin\theta^2 d\phi^2\right)$$

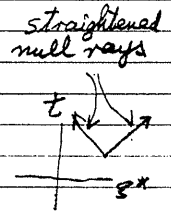
The metric of flat Minkowski space time relative to Rindler coordinates (t and ξ)

$$ds^2 = -\xi^2\, dt^2 + d\xi^2 + dY^2 + dZ^2$$

has an analogous feature. By introducing what for Schwarzschild spacetime

corresponds to the "tortoise" coordinate

$$\frac{d\xi}{\xi} = d\xi^* \; ; \quad \xi = e^{\xi^*},$$



the metric becomes

$$ds^2 = -e^{\xi^*}\left[dt^2 - d\xi^{*2}\right] + dY^2 + dZ^2$$

Thus photon worldlines of light rays along the direction of acceleration have been straightened out, just like the photon worldlines of radial light rays in the Schwarzschild geometry have been straightened out.

Thus there is a qualitative correspondence between the null cone structure of radial light rays viewed relative to Schwarzschild coordinates and the null cone structure of light rays along the direction of acceleration in a Rindler coordinate frame.

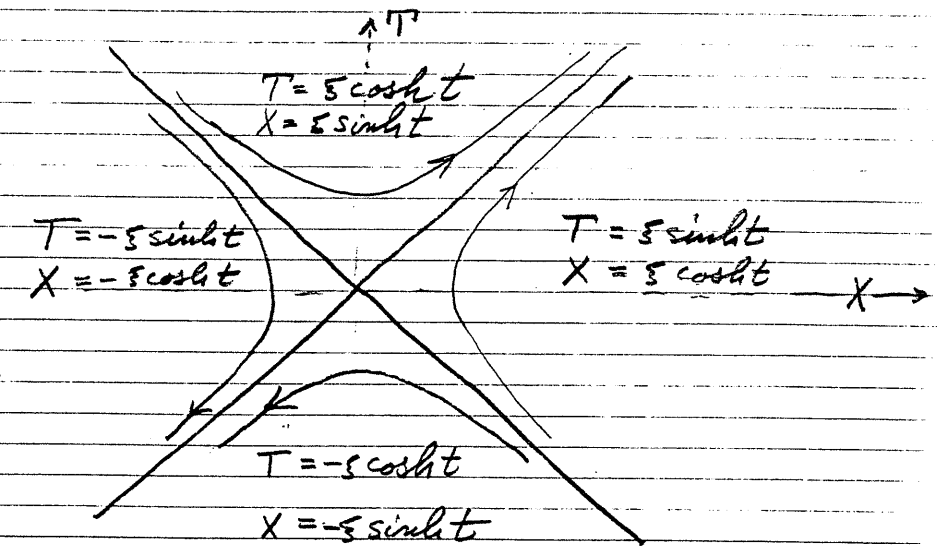
We know that a Rindler coordinate (t, ξ) frame

$$\left.\begin{aligned} T &= \xi \sinh t \\ X &= \xi \cosh t \end{aligned}\right\} \quad \begin{aligned} 0 < \xi < \infty \\ -\infty < t < \infty \end{aligned}$$

covers only the limited spacetime sector $|T| < X$ of Minkowski space time, whose metric is

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2.$$

The remaining three sectors are covered by related coordinate transformations



with $0 < \xi < 0$, $-\infty < t < \infty$ in each coordinate sector.

Given that the Schwarzschild coordinates (t, r) for $r > 2M$ play the same role that the Rindler coordinates (t, ξ) play for Minkowski spacetime, the natural question that arises is this: Does there exist a corresponding coordinate transformation, say

$$T = f(r) \sinh \alpha t$$
$$R = f(r) \cosh \alpha t,$$

which can be extended to a global spacetime in the same way that the Rindler coordinates can be extended to Minkowski spacetime?

Such a coordinate system (T, R) can be shown to exist, if one can exhibit functions $f(r)$ and a constant α such that the metric tensor is nonsingular relative this new global (T, R) coordinate system.

The coordinate transformation from the inextendible Schwarzschild (t, r) coordinates -- inextendible because the metric is singular at $r = 2M$ -- to the to-be-constructed extendible Kruskal (T, R) coordinates is determined by the differential equation for $f(r)$. The computation is simple and is parallel to that for Minkowski spacetime:

$$dT = f'(r) \sinh \alpha t \, dr + \alpha f(r) \cosh t \, dt$$
$$dR = f'(r) \cosh \alpha t \, dr + \alpha f(r) \sinh t \, dt$$

$$dR^2 - dT^2 = (f')^2 dr^2 - f^2 \alpha^2 dt^2$$

Because it is not flat, the Schwarschild metric has a form for its time and radial part which is given by

$$-h(R,T)\left[dT^2-dR^2\right]=ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}.$$

In order that the (T,R) coordinates be extendible we require that h be continuous at $r=2M$, i.e.

$$\boxed{\lim_{r\to 2M} h = \text{finite and nonzero!}}$$

The transformation which relates (T,R) and (t,r) yields

$$-h\,f^2\alpha^2\left[dt^2-\left(\frac{f'}{\alpha f}\right)^2 dr^2\right]=-\left(1-\frac{2M}{r}\right)\left[dt^2-\frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right]$$

The equations implied by this requirement are

$$\boxed{\begin{aligned}\frac{f'}{f} &= \frac{\alpha}{1-\frac{2M}{r}} = \alpha\left(1+\frac{2M}{r-2M}\right) \qquad \text{for } 2M<r\\ h f^2\alpha^2 &= 1-\frac{2M}{r}\end{aligned}}$$

The last three boxed equations are three equations for $f(r)$, $h(R,T)$ and the constant α.

With an appropriate integration constant the solution for f is

$$f=\pm e^{\alpha r}\left(\frac{r-2M}{2M}\right)^{2M\alpha} \qquad \text{(dimensionless)}$$

The limit requirement on

$$h=\frac{1}{\alpha^2}\,\frac{r-2M}{r}\,\frac{(2M)^{4M\alpha}}{e^{2\alpha r}(r-2M)^{4M\alpha}}$$

demands that the only allowed value for α is

$$\boxed{\alpha=\frac{1}{4M}}$$

This must hold with razor sharp precision! If α deviates ever so slightly from the value $\frac{1}{4M}$, then $h(R,T)$ would either $\to 0$ or ∞ as $r\to 2M$.

For ~~on~~ this ~~the~~ value of α

$$f(r) = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2}$$

$$h(R,T) = \frac{32M^3}{r} e^{-r/2M}$$

The corresponding Kruskal-Szekeres form of the Schwarzschild metric is

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(dR^2 - dT^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The coordinate transformations giving rise to this form are

$$T = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \sinh \frac{t}{4M} \qquad 2M < r$$

$$R = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \cosh \frac{t}{4M}$$

These coordinates are dimensionless, and they imply that

$$e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) = R^2 - T^2$$

Thus we see that, inspite of its r-dependence, the metric coefficient function $h(R,T)$ is a function of R and T; in fact, it is a function of $R^2 - T^2$.

The computed coordinate transformations for $r > 2M$ are not enough because we need another pair which applies to $r < 2M$. To obtain it we repeat the same procedure: following the lead from a uniformly accelerated coordinate frame in Minkowski spacetime, we require that

$$T = g(r) \cosh \beta t$$

$$R = g(r) \sinh \beta t$$

$$dR^2 - dT^2 = (g')\, dr^2 - g^2 \beta^2\, dt^2$$

These are to apply in the spacetime region where $r < 2M$. This requirement, together with

$$\boxed{\lim_{r \to 2M} h = \text{finite and nonzero!}}$$

yields $\beta = \frac{1}{4M}$

$$g(r) = \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2}$$

$$h = \frac{32M^3}{r} e^{-\frac{r}{2M}}$$

Thus we again obtain another pair of (dimensionless) coordinates

$$\boxed{\begin{aligned} T &= \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \cosh \frac{t}{4M} \\ R &= \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \sinh \frac{t}{4M} \end{aligned}} \qquad r < 2M$$

relative to which the metric has the same desired form as before, namely

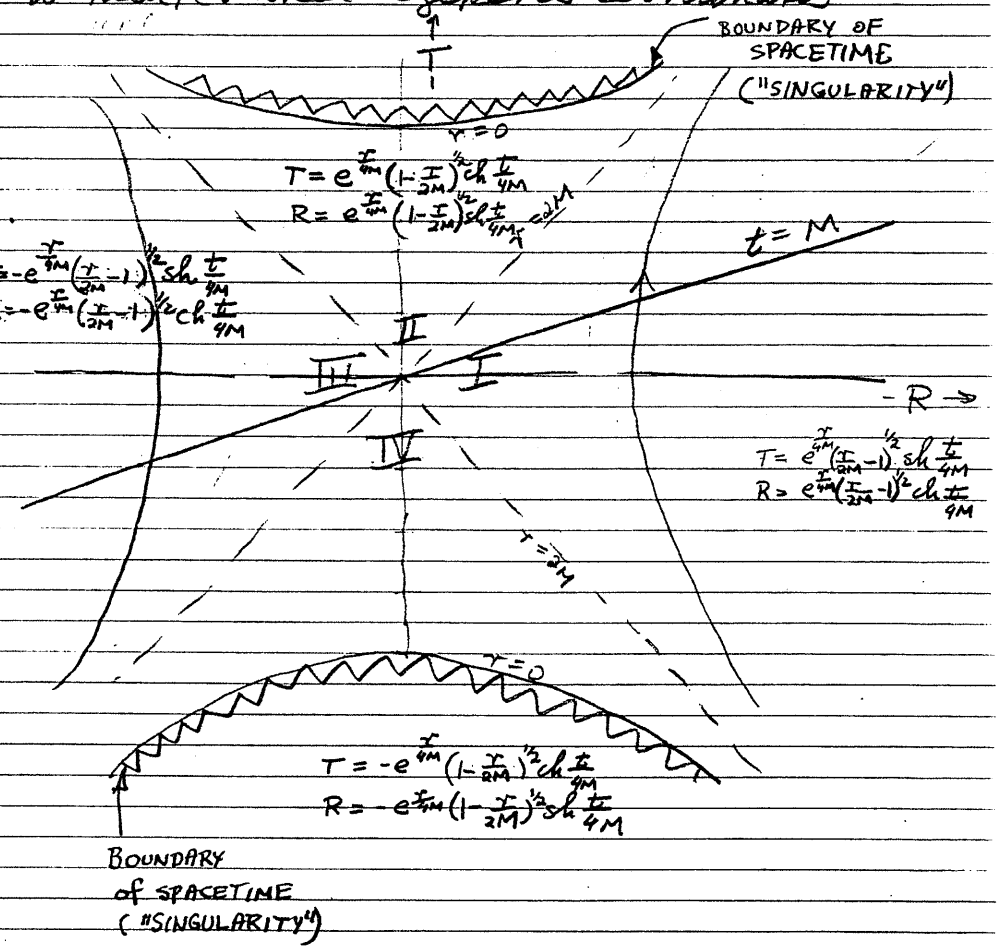
$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(dR^2 - dT^2\right) + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

This is the globally defined Kruskal-Szekeres form of the metric on the Schwarzschild spacetime. This form is analogous to the Minkowski form of the metric

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$$

for flat spacetime.

The global structure of Schwarzschild spacetime is most transparent relative to these Kruskal Szekeres coordinates



The Schwarzschild spacetime has an <u>interior</u> region, $r<2M$, and an <u>exterior</u> region, $r>2M$. These regions are different geometrically and physically:

1. The Schwarzschild coordinates (t, r) yield hyperbolas and straight lines relative to the K-S coordinates (T, R). Indeed in the exterior

$$R^2 - T^2 = e^{\frac{r}{2M}}\left(\frac{r}{2M} - 1\right) \qquad 2M<r \text{ : in I and III}$$

$$\frac{T}{R} = \tanh\frac{t}{4M}$$

while in the interior

$$T^2 - R^2 = e^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right) \qquad r<2M \text{ : in II and IV}$$

$$\frac{T}{R} = \coth\frac{t}{4M}$$

2. The exterior $(2M<r)$ is <u>static</u>, while the interior $(r<2M)$ is <u>dynamic</u>: Indeed, for $r>2M$ there exist timelike (observer) worldlines, namely

$$r = \text{const},$$

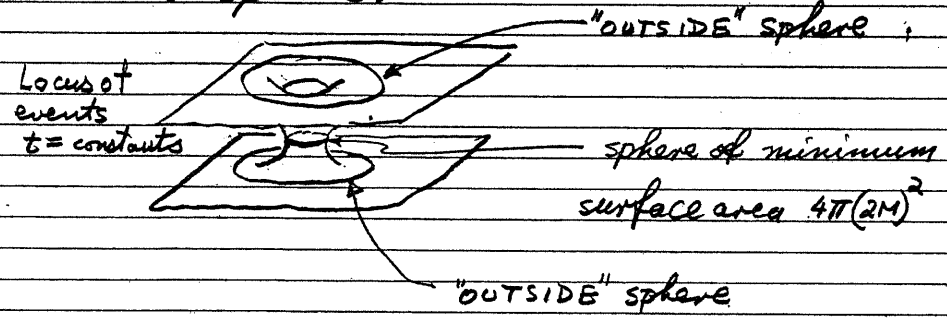
relative to which the metric

is static, i.e independent of t.

However for $r<2M$, in the interior, there do <u>not</u> exist any timelike worldlines relative to which the metric is static. For example, $r = \text{const}$, are spacelike curves which do not represent the worldline of any observer.

3. The geometry on each of the spacelike hypersurfaces $t = c$, $-\infty < c < \infty$, is that of a statatic Schwarzschild throat. Each hypersurface $t = \text{constant}$ consists of a sequence of nested spheres. Their surface area $4\pi r^2$ is not a monotonic function of their proper separation. Instead there is a sphere of minimum area $4\pi(2M)^2 = 16\pi M^2$. <u>This sphere</u>, as well as all the others, <u>have no center.</u>

The spheres on one side of this minimum area sphere, as well as those on its other side, form a sequence with ever increasing surface area. Both sequences of spheres are on the "outside" of the minimum area sphere.



Together the two sequences of spheres form a three-dimensional passage ("Schwarzschild throat") between two asymptotically flat spatial regions.

(i) One sees from the Kruskal diagram that these two regions are causally disjoint: an event in one region can not influence

or be influenced by an event in the other region.

(ii) One also sees that if one follows the evolution of this Schwarzschild throat on successive hypersurfaces $T = \text{const}$ over the coordinate time interval $0 \leq T \leq 1$, the throat evolves from its (maximum) area $16\pi M^2$ at $T=0$ to zero area at $T=1$; in other words the throat pinches off in a finite interval of time.