Interior vs exterior domain of a black hole, mathematized by means of the Finkelstein coordinates.

Global Rindler coordinate atlas as a guide to the globally defined Kruskal-Szekeres coordinate atlas (MTW § 31.5).

Event Horizons Via Finkelstein-Rindler and Kruskal Charts.
II. Dynamics

The causal structure of the Schwarzschild solution manifests itself through the shape and distribution of light cones through out spacetime.

Because of spherical symmetry, it is sufficient to set \( \theta = \text{const} \), \( \phi = \text{const} \) and consider only the causal structure on \( M^2 = M^2/c^2 \)

\[ ds^2 = -(1 - \frac{2M}{r}) \, dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

Light cones are generated by tangents to photon world lines.

\[ (ds)^2 = 0 : \quad \frac{dr}{dt} = \left( -\frac{2M}{r} \right) \, (\text{outgoing}) \]

\[ \quad \frac{dt}{dr} = \left( -\frac{2M}{r} \right) \, (\text{ingoing}) \]

The problem with the Schwarzschild coordinates is that the time coordinate \( t \) is a bad coordinate. It prevents geodesics from being continued across the locus of events \( r = 2M \). Geodesics ingoing from the outside \( (r > 2M) \) as well those ingoing from the inside \( (r < 2M) \) converge towards \( (t = \infty, r = 2M) \).

Figuratively speaking, the time coordinate pulls geodesics in a finite amount of proper time towards \( t = \infty \) off the Schwarzschild \((t, r)\) coordinate chart.

This deficiency can be overcome by straightening out the null geodesics, which generate the null cone structure.

(i) Straightening out the ingoing null geodesics implies the introduction of the ingoing Finkelstein-Eddington coordinates. They cover two Schwarzschild neighborhoods.
(i) Straightening out the outgoing null geodesics implies the introduction of the outgoing Eddington-Finkelstein coordinates. They also cover two Schwarzschild neighborhoods.

(ii) The introduction of the Kruskal-Szekeres coordinates yields a global coordinate chart which covers the whole inextendible Schwarzschild spacetime. The Kruskal chart is maximal; it has no nonsingular boundaries across which one can enlarge the spacetime by introducing additional coordinate charts.

The K-S coordinates also straighten out ingoing and outgoing radial null geodesics. One can also straighten out radial timelike geodesics, this gives rise to the maximal Novikov coordinate chart. See MTW §3.14

We shall only discuss (i) and (ii).

(iii) **Ingoing Eddington-Finkelstein Coordinates**

In order to straighten out the ingoing radial photon worldlines we integrate their differential equation

\[(\Delta s)^2 = 0: \quad \frac{dr}{dt} = \pm \left(-\frac{2M}{r}\right)\]

By separating variables one obtains

\[dt = \pm \frac{r dr}{r - 2M}\]

whose solution is

\[t = r - 2M \ln \left(\frac{r - 2M}{2M}\right) + \tilde{t}\]

\[t = -r + 2M \ln \left(\frac{2M}{r - 2M}\right) + \tilde{t}\]

Here \(\tilde{t}\) = constant is a coordinate surface containing a photon. This coordinate function is called the "advanced" time. The coordinates \((\tilde{t}, r)\) are the ingoing Eddington-Finkelstein coordinates.

The coordinates \((\tilde{t}, r)\) are the outgoing E-F coordinates.
a) Metric Relative to Eddington–Finkelstein coordinates

By implementing the coordinate transformation

\[ dr = dr' \]

\[ dt = - \frac{dr^2}{1 - \frac{2m}{r}} + d\tilde{v} \]

Relative to these coordinates the metric tensor assumes the Eddington–Finkelstein form

\[ ds^2 = -(1 - \frac{2m}{r}) \left[ dt^2 - \frac{dr^2}{(1 - \frac{2m}{r})^2} \right] + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ = -(1 - \frac{2m}{r}) \left[ dt + \frac{dr}{1 - \frac{2m}{r}} \right]^2 + r^2 \left( d\tilde{v}^2 + \frac{(d\theta^2 + \sin^2 \theta d\phi^2)}{1 - \frac{2m}{r}} \right) \]

The utility of using these coordinates is that in the straightening out process we have enlarged the spacetime domain of the coordinate chart.

This enlargement becomes evident when one exhibits the null cone structure.

b) Spacetime Picture

In order to make the ingoing (\( \tilde{v} = \text{const} \)) photon worldlines appear as 45° lines, we introduce the timelike coordinate \( \tilde{v}' = \tilde{v} - r \)

and hence

\[ d\tilde{v} = d\tilde{v}' + dr \]

The metric

\[ ds^2 = -d\tilde{v}' (d\tilde{v}' - 2d\tilde{r}) + \frac{2m}{\tilde{r}} d\tilde{v}'^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

becomes for these ingoing E–F coordinates

\[ ds^2 = -(d\tilde{v}' dr) (d\tilde{v}' - dr) + \frac{2m}{\tilde{r}} (d\tilde{v}' dr)^2 + r^2 ds^2 \]

Comment:
Had we introduced the outgoing E–F coordinate \( \tilde{v}' \)

\[ dt = \frac{dr}{1 - \frac{2m}{r}} - d\tilde{v} \]

and then \( d\tilde{v} = d\tilde{v}' - dr \)

the metric would have had the form

\[ ds^2 = -(1 - \frac{2m}{r}) d\tilde{v}'^2 - 2d\tilde{v}' dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
\[
= -\frac{\partial u}{(d \theta + 2d \varphi)^2} + \frac{2M}{\varrho^2} d \theta^2 + \varrho^2 d \varphi^2
\]

End of Comment.

The null cone structure is determined by the condition

\[(\Delta s)^2 = 0\]

Relative to the ingoing E-F coordinates \((V, r)\), one therefore obtain the radially directed tangents to the null cones:

\[
\frac{dV}{dr} = -1 ; \quad \frac{dV}{dC} = \frac{1 + \frac{3M}{r}}{1 - \frac{3M}{r}} = \frac{r + 3M}{r - 2M}
\]

The corresponding null cone structure is therefore as follows:

\begin{itemize}
  \item To obtain picture relative to outgoing E-F, turn the picture upside down.
  \item Once inside \( r = 2M\), all particle (timelike) and photon (null) geodesic terminate at \( r = 0\), a natural boundary of spacetime where the curvature becomes singular.
\end{itemize}
Kruskal-Szekeres Coordinates: a Maximal Chart.

The Schwarzschild space-time has a metric

\[ ds^2 = -(1 - \frac{2M}{r}) \left[ dt^2 - \frac{dr^2}{(1 - \frac{2M}{r})^2} \right] + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

whose radial null lines can be straightened out easily by introducing the Regge-Wheeler ("tortoise") coordinate \( \tau^+ \):

\[ d\tau^+ = \frac{dt}{1 - \frac{2M}{r}}; \quad \tau^+ = \tau + 2M \ln \left( \frac{r}{2M} - 1 \right) \]

In terms of this coordinate, the radial null lines are straight because the metric has the form

\[ ds^2 = -(1 - \frac{2M}{\tau(\tau^+)} ) \left[ dt^2 - d\tau^+ 2 \right] + \tau(\tau^+) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

The metric of flat Minkowskian space-time relative to Rindler coordinates (t, x) is

\[ ds^2 = -s^2 dt^2 + dx^2 + dv^2 + dz^2 \]

has an analogous feature. By introducing what for Schwarzschild spacetime
corresponds to the "tortoise" coordinate
\[ ds^2 = e^z \left[ dt^2 - dz^2 \right] + d\lambda^2 + d\zeta^2 \]

the metric becomes
\[ ds^2 = -e^z \left[ dt^2 - dz^2 \right] + d\lambda^2 + d\zeta^2 \]

Thus photon worldlines of light rays along the direction of acceleration have been straightened out just like the photon worldlines of radial light rays in the Schwartzschild geometry have been straightened out.

Thus there is a qualitative correspondence between the null cone structure of radial light rays viewed relative to Schwartzschild coordinates and the null cone structure of light rays along the direction of acceleration in a Rindler coordinate frame.

We know that a Rindler coordinate (5.3) from
\[ T = \pm \sinh t \quad 0 < t < \infty \]
\[ X = \pm \cosh \frac{\lambda}{\zeta} \quad -\infty < t < \infty \]

covers only the limited spacetime sector \( I < X < \infty \) of Minkowski space time, whose metric is
\[ ds^2 = -dT^2 + dX^2 + d\lambda^2 + d\zeta^2 \]

The remaining three sectors are covered by related coordinate transformations.

With \( 0 < t < \infty \), \(-\infty < t < \infty \) in each coordinate sector.
Given that the Schwarzschild coordinates $(t, r)$ for $r > 2M$ play the same role that the Rindler coordinates $(t, \xi)$ play for Minkowski spacetime, the natural question that arises is this: Does there exist a corresponding coordinate transformation, say
\[ T = f(r) \sinh \xi \]
\[ R = f(r) \cosh \xi \]
which can be extended to a global spacetime in the same way that the Rindler coordinates can be extended to Minkowski spacetime? Such a coordinate system can be shown to exist if one can exhibit functions $f(r)$ and a constant $\alpha$ such that the metric tensor is nonsingular relative to this new global $(T, R)$ coordinate system.

The coordinate transformation from the inextendible Schwarzschild $(t, r)$ coordinates -- inextendible because the metric is singular at $r = 2M$ -- to the to-be-constructed extendible Kruskal $(T, R)$ coordinates is determined by the differential equation for $f(r)$. The computation is simple and is parallel to that for Minkowski spacetime:
\[ dT = f(r) \sinh \xi \, dr + \alpha f(r) \cosh \xi \, dt \]
\[ dR = f(r) \cosh \xi \, dr + \alpha f(r) \sinh \xi \, dt \]
\[ dR^2 - dT^2 = (f(r))^2 \, dr^2 - f(r)^2 \, dt^2 \]
Because it is not flat, the Schwarzschild metric has a form for its time and radial part which is given by

\[ h(R,T) \left[ dT^2 - dR^2 \right] = ds^2 = -(1 - \frac{2M}{r}) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}}. \]

In order that the \((T,R)\) coordinates be extensible and the constants \(a, b, c, \ldots\) in the last three boxed equations are three equations for \(f(0), h(\infty)\), we require that \(h\) be continuous at \(r = 2M\), i.e.

\[ \lim_{r \to 2M} h = \text{finite and nonzero}. \]

The transformation which relates \((T,R)\) and \((t,r)\) yields

\[ h f^2 \left[ dt^2 - \left( \frac{f}{a} \right)^2 dr^2 \right] = -(1 - \frac{2M}{r}) \left[ dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} \right]. \]

The equations implied by this requirement are

\[ f^2 = \frac{a}{1 - \frac{2M}{r}}, \quad f = \frac{a}{1 - \frac{2M}{r}}. \]

The last three boxed equations are three equations for \(f(0), h(\infty)\), the constants \(a, b, c, \ldots\) in the above, and the solution for \(f\) is

\[ f = \pm e^{\frac{a}{r} \left( \frac{r - 2M}{2M} \right)^2}. \]

With an appropriate integration constant the solution for \(f\) is

\[ f = \pm e^{\frac{a}{r} \left( \frac{r - 2M}{2M} \right)^2}. \]

The limit requirement on \(h\)

\[ h = \frac{1}{a^2} \frac{r - 2M}{2M} \left( \frac{2M}{r - 2M} \right)^4. \]

demands that the only allowed value for \(a\) is

\[ a = \frac{1}{4M}. \]

This must hold with rigorous precision! If \(a\) deviates ever so slightly from the value \(\frac{1}{4M}\), then \(h(\infty)\) would either go to 0 or to \(\infty\) as \(r \to 2M\).
For this initial value of $\alpha$

$$f(r) = \pm e^{\frac{r}{2m} \left( \frac{5m}{2m} - 1 \right)}$$

$$h(rT) = \frac{32m^3}{r} e^{-\frac{r}{2m}}$$

The corresponding, Kruszkal-Szekeres form of the Schwarzschild metric is

$$ds^2 = \frac{32m^3}{r} e^{-\frac{r}{2m}} (dR^2 - dT^2) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

The coordinate transformations giving rise to this form are

$$T = \pm e^{\frac{r}{2m} \left( \frac{5m}{2m} - 1 \right) \sinh \frac{r}{4m}}$$

$$T = \frac{2m}{r}$$

$$R = \pm e^{\frac{r}{2m} \left( \frac{5m}{2m} - 1 \right) \cosh \frac{r}{4m}}$$

These coordinates are dimensionless, and they imply that

$$e^{\frac{r}{2m} \left( \frac{5m}{2m} - 1 \right)} = R^2 - T^2$$

Thus we see that, despite of its $r$-dependence, the metric coefficient function $h(rT)$ is a function of $R$ and $T$; in fact, it is a function of $R^2 - T^2$.

The computed coordinate transformations for $r > 2M$ are not enough because we need another pair which applies to $r < 2M$.

To obtain it we repeat the same procedure:

following the lead from a uniformly accelerated coordinate frame in Minkowski spacetime, we require that

$$T = g(r) \cos \beta t$$

$$R = g(r) \sin \beta t$$

$$dR^2 = dT^2 = (g')^2 dr^2 - g^2 \beta^2 dt^2$$

These are to apply in the spacetime region where $r < 2M$.

This requirement, together with

$$\lim_{r \to 2M} h = \text{finite and nonsens}!$$
yields \( \beta = \frac{1}{4M} \)

\[
q(r) = \pm e^{\frac{r}{2M}} \left( 1 - \frac{T}{2M} \right)^{\frac{1}{2}}
\]

\[
h = \frac{32M^3}{\gamma} e^{-\frac{r}{2M}}
\]

Thus we again obtain another pair of (dimensional) coordinates

\[
T = \pm e^{\frac{r}{2M}} \left( 1 - \frac{T}{2M} \right)^{\frac{1}{2}} \cos \frac{\gamma}{4M}
\]

\[
R = \pm e^{\frac{r}{2M}} \left( 1 - \frac{T}{2M} \right)^{\frac{1}{2}} \sin \frac{\gamma}{4M}
\]

relative to which the metric has the same desired form as before, namely

\[
ds^2 = \frac{32M^3}{\gamma} e^{-\frac{r}{2M}} (dT^2 - dR^2) + T^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

This is a globally defined Kruskal-Szekeres form of the metric on the Schwarzschild spacetime, the analog of the Minkowski form of the metric for flat spacetime.

\[
ds^2 = -dT^2 + dx^2 + dy^2 + dz^2
\]

The global structure of Schwarzschild spacetime is most transparent relative to these Kruskal-Szekeres coordinates.
The Schwarzschild spacetime has an interior region, \( r < 2M \), and an exterior region, \( r > 2M \). These regions are different geometrically and physically.

1. The Schwarzschild coordinates \((\tau, r)\) yield hyperbolae and straight lines relative to the K-S coordinates \((T, R)\).

Indeed in the exterior
\[
R^2 - T^2 = e^{2\frac{r}{M}} \left( \frac{T}{M} - 1 \right)
\]

\(2M < T\) in I and II.

2. The exterior \((2M < r)\) is static while the interior \((r < 2M)\) is dynamic.

Indeed, for \( r > 2M \) there exist timelike (observer) worldlines, namely

\( \tau = \text{const}\)

relative to which the metric

is static, i.e. independent of \( \tau \).

However for \( r < 2M \), in the interior, there do not exist any timelike world lines relative to which the metric is static.

For example, \( r = \text{const}\) are space-like curves which do not represent the worldline of any observer.

\[
T = \tanh \frac{T}{M}
\]

while in the interior
\[
T^2 - R^2 = e^{2\frac{T}{M}} \left( 1 - \frac{T}{2M} \right)
\]

\( r < 2M \); III and IV

\[
T = \coth \frac{T}{M}
\]

2M < \(T\) in III and IV.

3. The geometry on each of the spacelike hypersurfaces \( t = c \), \(-\infty < c < \infty\), is that of a static Schwarzschild throat. Each hypersurface \( t = \text{constant}\) consists of a sequence of nested spheres. Their surface area \( 4\pi r^2\)

is not a monotonic function of their proper separation. Instead there is a sphere of minimum area \( 4\pi (2M)^2 = 16\pi M^2 \)

This sphere as well as all the others have no center.
The spheres on one side of this minimum area sphere, as well as those on its other side, form a sequence with ever increasing surface area. Both sequences of spheres are on the "outside" of the minimum area sphere.

or be influenced by an event in the other region.

(ii) One also sees that if one follows the evolution of this Schwarzschild throat on successive hypersurfaces $T = \text{const.}$ over the coordinate time interval $\delta T \leq 2$, the throat evolves from its (maximum) area $16\pi M^2$ at $T = 0$ to zero area at $T = 1$; in other words the throat pinches off in a finite interval of time.

Together the two sequences of spheres form a three-dimensional passage (Schwarzschild throat) between two asymptotically flat spatial regions.

(i) One sees from the Kruskal diagram that these two regions are causally disjoint: an event in one region can not influence...