Lecture 38.

Globally defined coordinate system for Schwarzschild spacetime:

Kruskal Szekeres coordinates [MTW §31.5]

Kruskal-Szekeres Coordinates: a Maximal Chart.

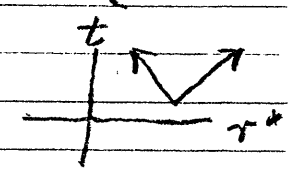
The Schwarzschild space-time has a metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

whose radial null lines can be straightened out easily by introducing the Regge-Wheeler ("tortoise") coordinate r^*:

$$dr^* \equiv \frac{dr}{1-\frac{2M}{r}} \; : \; r^* = r + 2M\ln\left(\frac{r}{2M}-1\right)$$

In terms of this coordinate, the radial null lines are straight because the metric has the form



$$ds^2 = -\left(1-\frac{2M}{r(r^*)}\right)\left[dt^2 - dr^{*2}\right] + r^2(r^*)\left(d\theta^2 + \sin\theta^2 d\phi^2\right)$$

The metric of flat Minkowski space time relative to Rindler coordinates (t and ξ)

$$ds^2 = -\xi^2\, dt^2 + d\xi^2 + dY^2 + dZ^2$$

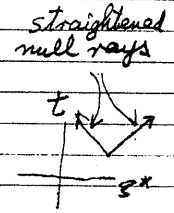
has an analogous feature. By introducing what for Schwarzschild spacetime

corresponds to the "tortoise" coordinate

$$\frac{d\xi}{\xi} = d\xi^* \quad ; \quad \xi = e^{\xi^*},$$

the metric becomes

$$ds^2 = -e^{\xi^*}\left[dt^2 - d\xi^{*2}\right] + dY^2 + dZ^2$$



Thus photon worldlines of light rays along the direction of acceleration have been straightened out, just like the photon worldlines of radial light rays in the Schwarzschild geometry have been straightened out.

Thus there is a qualitative correspondence between the null cone structure of radial light rays viewed relative to Schwarzschild coordinates and the null cone structure of light rays along the direction of acceleration in a Rindler coordinate frame.

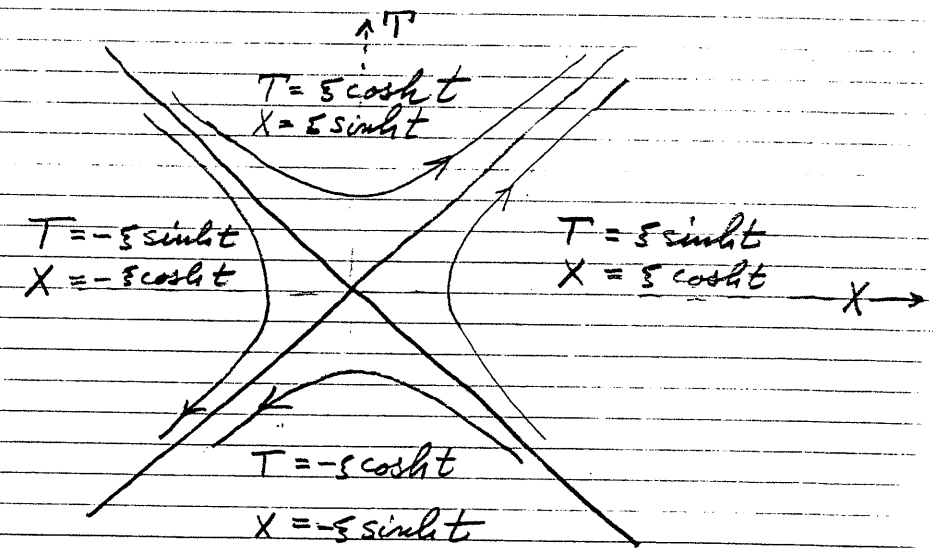
We know that a Rindler coordinate (t, ξ) frame

$$\left.\begin{aligned} T &= \xi \sinh t \\ X &= \xi \cosh t \end{aligned}\right\} \begin{aligned} &0 < \xi < \infty \\ &-\infty < t < \infty \end{aligned}$$

covers only the limited spacetime sector $|T| < X$ of Minkowski space time, whose metric is

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2.$$

The remaining three sectors are covered by related coordinate transformations



with $0 < \xi < 0$, $-\infty < t < \infty$ in each coordinate sector.

Given that the Schwarzschild coordinates (t, r) for $r > 2M$ play the same role that the Rindler coordinates (t, ξ) play for Minkowski spacetime, the natural question that arises is this: Does there exist a corresponding coordinate transformation, say

$$T = f(r) \sinh \alpha t$$
$$R = f(r) \cosh \alpha t,$$

which can be extended to a global spacetime in the same way that the Rindler coordinates can be extended to Minkowski spacetime?

Such a coordinate system (T, R) can be shown to exist, if one can exhibit functions $f(r)$ and a constant α such that the metric tensor is nonsingular relative this new global (T, R) coordinate system.

The coordinate transformation from the inextendible Schwarzschild (t, r) coordinates -- inextendible because the metric is singular at $r = 2M$ -- to the to-be-constructed extendible Kruskal (T, R) coordinates is determined by the differential equation for $f(r)$. The computation is simple and is parallel to that for Minkowski spacetime:

$$dT = f'(r) \sinh \alpha t \, dr + \alpha f(r) \cosh t \, dt$$
$$dR = f'(r) \cosh \alpha t \, dr + \alpha f(r) \sinh t \, dt$$

$$dR^2 - dT^2 = (f')^2 dr^2 - f^2 \alpha^2 dt^2$$

$r \to 2M$

Because it is not flat, the Schwarschild metric has a form for its time and radial part which is given by

$$-h(R,T)\left[dT^2-dR^2\right]=ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}$$

In order that the (T,R) coordinates be extendible we require that h be continuous at $r=2M$, i.e.

$$\boxed{\lim_{r\to 2M} h = \text{finite and nonzero!}}$$

The transformation which relates (T,R) and (t,r) yields

$$-h\,f^2\alpha^2\left[dt^2-\left(\frac{f'}{\alpha f}\right)^2 dr^2\right]=-\left(1-\frac{2M}{r}\right)\left[dt^2-\frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right]$$

The equations implied by this requirement are

$$\boxed{\begin{aligned}\frac{f'}{f} &= \frac{\alpha}{1-\frac{2M}{r}} = \alpha\left(1+\frac{2M}{r-2M}\right) \qquad \text{for } 2M<r\\ h f^2\alpha^2 &= 1-\frac{2M}{r}\end{aligned}}$$

The last three boxed equations are three equations for $f(r)$, $h(R,T)$ and the constant α.

With an appropriate integration constant the solution for f is

$$f=\pm e^{\alpha r}\left(\frac{r-2M}{2M}\right)^{2M\alpha} \qquad \text{(dimensionless)}$$

The limit requirement on

$$h=\frac{1}{\alpha^2}\,\frac{r-2M}{r}\,\frac{(2M)^{4M\alpha}}{e^{2\alpha r}(r-2M)^{4M\alpha}}$$

demands that the only allowed value for α is

$$\boxed{\alpha=\frac{1}{4M}}$$

This must hold with razor sharp precision! If α deviates ever so slightly from the value $\frac{1}{4M}$, then $h(R,T)$ would either $\to 0$ or ∞ as $r\to 2M$.

For ~~on this~~ the value of α

$$f(r) = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2}$$

$$h(R,T) = \frac{32M^3}{r} e^{-r/2M}$$

The corresponding Kruskal-Szekeres form of the Schwarzschild metric is

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(dR^2 - dT^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The coordinate transformations giving rise to this form are

$$T = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \sinh \frac{t}{4M} \qquad 2M < r$$

$$R = \pm e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{1/2} \cosh \frac{t}{4M}$$

These coordinates are dimensionless, and they imply that

$$e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) = R^2 - T^2$$

Thus we see that, in spite of its r-dependence, the metric coefficient function $h(R,T)$ is a function of R and T; in fact, it is a function of $R^2 - T^2$.

The computed coordinate transformations for $r > 2M$ are not enough because we need <u>another</u> pair which applies to <u>$r < 2M$</u>. To obtain it we repeat the same procedure: following the lead from a uniformly accelerated coordinate frame in Minkowski spacetime, we require that

$$T = g(r) \cosh \beta t$$

$$R = g(r) \sinh \beta t$$

$$dR^2 - dT^2 = (g')\, dr^2 - g^2 \beta^2\, dt^2.$$

These are to apply in the spacetime region where $r < 2M$. This requirement, together with

$$\boxed{\lim_{r \to 2M} h = \text{finite and nonzero!}}$$

yields $\beta = \frac{1}{4M}$

$$g(r) = \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2}$$

$$h = \frac{32M^3}{r} e^{-\frac{r}{2M}}$$

Thus we again obtain another pair of (dimensionless) coordinates

$$\boxed{\begin{aligned} T &= \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \cosh \frac{t}{4M} \\ R &= \pm e^{\frac{r}{4M}} \left(1 - \frac{r}{2M}\right)^{1/2} \sinh \frac{t}{4M} \end{aligned}} \qquad r < 2M$$

relative to which the metric has the same desired form as before, namely

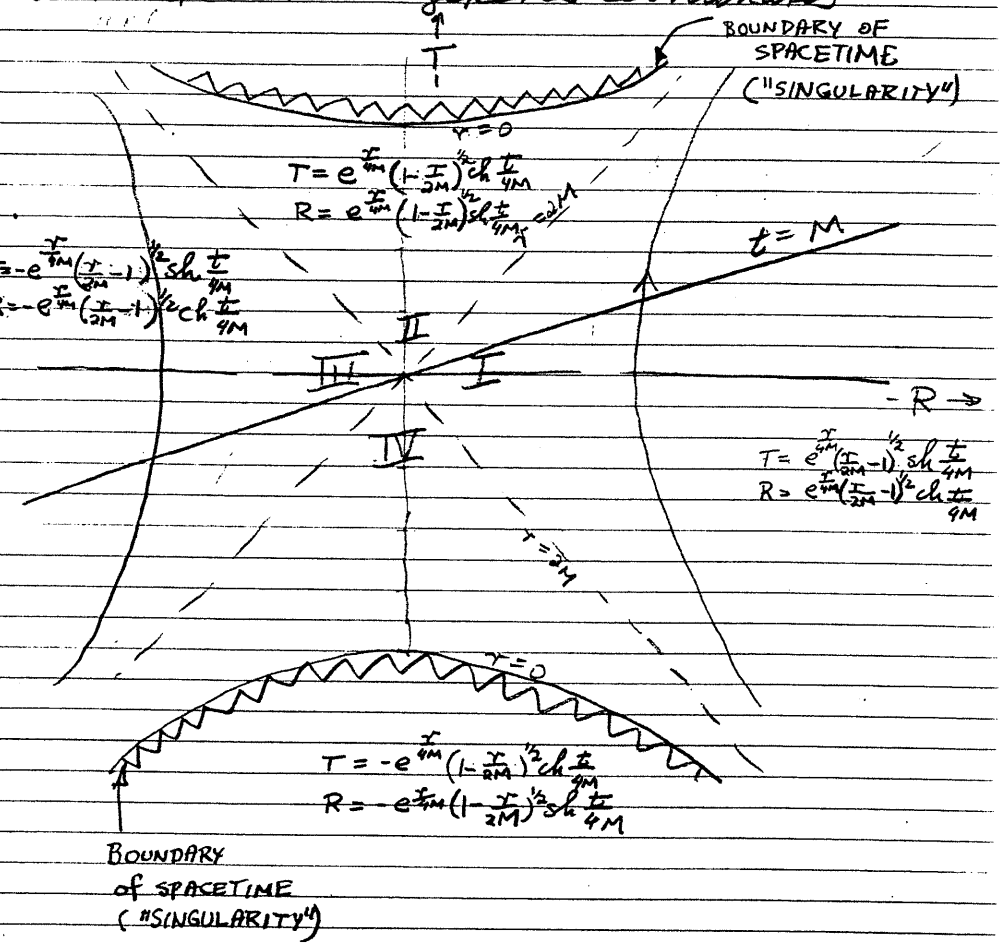
$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(dR^2 - dT^2\right) + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

This is the globally defined Kruskal-Szekeres form of the metric on the Schwarzschild spacetime. This form is analogous to the Minkowski form of the metric

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$$

for flat spacetime.

The global structure of Schwarzschild spacetime is most transparent relative to these Kruskal Szekeres coordinates



The Schwarzschild spacetime has an <u>interior</u> region, $r<2M$, and an <u>exterior</u> region, $r>2M$. These regions are different geometrically and physically:

1. The Schwarzschild coordinates (t, r) yield hyperbolas and straight lines relative to the K-S coordinates (T, R). Indeed in the exterior

$$R^2 - T^2 = e^{\frac{r}{2M}}\left(\frac{r}{2M} - 1\right) \qquad 2M<r \text{ ; in I and III}$$

$$\frac{T}{R} = \tanh\frac{t}{4M}$$

while in the interior

$$T^2 - R^2 = e^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right) \qquad r<2M \text{ ; in II and IV}$$

$$\frac{T}{R} = \coth\frac{t}{4M}$$

2. The exterior $(2M<r)$ is <u>static</u>, while the interior $(r<2M)$ is <u>dynamic</u>: Indeed, for $r>2M$ there exist timelike (observer) worldlines, namely

$$r = \text{const},$$

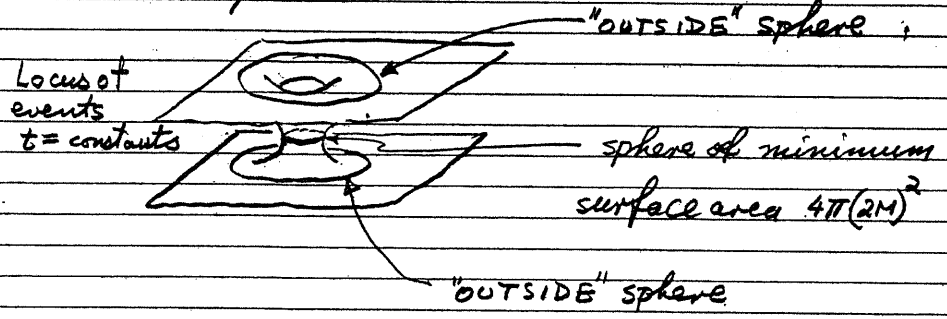
relative to which the metric

is static, i.e independent of t.

However for $r<2M$, in the interior, there do <u>not</u> exist any timelike worldlines relative to which the metric is static. For example, $r = \text{const}$, are spacelike curves which do not represent the worldline of any observer.

3. The geometry on each of the spacelike hypersurfaces $t = c$, $-\infty < c < \infty$, is that of a statatic Schwarzschild throat. Each hypersurface $t = \text{constant}$ consists of a sequence of nested spheres. Their surface area $4\pi r^2$ is not a monotonic function of their proper separation. Instead there is a sphere of minimum area $4\pi(2M)^2 = 16\pi M^2$. <u>This sphere</u>, as well as all the others, <u>have no center</u>.

The spheres on one side of this minimum area sphere, as well as those on its other side, form a sequence with ever increasing surface area. Both sequences of spheres are on the "outside" of the minimum area sphere.



Together the two sequences of spheres form a three-dimensional passage ("Schwarzschild throat") between two asymptotically flat spatial regions.

(i) One sees from the Kruskal diagram that these two regions are causally disjoint: an event in one region can not influence

or be influenced by an event in the other region.

(ii) One also sees that if one follows the evolution of this Schwarzschild throat on successive hypersurfaces $T = \text{const}$ over the coordinate time interval $0 \le T \le 1$, the throat evolves from its (maximum) area $16\pi M^2$ at $T=0$ to zero area at $T=1$; in other words the throat pinches off in a finite interval of time.