Lecture 38.

Globally defined coordinate system for Schwarzschild spacetime:

Kruskal-Szekeres coordinates [MTW 8.3.5]
Kruskal-Szekeres Coordinates: a Maximal Chart.

The Schwarzschild space-time has a metric:

\[ ds^2 = -(1 - \frac{2M}{r}) \left[ dt^2 - \frac{dr^2}{(1 - \frac{2M}{r})^2} \right] + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

whose radial null lines can be straightened out easily by introducing the Regge-Wheeler ("tortoise") coordinate \( r^* \):

\[ dr^* = \frac{dr}{1 - \frac{2M}{r}} \quad r^* = r + 2M \ln \left( \frac{r}{2M} - 1 \right) \]

In terms of this coordinate, the radial null lines are straight because the metric has the form:

\[ ds^2 = -(1 - \frac{2M}{r(r^*)}) \left[ dt^2 - dr^* \right] + r^2(r^*)^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

The metric of flat Minkowski space-time relative to Rindler coordinates (\( t, \phi, \tau, \phi \))

\[ ds^2 = -\tau^2 dt^2 + d\tau^2 + dy^2 + dz^2 \]

has an analogous feature. By introducing \( \theta \) and \( \phi \) as for Schwarzschild space-time
corresponds to the "tortoise" coordinate \( \xi = e^x \). The metric becomes
\[
\frac{ds^2}{\xi} = \xi^2 \left[ dt^2 - d\xi^2 \right] + dv^2 + dr^2
\]
Thus photon worldlines of light rays along the direction of acceleration have been straightened out, just like the photon worldlines of radial light rays in the Schwarzschild geometry have been straightened out.

Thus there is a qualitative correspondence between the null cone structure of radial light rays viewed relative to Schwarzschild coordinates and the null cone structure of light rays along the direction of acceleration in a Rindler coordinate frame.

We know that a Rindler coordinate \((\xi, \eta)\) from
\[
\begin{align*}
T &= \xi \sinh \eta \quad &0 < \xi < \infty \\
X &= \xi \cosh \eta \quad &-\infty < \xi < \infty
\end{align*}
\]
covers only the limited spacetime sector \( \Pi < X \) of Minkowski space-time, whose metric is
\[
d\eta^2 = -dT^2 + dX^2 + dY^2 + dZ^2
\]
The remaining three sectors are covered by related coordinate transformations with \( 0 < \xi < \infty, -\infty < \xi < \infty \) in each coordinate sector.
Given that the Schwarzschild coordinates $(t, r)$ for $r > 2M$ play the same role that the Rindler coordinates $(t, s)$ play for Minkowski spacetime, the natural question that arises is this: Does there exist a corresponding coordinate transformation, say

\[ T = f(r) \sinh x t \]
\[ R = f(r) \cosh x t, \]

which can be extended to a global spacetime in the same way that the Rindler coordinates can be extended to Minkowski spacetime? Such a coordinate system can be shown to exist if one can exhibit functions $f(r)$ and a constant $x$ such that the metric tensor is nonsingular relative to this new global $(T, R)$ coordinate system.

The coordinate transformation from the inextendible Schwarzschild $(t, r)$ coordinates -- inextendible because the metric is singular at $r = 2M$ -- to the to-be-constructed extendible Kruskal $(T, R)$ coordinates is determined by the differential equation for $f(r)$. The computation is simple and is parallel to that for Minkowski spacetime:

\[ dT = f(r) \sinh x t \, dr + r f'(r) \cosh x t \, dt \]
\[ dR = f(r) \cosh x t \, dr + r f'(r) \sinh x t \, dt \]

\[ dR^2 - dT^2 = f(r)^2 \, dr^2 - f^2 \, dx^2 \, dt^2 \]
Because it must not fit the Schwarzschild, the equations implied by this requirement:

\[ f(r) = \sqrt{\frac{\alpha}{r - 2M}} \]

for \( M = \frac{\alpha}{2} \) have a form.

The limit requirement on \( h \) demands that the only allowed value for \( x \) is

\[ x = \frac{1}{2} \]

This must hold with regard to the precision of the value, not the more exact.

\[ \alpha = 0 \text{ or } \infty \]

The transformation which holds (provided \( (\phi) \equiv \frac{1}{2} \alpha \)) yields:

\[ -\frac{1}{\alpha^2} + \frac{\alpha^2}{2} = -\frac{1}{(2\pi)} \frac{d\phi}{dr} \]

The metric has a form for its first and second derivative, consistent with the solution for \( \alpha \).
For a fixed value of $x$:

\[ f(x) = \pm e^{\frac{x}{2M}} \left( \frac{3M}{2M} - 1 \right)^{\frac{1}{2}} \]

\[ h(R, T) = \frac{32M^3}{7} e^{-\frac{x}{2M}} \]

The corresponding Kruskal-Szekeres form of the Schwarzschild metric is:

\[ ds^2 = \frac{32M^3}{7} e^{-\frac{x}{2M}} \left( dR^2 - dT^2 \right) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

The coordinate transformations giving rise to this form are:

\[ T = \pm e^{\frac{x}{2M}} \left( \frac{3M}{2M} - 1 \right)^{\frac{1}{2}} \sinh \frac{r}{4M} \quad 2M \leq r \]

\[ R = e^{\frac{x}{2M}} \left( \frac{3M}{2M} - 1 \right)^{\frac{1}{2}} \cosh \frac{r}{4M} \]

These coordinates are dimensionless, and they imply that:

\[ e^{\frac{x}{2M}} \left( \frac{3M}{2M} - 1 \right) = R^2 - T^2 \]

Thus, we see that in spite of its $r$-dependence, the metric coefficient function $h(R, T)$ is a function of $R$ and $T$. In fact, it is a function of $R^2 - T^2$.

The computed coordinate transformations for $r > 2M$ are not enough because we need another pair which applies to $r < 2M$.

To obtain it we repeat the same procedure:

Following the lead from a uniformly accelerated coordinate frame in Minkowski spacetime, we require that:

\[ T = g(r) \cosh \beta t \]

\[ R = g(r) \sinh \beta t \]

\[ dR^2 - dT^2 = (g')_0 dr^2 - g_0 \beta^2 dt^2 \]

These are to apply in the spacetime region where $r < 2M$.

This requirement, together with

\[ \lim_{r \to 2M} h = \text{finite and nonzero}! \]
The global structure of Schwarzschild spacetime is most transparent relative to the Kruskal-Szekeres coordinates.

Thus we again obtain another pair of (dimensional) coordinates

\[
\begin{align*}
T &= \pm e^{\frac{T}{2M}} (1 - \frac{T}{2M})^2 \cosh \frac{T}{M} \\
R &= \pm e^{\frac{T}{2M}} (1 - \frac{T}{2M})^2 \sinh \frac{T}{M}
\end{align*}
\]

relative to which the metric has the same desired form as before, namely

\[
ds^2 = \frac{2M^2}{T} e^{\frac{T}{2M}} (dR^2 + dT^2) + R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)
\]

This is a globally defined Kruskal-Szekeres form of the metric on the Schwarzschild spacetime, this being analogous to the Minkowski form of the metric

\[
ds^2 = -dT^2 + dx^2 + dy^2 + dz^2
\]

for flat spacetime.
The Schwarzschild spacetime has an interior region, \( r < 2M \), and an exterior region, \( r > 2M \). These regions are different geometrically and physically:

1. The Schwarzschild coordinates \((t, r)\) yield hyperbolae and straight lines relative to the K-S coordinates \((T, R)\).

Indeed in the exterior

\[ R^2 - T^2 = e^{2T} \left( \frac{T}{2M} - 1 \right) \]

\( 2M \leq T \) in I and III.

2. The exterior \((2M < r)\) is static, while the interior \((r < 2M)\) is dynamic.

\[ T = \tanh \frac{t}{2M} \] \( r > 2M \); in II and IV.

\[ T = \coth \frac{t}{2M} \] \( r < 2M \).

Thus, the geometry on each of the spacelike hypersurfaces \( t = \text{const.} \), \(-\infty < t < \infty\), is that of a static Schwarzschild throat. Each hypersurface \( t = \text{const.} \) consists of a sequence of nested spheres. Their surface area \( 4\pi r^2 \) is not a monotonic function of their proper separation. Instead there is a sphere of minimum area \( 4\pi (2M)^2 = 16\pi M^2 \) (observer) worldlines, namely

\[ r = \text{const.} \]

For example, \( r = \text{const.} \), are space-like curves which do not represent the worldline of any observer.
The spheres on one side of this minimum area sphere, as well as those on its other side, form a sequence with ever increasing surface area. Both sequences of spheres are on the "outside" of the minimum area sphere.

(ii) One also sees that if one follows the evolution of this Schwarzschild throat on successive hypersurfaces $T = \text{const}$ over the coordinate time interval, $0 \leq T \leq \frac{3}{2}$, the throat evolves from its (maximum) area $16\pi M^2$ at $T=0$ to zero area at $T=\frac{3}{2}$; in other words, the throat pinches off in a finite interval of time.

Together the two sequences of spheres form a three-dimensional passage ("Schwarzschild throat") between two asymptotically flat spatial regions.

(i) One sees from the Kruskal diagram that these two regions are causally disjoint; an event in one region cannot influence