LECTURE 4

1. Free-particle motion in a rotating frame and the necessity of its mathematical reformulation in geometrical terms.

2. Free-particle motion in an accelerated frame, its mathematical reformulation in geometrical terms.

3. Inertial force and mass vs. gravitational force and mass. Their equality as inferred from the equivalence principle.

4. Gravitation = "Geometry"

The two terms on the r.h.s. of the equation of motion

$$m\frac{d^2x^i}{dt^2} = -2m\left[\omega\times\vec{v}_{rot}\right]^i - m\left[\omega\times(\omega\times r)\right]^i$$

are the "Coriolis force" and the "centrifugal force", and the curvilinear coordinates x^0, x^1, x^2, x^3 (of the rotating frame) are related to the velocity components relative to the rotating frame by

$$\left.\begin{aligned}\left[\vec{v}_{rot}\right]^i &= \frac{dx^i}{d\tau}\\ 1 &= \frac{dt}{d\tau} = \frac{dx^0}{d\tau}\end{aligned}\right\}\text{IN THE ROTATING FRAME,}$$

Now compare the first component of the equation for the free particle with the first component of the equation for a geodesic:

$$m\left[a_{1r} + 2\left(\omega_2\frac{dx^3}{d\tau} - \omega_3\frac{dx^2}{d\tau}\right) + \vec{\omega}\cdot\vec{r}\,\omega_1 - \omega\cdot\omega x^1\right] = 0$$

vs

$$m\left[\frac{d^2x^1}{d\tau^2} + 2\Gamma^1_{0j}\frac{dx^j}{d\tau} + \Gamma^1_{00}\right] = 0$$

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If, as Einstein pointed out, one compares these equations of motion with those of a spacetime geodesic relative to curvilinear coordinates, (P3.1)

$$m\frac{d^2x^\mu}{d\tau^2} = -m\sum_\alpha\sum_\beta \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \qquad \mu = 1,2,3$$

then we are forced to make the following identification

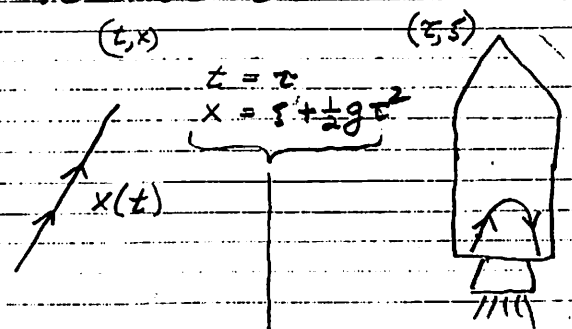
$$\boxed{\begin{array}{l} \Gamma^i_{0j} \neq 0 \iff \text{Coriolis "force"} \neq 0 \\ \Gamma^i_{00} \neq 0 \iff \text{centrifugal "force"} \neq 0 \end{array}}$$

This observation promises to open new vistas in physics. The next example fullfills this promise.

GEOMETRICA INTERPRETATION OF ~~3.14~~ 4.4 GRAVITATIONAL FORCE

IIa) GRAVITATIONAL FORCE

Consider the motion of a free particle relative to an accelerated frame



$\xi(t) = x - \frac{1}{2} g t^2$

$t = \tau$

$x = \underbrace{\xi + \frac{1}{2} g \tau^2}$

Free particle in an acc'd frame.

$$0 = \frac{d^2 x}{dt^2} = \frac{d^2}{d\tau^2}\left[\xi(\tau) + \frac{1}{2} g \tau^2\right] = \frac{d^2 \xi}{d\tau^2} + g$$

Free particle in a FREE float frame.

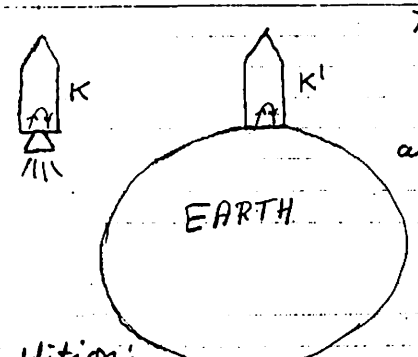
$$\Downarrow$$

$$\frac{d^2 \xi}{d\tau^2} = -g = -\Gamma^1_{00}$$

$$\Downarrow$$

Let $x^0 = \tau$, $x^1 = \xi$ $\quad \Gamma^1_{00} = +g \neq 0$.

IIb) Equivalence principle:



There is no difference between non inertial "force" and gravitational "force" i.e. $K = K'$ $\forall$ all particles

Conclusion:

$$\boxed{\Gamma^i_{00} \neq 0 \Leftrightarrow \text{gravitational "force"}}$$

In addition:

$0 = \Gamma^\mu_{\alpha\beta} \Leftrightarrow \frac{\partial g_{\alpha\beta}}{\partial x^\mu} = 0 \Rightarrow (ds)^2 = -1(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

RECTILINER COORDIATES.

III. Let us extend the geometrical interpretation to an arbitrary static gravitational field.

Recall that

$$\Gamma^i_{00} = \frac{1}{2} \underbrace{g^{i\alpha}}\left(\underbrace{g_{\alpha 0,0}}_{zero} + \underbrace{g_{\alpha 0,0}}_{zero} - g_{00,\alpha}\right)$$

For a static (time independent) gravitational force all time derivatives $\left(\frac{\partial}{\partial x^0}\right)$ vanish.

For a coordinate frame which is nearly Lorentzian

$\{ g^{ik} = \delta_{ik} + \text{negleglible quantity}$

Consequently

$$\Gamma^i_{00} = -\frac{1}{2} \frac{\partial g_{00}}{\partial x^i}.$$

$\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \quad i = 1,2,3$

Thus the equation for a geodisic are

$$\boxed{\frac{d^2 x^i}{dx^{0\,2}} = -\Gamma^i_{00} \approx \frac{1}{2} \frac{\partial g_{00}}{\partial x^i}}.$$

Consider Newton's equation for a particle in a gravitational potential Φ:

$$\boxed{(\text{mass})_{inert} \frac{d^2 x^i}{d t^2_{conv}} = (\text{mass})_{grav} \left(-\frac{\partial \Phi}{\partial x^i}\right)} \quad c t_{conv} = x^0$$

These were the equations of motion before Einstein.

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Einstein always wondered why it is that one always finds that

Why $\boxed{(\text{mass})_{\text{inertial}} = (\text{mass})_{\text{gravitational}}}$?

Then, one day it occurred to him.

i) In a free float frame (including in a freely falling one) all particles move in the same way, namely along straight world lines.

The material composition of these particles is irrelevant

ii) This locally rectilinear motion in spacetime becomes a non-rectilinear motion involving acceleration, when reexpressed in a non-free float frame (such as an accelerated one, static relative to a gravitating body).

NOTE: The student is asked to ignore the ensuing paragraph; it constitutes private rumminations.

[PRIVATE THOUGHT: Observations made relative to locally inertial vs observations relative to accelerated frames are complementary

(Bohr: "mutually exclusive (?)")

(Question: Is this really true? Answer: ??)

4.7 ~~3.17~~

just as quantum mechanically the observations of momentum and position are complementary. One can make observations to arbitrary accuracy of one type, or the other, but not both.

Note: Quantum mechanically straight world lines become plane waves, while non-rectilinear world lines become distorted waves in a non-free float frame.

END of PRIVATE THOUGHT]

(iii) This non-rectilinear motion manifests itself as the apparent inertial Coriolis and centrifugal forces in a rotating frame.

Similarly the gravitational forces are an apparent manifestation of a non-free float frame, an accelerated one.

END Intermission

It follows that what used to be called gravitational mass, $(\text{mass})_{grav}$, is simply the inertial mass

$$\boxed{(\text{mass})_{grav} = (\text{mass})_{inert}}$$

which one must introduce inorder to relate the non-rectilinear motion to the apparent (= frame dependent!) force.

[Another PRIVATE THOUGHT: Force and acceleration are secondary concepts in quantum mechanics. So how would Einstein have carried out building his bridge to geometry, if he had to do it strictly in terms of quantum mechanical ideas, i.e. using systems (particles) with very low action, without resorting to classical mechanics. Quantum mechanical measurements, how are they used to establish

$$(\text{mass})_{grav} = (\text{mass})_{inertial} \;?$$ (see end of lecture on history)

End of PRIVATE THOUGHT.]

Thus we conclude that

$m_{grav} = m_{inert}$ if and only if Newton's equations of motion, after cancellation, become geometrical, i.e become the equations for a geodesic.

This gives rise to an astonishing conclusion: Letting $x^0 = c\, t_{conv}$, one has

$$\frac{d^2x^i}{dx^{02}} = -\frac{\partial}{\partial x^i}\left(\frac{\Phi}{c^2}\right),$$

which are to be compared with

$$\frac{d^2x^i}{dx^{02}} = \frac{1}{2}\frac{\partial g_{00}}{\partial x^i}.$$

Thus, based solely on the strength of the equivalence principle, every motion of every particle implies the same fact

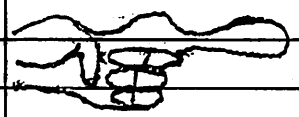
$$g_{00} = \text{const.} - \frac{2\Phi}{c^2}$$

Boundary condition: $\Phi = 0 \Rightarrow ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

Imposing the boundary condition that the relation between spacetime events is Lorentzian in the absence of gravitation, i.e.

$$g_{00} = -1 \Longleftrightarrow \Phi = 0$$

one obtains



$$\boxed{g_{00} = -1 + \frac{2}{c^2}\overbrace{\left(\begin{array}{c}\text{Newtonian}\\ \text{gravitational}\\ \text{potential}\end{array}\right)}^{\Phi}}$$

This is a conceptual/mathematical breakthrough of the first order. Its qualitative implication is that one should not think of gravitation being expressed mathematically in terms of a single gravitational potential, but that instead

$$\left(\begin{array}{c}\text{set of generalized}\\ \text{gravitational}\\ \text{potentials}\end{array}\right) = \left(\begin{array}{c}\text{set of components}\\ \text{of a metric}\end{array}\right)$$

or

$$\boxed{\text{"gravitation"} = \text{"geometry"} \quad (\text{properly understood})}$$

The establishment of this unifying link between seemingly disparate aspects of Nature serves to open new vistas

Let us summarize the newly introduced geometrical framework by highlighting its observational basis:

{The state of motion of a free particle in the presence (or absence) of gravitation is a geodesic state of motion governed by a spacetime metric}

Formulated mathematically one has

$$\left\{ m_{inertial} \frac{d^2 x^\mu}{d\tau^2} = m_{grav} \frac{1}{2} g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right\}$$

cancel!

$\Leftarrow$ $\{ m_{inertial} = m_{gravitational}$

$\forall$ trajectories relative to all accelerated (& non-acc'd) frames$\}$

$\cup$ {all of Newtonian / Lagrangian mechanics}

The implication "$\Leftarrow$" at the bottom of p 4.1 is example of scientific induction.

[Nota Bene: Induction is the process of inferring generalizations from particular instances.]

Einstein performed this in a paper he published in 1912. He was able to achieve this from a single observation, namely from

(*) $\boxed{m_{inertial} = m_{gravitational}}$

Einstein concluded

(**) $\boxed{\text{gravitation} = \text{"geometry"} \ \left(\text{properly understood}\right)}$

However, please notice that there was

an enormous set of necessary concepts, principles, and theories that Einstein needed in order to induce (**) from (*), namely

(i) "accelerated frame" = 1-parameter family of inertial frames

(ii) Newtonian mechanics

(iii) Newtonia gravitation

(iv) Hamiltonian/Lagrangian formulation of mechanics

(v) special relativity

Without (i)-(v) Einstein would not have been able to arrive at his inductive conclusion

(*) implies (**)

The necessity of having at one's command all the relevant knowledge is the key to valid inductive reasoning and it also is the reason why induction is so much more difficult than the complementary process, deduction — applying generalizations to new instances [e.g. calculation the precession of the perihelion of Mercury from the equation of a geodesic].

It is worth mentioning that there are other notable historical examples where one arrives at a generalization by means of inductive reasoning

from a single observation. One of these is due to Benjamin Franklin. Flying a kite in a thunderstorm, he observed

(☆) {sparks fly from a metal key to the wire in a Leyden jar}

From this he was able to conclude that

(☆☆) {lightning = electric discharge}

This and other important instances of valid inductive reasoning in physics, astrononomy, chemistry, and philosophy are chronicles in David Harriman's "THE LOGICAL LEAP: Induction in Physics."

It is understood, but worth mentioning that the domain of applicability of a generalization such as "gravitation" = "geometry" is based both on the scope and the accuracy of the observational evidence underlying. The scope includes classical (i.e. non-quantum) mechanics (whose mathematical formulation is usually in terms of systems of o.d.e.) but does not include elementary phenomena such as the accelerated frame-induced black body radiation characterized by the Unruh-Davies

temperature T_i

$$kT = \frac{\hbar}{c}\frac{(acc'n)}{2\pi}.$$

The accuracy is determined by the the technological landscape of the observations.

In summary, the validity of observational framework is constrained by and applies to the range of the domain of observations and their degree of accuracy. Both are limited to what is actually observed. Because of technological advances, the first has increased while the second has decreased.