LECTURE 4

1. Free-particle motion in a rotating frame and the necessity of its mathematical reformulation in geometrical terms.

2. Free-particle motion in an accelerated frame, its mathematical reformulation in geometrical terms.

3. Inertial force and mass vs gravitational force and mass. Their equality as inferred from the equivalence principle.

4. Gravitation = "Geometry"
The two terms on the r.h.s. of the equation of motion
\[ m \frac{d^2x_i}{dt^2} = -m \left[ \omega \times \omega_{rot} \right] \cdot \nabla x_i - m \left[ \omega \times (\omega \times x_i) \right] \]
are the "Coriolis force" and the "centrifugal force," and the curvilinear-coordinates \( x^0, x^1, x^2, x^3 \) (of the rotating frame) are related to the velocity components relative to the rotating frame by
\[ \left[ x^{\mu \rho}_{rot} \right] = \frac{dx^\rho}{dt} \]
\[ 1 = \frac{dt^\mu}{d\xi} = \frac{dx^\rho}{d\xi} \]
Now compare the first component of the equation for the free particle with the first component of the equation for a geodesic:
\[ m \left[ a_{x_1} + \alpha_\omega \left( \frac{dx_1}{dt} - \omega \times x_1 \right) + \omega \times \omega_\omega - \omega \times x_1 \right] = 0 \]
\[ m \left[ \frac{d^2x_1}{dt^2} + 2 \Gamma^1_{0\rho} \frac{dx_1}{dt} + \Gamma^1_{00} \right] = 0 \]
Go to next page.
If, as Einstein pointed out, one compares these equations of motion with those of a spacetime geodesic relative to curvilinear coordinates, (P3.1) 
\[ m \frac{d^2 x^\mu}{d\tau^2} = -m \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}, \mu = 1, 2, 3 \]
then we are forced to make the following identification:

- \( \Gamma_{ij} \neq 0 \iff \text{Coriolis "force"} \neq 0 \)
- \( \Gamma_{00} \neq 0 \iff \text{centrifugal "force"} \neq 0 \)

This observation promises to open new vistas in physics. The next example fulfills this promise.
Geometrical Interpretation of \( \frac{3}{4} \) of 4.4

Consider the motion of a free particle relative to an accelerated frame.

\[ s(t) = x - \frac{1}{2} g t^2 \]

Recall that

\[ \Gamma_{\alpha\beta} \equiv \frac{1}{2} g_{\alpha\beta} (g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\alpha\nu}) \]

For a static (time independent) gravitational field, all time derivatives \( \frac{\partial}{\partial t} \) vanish.

For a coordinate frame which is nearly Lorentzian,

\[ g_{\alpha\beta} = \delta_{\alpha\beta} + \text{negligible quantity} \]

Consequently,

\[ \Gamma^{\alpha}_{\beta\gamma} = -\frac{1}{2} \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} \]

Thus, the equation for a good observer

\[ \frac{d^2x}{dt^2} = -\Gamma^{\alpha}_{\beta\gamma} \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} \]

Consider Newton's equation for a particle on a gravitational potential \( \Phi \):

\[ (\text{mass}) \frac{d^2r}{dt^2} = (\text{mass}) g_{\alpha\beta} \Phi \]

These were the equations of motion before Einstein.

The equivalence principle:

There is no difference between noninertial "force" and gravitational "force," i.e., \( K = K' \).

Conclusion:

\[ \Gamma_{\alpha\beta} = 0 \Rightarrow \text{gravitational "force"} \]

In addition:
\[ \frac{d^2r}{dt^2} = -\Gamma^{\alpha}_{\beta\gamma} \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} \]

Rectilinear Coordinates.
Einstein always wondered why it is that one always finds that \[(\text{mass})_{\text{measured}} = (\text{mass})_{\text{gravitational}}\] just as quantum mechanically the observations of momentum and position are complementary. One can make observations to arbitrary accuracy of one type, or the other, but not both. The material composition of these particles is irrelevant.

In a free float frame (including in a freely falling one) all particles move in the same way, namely along straight world lines. Quantum mechanically, straight worldlines become plane waves, while non-rectilinear worldlines become distorted waves in a non-free float frame.

**END of PRIVATE THOUGHT.**

This locally rectilinear motion in spacetime becomes a non-rectilinear motion involving acceleration, which reexpressed in a non-free float frame (such as an accelerated one static relative to a gravitating body).

**PRIVATE THOUGHT: Observations made relative to locally inertial vs observations relative to accelerated frames are complementary. (Bohr: "mutually exclusive?"; Question: Is this really true? Answer: ?)"
Thus we conclude that Newton's equations of motion, after cancellation

\[ \dot{x} \times = \ddot{x} \times = 0 \]

become geometrical, i.e., become the equations for a geodesic. This gives rise to an astonishing conclusion: letting \( x = c \) one has

\[ \ddot{x} = \ddot{c} \times 0 \]

simply the inertial mass (now) proof is

\[ m(p) = (\text{mass}) \]

\[ \text{gravitational mass} \]

\[ \text{gravitational mass} \]

...
Imposing the boundary condition that
the relation between spacetime events
is Lorentzian in the absence of
gravitation, i.e.
\[ g_{00} = -1 \iff \phi = 0 \]
one obtains
\[ g_{00} = -\frac{1}{1 + \frac{2}{\phi}} \] (Newtonian gravitational potential)

This is a conceptual/mathematical
breakthrough of the first order.

Its qualitative implication is that
one should not think of gravitation
being expressed mathematically
in terms of a single gravitational
potential, but that instead

The establishment of this unifying
link between seemingly disparate
aspects of Nature serves to open new
vistas.
Let us summarize the newly introduced geometrical framework by highlighting its observational basis:

The state of motion of a free particle in the presence (or absence) of gravitation is a geodesic state of motion governed by a spacetime metric $g$.

Formulated mathematically, one has

$$\text{inertial } \frac{\partial}{\partial t} = m_{\text{inertial}} \frac{\partial}{\partial t} < g_{\rho\sigma}(\phi_{\mu\nu} + \phi_{\rho\sigma} - \phi_{\nu\rho}, \phi_{\sigma\mu}, \phi_{\tau\tau}) \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\sigma}$$

cancel.

$\Leftrightarrow \text{inertial } = m_{\text{gravitational}}$

A trajectory relative to all accelerated (non-inertial) frames:

$\text{all of Newtonian/Lagrangian mechanics?}$

The implication $\Leftarrow$ at the bottom of p. 41 is an example of scientific induction.

Nota Bene: Induction is the process of inferring generalizations from particular instances.

Einstein performed this in a paper he published in 1912. He was able to achieve this from a single observation, namely from:

(2) inertial = $m_{\text{gravitational}}$

Einstein concluded

gravitation = "geometry" (properly understood) (1)

However please notice that there was
The necessity of having at one's command all the relevant knowledge is the key to valid inductive reasoning, and it also is the reason why induction is so much more difficult than the complementary process, deduction—applying generalizations to new instances [e.g., calculating the precession of the perihelion of Mercury from the equation of a geodesic].

It is worth mentioning that there are other notable historical examples where one arrives at a generalization by means of inductive reasoning.

an enormous set of necessary concepts, principles, and theories that Einstein needed in order to induce (**) from (**), namely:

(i) "accelerated frame" = 1-parameter family of inertial frames

(ii) Newtonian mechanics

(iii) Newtonian gravitation

(iv) Hamiltonian/Lagrangian formulation of mechanics

(v) special relativity

Without (i)-(v) Einstein would not have been able to arrive at his inductive conclusion (** implies (**))
from a single observation. One of these is due to Benjamin Franklin. Flying a kite in a thunderstorm, he observed that sparks fly from a metal key to the wire on a Leyden jar.

From this he was able to conclude that lightning = electric discharge.

This and other important instances of valid inductive reasoning in physics, astronomy, chemistry, and philosophy are chronicled in David Harriman's "The Logical Leap: Induction in Physics."

It is understood, but worth mentioning that the domain of applicability of "gravitation" = "geometry" is based on the scope and the accuracy of the observational evidence underlying the scope includes classical (i.e. non-quantum) mechanics (whose mathematical formulation is usually in terms of systems of o.d.e.) but does not include elementary phenomena such as the accelerated frame-induced black body radiation, characterized by the North-Davies...
Temperature $T_i = \frac{e}{2 \pi}$.

The accuracy is determined by the technological landscape of the observations.

In summary, the validity of observational framework is constrained by and applies to the range of the domain of observations and their degree of accuracy. Both are limited to what is actually observed because of technological advances, the first has increased while the second has decreased.