Lecture 5

The mathematical bridge from Newton’s equations of motion to geometry.
Lecture 5:

consequences of

Extremal proper times (i) compatibility

between metric and parallel transport

[MTW §13.3] and (ii) geodesic deviation and curvature.

Jacobi's eqn of geodesic deviation

[MTW 11.3, Box 11.4];

(iii) the geodesic Lagrangian method

[MTW 14.4]
5.1 Mathematical comment about "geometry"

1) Hamilton's principle yields extremal curves whose tangents have components that satisfy
\[ \frac{du^\mu}{dt} + \Gamma^\mu_{\nu \lambda} u^\nu u^\lambda = 0, \quad \text{where} \quad u^\nu = \frac{dx^\nu}{dt} \]

or
\[ e_\nu \frac{d}{dx^\nu} u^\lambda + e_\nu \Gamma^\lambda_{\nu \mu} u^\mu = 0 \]

\[ e_\nu \nabla_\mu u^\lambda + \left( \nabla_\nu e_\lambda \right) u^\mu = 0 \]

\[ \nabla_\mu u^\nu e_\mu = 0 \]

Thus we have
\[ \nabla_\mu u^\nu = 0 \]

Thus we have

\[ \text{Schwinger's theorem (metric compatibility)} \]

\[
\begin{align*}
\text{metric} & \rightarrow \text{geodesics} \rightarrow \text{parallel transport} \\
g_{\mu \nu} & \rightarrow \frac{\delta L}{\delta x^\mu} = 0 \\
& \rightarrow \Gamma^\alpha_{\mu \nu} g_{\alpha \mu} + g_{\mu \alpha} - g_{\mu \nu} \Gamma^\alpha_{\nu \beta} \\
& \Rightarrow 0 = g_{\alpha, \mu} - g_{\mu, \alpha} \Gamma^\alpha_{\nu \beta} - g_{\mu, \alpha} \Gamma^\alpha_{\nu \beta} = \Gamma^\alpha_{\nu \beta} \\
& \Rightarrow \nabla_\mu (e_\alpha \cdot e_\beta) = e_\alpha \cdot \nabla_\mu e_\beta + (\nabla_\mu e_\alpha) \cdot e_\beta
\end{align*}
\]

5.2 These are the compatibility conditions between a metric and its parallel transport.

2) The nature of this parallel transport is determined by the Riemannian curvature.

From the viewpoint of physics, curvature is gravitation, i.e.
\[ \text{curvature} = 0 \quad \Rightarrow \quad \text{gravitation is absent} \]

or
\[ \text{curvature} \neq 0 \quad \Rightarrow \quad \text{gravitation is present} \]

Gravitation leaves its imprints on the states of geodesic motion:
\[ \nabla_\mu u^\nu = 0, \]

via Jacobi's equation of geodesic deviation
\[ \nabla_\mu \nabla_\nu u^\rho + R(\nu, \mu) u^\rho = 0, \]

where
\[ R(\nu, \mu) = [\nabla_\nu, \nabla_\mu] - \nabla_{\nu \mu} \]

is the Riemann curvature.

Let us recall the line of reasoning leading to this equation, and thereby determine its physical and/or geometrical significance.
Geodesic Deviation

(The effect of a gravitational field gradient on freely orbiting particles)

1. Consider a one parameter family of geodesics, \( \Gamma(\lambda, n) \), such that the curves corresponding to fixed parameter \( n \) are geodesics whose tangents are

\[ U = \frac{\partial}{\partial \lambda} \]

They satisfy the geodesic equation

\[ \nabla U = 0 \]

2. Let

\[ n = \frac{\partial}{\partial n} \lambda \]

be the vector which measures differences in coordinates

\[ n[x^i] = \frac{\partial x^i}{\partial n} \lambda \]

or in any other function

\[ n[f] = \frac{\partial f}{\partial n} \lambda \]

between corresponding (same \( \lambda \)) points on geodesics. This vector \( n \) is called the deviation vector.

From the definition of the vectors \( n \) and \( U \) as partial derivatives it follows that \( n \) and \( U \) commute,

\[ [n, U] = 0 \]

3. Let us now consider the curvature in
the plane spanned by \( \lambda \) and \( \nu \).

In this plane

\[ \nabla_\lambda \nabla_\nu - \nabla_\nu \nabla_\lambda = 0 \]

Consequently,

\[ R(\lambda, \nu) \nu = \nabla_\lambda \nabla_\nu \nu - \nabla_\nu \nabla_\lambda \nu = 0. \]

For geodesics one has \( \nabla_\nu \nu = 0 \). Thus

\[ R(\mu, \nu) \mu = -\nabla_\nu \nabla_\mu \mu. \]

We are assuming that the manifold for our geodesics has no torsion; thus

\[ \nabla_\nu \tilde{\nu} - \nabla_{\tilde{\nu}} \nu = [\nu, \tilde{\nu}] = 0. \]

The fact that \( \nu \) and \( \tilde{\nu} \) commute implies therefore that

\[ \nabla_\nu \tilde{\nu} = \nabla_{\tilde{\nu}} \nu \]

One obtains therefore the final result

\[ \nabla_\nu \nabla_\nu \nu + R(\mu, \nu) \mu = 0 \]

the equation of geodesic deviation.

Calamai's Variants: "Jacobi Eqn."

In Calculus of Variations this equation is also called the Jacobi equation. It is the equation satisfied by the difference between two neighboring ("infinitely close") extremals.

See e.g. Gelfand & Fomin, Sect. 26

and also MTW, Ch. 11

The best way of seeing the content of the equation of geodesic deviations is to choose an appropriate vector basis at each point.

At one point of any of the geodesics (say on the \( \mu = 0 \), geodesic) choose any basis

\[ \{e_1, e_2, \ldots, e_n\} \]

which may contain \( \nu \) as one of the basis vectors.

Extend this basis to all points of the geodesic by using these basis vectors as initial conditions for solving the parallel transport differential equation,

\[ \nabla_{\tilde{\nu}} (e_i) = 0 \Rightarrow e_i(\lambda) \]

along the curve \( \gamma(\lambda) \)
Using these basis of vectors, which are parallel (constant) along the curve, write
\[ \sigma^2 = n^i e_i. \]

Thus
\[ \nabla_\lambda R = \nabla_\lambda (n^i) e_i = \frac{d}{d\lambda} (n^i) e_i, \]
and the equation of geodesic deviation becomes
\[ R(\lambda, \lambda) e_i = \nabla_\lambda \nabla_\lambda n^i = R^j_k \epsilon^i_{jk} e_i. \]

In the absence of curvature, we see that the deviation vector depends linearly on x,
\[ n^i = a^i \lambda + b^i. \]

Evidently, the curvature term in the equation measures precisely the extent to which this linear deviation law for straight lines fails.

This term evidently measures the curvature of nearly (small n=16.1) geodesics relative to the one taken as a base line (n=16.1).

In Newtonian theory, the time and the equation yield the relative acceleration between adjacent freely orbiting particles. One therefore infers from this fact that curvature is an expression of gravitational field gradient.

Consider
\[ n^i = r^i, \]
with \( \frac{dr^i}{d\lambda} = 0 \) \( \forall \lambda \).

Let \( f^i = -k^i_\delta \delta^\delta \) be the restoring force. Then
\[ \frac{d^2 r^i}{d\lambda^2} + R^i_{jik}(\dot{r}^j \dot{r}^k) = -\frac{1}{m} k^i_\delta \delta^\delta, \]
so that
\[ \frac{d^2 r^i}{d\lambda^2} + (\frac{1}{m} k^i_\delta + R^i_{jik} \dot{r}^j \dot{r}^k) \delta^\delta = -R^i_{jik} \dot{r}^j \dot{r}^k \delta^\delta, \] is the eq'n governing the displacement \( \delta^\delta \).
Rotation = Tidal Acceleration

Consider two close-by geodesics and the evolution of the connecting vector \( \mathbf{a} \) between them.

\[
\mathbf{a}_{i+1} = \mathbf{a}_i + \Delta \mathbf{a}_i + \frac{1}{2} \Delta \mathbf{a}_{i+1} \mathbf{\nabla}_{\mathbf{a}_i} \mathbf{\nabla}_{\mathbf{a}_i} \mathbf{a}_{i+1} + \ldots
\]

\[
= \text{change in } \mathbf{a}_i \text{ as } s_i \to s_{i+1}
\]

or equivalently

Adding the two, one obtains the total change in \( \mathbf{a}_i \).

\[
\Delta \mathbf{a}_i = \frac{1}{2} \mathbf{\nabla}_{\mathbf{a}_i} \mathbf{\nabla}_{\mathbf{a}_i} \mathbf{a}_i
\]

\[
\mathbf{\nabla}_{\mathbf{a}_i} \mathbf{\nabla}_{\mathbf{a}_i} \mathbf{a}_i = \frac{1}{2} (\mathbf{a}_i \times \mathbf{\Delta a}_i) \times \mathbf{\Delta a}_i
\]
Without changing the relative acceleration between the two geodesics, we can tilt the right-hand geodesic clockwise around $\alpha_0$ until the two geodesics are parallel at $\alpha_0 = \alpha_0$. In that case, the picture at the top of this page becomes

\[ \nabla_{\alpha} \nu \nabla_{\alpha} \nu \eta = \delta^\alpha \nu \]

\[ -\delta^\alpha \nu [=-\mathcal{R}(\alpha \nu \nu, \alpha) \alpha \nu \nu = -\delta^\alpha \nu \]

\[ \alpha_0 = \alpha_0, \quad -\delta^\alpha \nu \]

\[ \text{parallel transport was done clockwise} \]

\[ \text{translate of } \alpha \nu \nu \text{ from } 1 \text{ along } \alpha_0 \]

\[ \alpha \nu \nu = \text{tangent at } \alpha_0 \]

\[ \text{tangent at } \alpha_0 \]

Comparing the two pictures on this page we obtain

\[ \frac{\Delta \nu}{\Delta \alpha} = -\mathcal{R}(\nu, \nu, \nu) \nu \]

or

\[ \frac{\Delta \nu}{\Delta \alpha} + \mathcal{R}(\nu, \nu, \nu) \nu = 0 \]
Properties of Curvature Tensors

Curvature tensors derived from metric induced connection have additional symmetries above and beyond those which all curvature tensors have.

Consider the curvature operator

\[ R(\alpha, \tau) = \left[ \nabla_\alpha, \nabla_\tau \right] - \nabla_{[\alpha, \tau]} \]

Letting it operate on functions and recalling that

\[ \nabla_\alpha f = \alpha[f] = \partial_\alpha f \]

we see that

\[ R(\alpha, \tau) f = 0 \]

In particular

\[ R(\alpha, \tau) u \cdot v = 0 \]

Expanding the left hand side with the help of the compatibility condition

\[ \nabla_\alpha u \cdot v = u \cdot \nabla_\alpha v + (\nabla_\alpha u) \cdot v \]

One obtains

\[ 0 = \{ \nabla_\alpha u \cdot \nabla_\tau v - \nabla_\tau u \cdot \nabla_\alpha v \} u \cdot v = \]

\[ = \nabla_\alpha \left[ u \cdot \nabla_\tau v + (\nabla_\tau u) \cdot v \right] - \nabla_\tau \left[ u \cdot \nabla_\alpha v + (\nabla_\alpha u) \cdot v \right] \]

\[ = u \cdot \nabla_\alpha \left[ \nabla_\tau v - \nabla_{[\alpha, \tau]} v \right] - (\nabla_{[\alpha, \tau]} u) \cdot v \]

\[ = (\nabla_\alpha u)(\nabla_\tau v) + u \cdot \nabla_\alpha \nabla_\tau v + (\nabla_\alpha \nabla_\tau u) v + u \cdot \nabla_{\nabla_\alpha \nabla_\tau} v \]

\[ - t \rightarrow s \]

\[ - u \cdot \nabla_{[\alpha, \tau]} v = (\nabla_{[\alpha, \tau]} u) \cdot v \]

The \( (\nabla_\alpha u)(\nabla_\tau v) \) term gets cancelled:

\[ = u \cdot \nabla_{[\alpha, \tau]} v + u \cdot \nabla_\alpha \nabla_\tau v + u \cdot \nabla_{\nabla_\alpha \nabla_\tau} v \]

\[ = u \cdot \nabla_\alpha \nabla_\tau v + u \cdot \nabla_{\nabla_\alpha \nabla_\tau} v \]

\[ = u \cdot \nabla_{\nabla_\alpha \nabla_\tau} v \]

\[ = u \cdot R_{\alpha \beta \tau}^{\gamma} k \cdot e \cdot \left( \omega^{\alpha} / \omega^{\beta} \right) \left( \omega^{\gamma} / \omega^{\tau} \right) + u \cdot v \]

\[ = (u \cdot \nabla_{\nabla_\alpha \nabla_\tau} v) \cdot v + u \cdot \nabla_{\nabla_\alpha \nabla_\tau} v \]

\[ = u \cdot v \cdot (R_{\alpha \beta \tau}^{\gamma} k \cdot e + R_{\alpha \beta \tau}^{\gamma} k \cdot e) \cdot \omega^{\alpha} / \omega^{\beta} \]

This holds, \( \forall \{ u^{\beta}, \{ \omega^{\beta}, \{ k^{\beta} \} \} \} \) and \( \{ e \} \) Consequently

\[ R_{\alpha \beta \gamma} = - R_{\alpha \gamma \beta} \]
Consolidating everything, yields the following tree of ideas:

- Newton's equations of motion
- Equivalence Principle
  - Geodesics and metric
    - Parallel transport = "GEOMETRY"
      - Curvature
Geodesic Lagrangian Method

of computing Christoffel symbols.

The most efficient way of computing connection coefficients relative to a coordinate basis is via the Geodesic Lagrangian Method.

Example (Connection on $S^2$)

Given: Metric on $S^2$: $ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2$

Step I: Set up the variational integral

$I = \int (a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2) d\lambda = \int \mathcal{L} d\lambda$
The most efficient way of computing connection coefficients relative to a coordinate basis:

Geodesic Lagrangian method of computing Christoffel Symbols

Given: Metric on $S^2$ 

$$ds^2 = a^2 (d\theta^2 + a^2 \sin^2 \theta d\phi^2)$$

Step I. Setup the variational integral

Step II. Vary the curve, one at a time, in their dependence on $\lambda$

a) First vary $\theta(\lambda)$ keeping $\phi(\lambda)$ fixed:

$$\theta(\lambda) \rightarrow \theta(\lambda) + \delta \theta(\lambda)$$

$$\phi(\lambda) \rightarrow \phi(\lambda) + \delta \phi(\lambda)$$

The requirement that $\delta I = 0$ for such a variation results in

$$\frac{d}{d\lambda} \frac{\delta L}{\delta \dot{\theta}} - \frac{\delta L}{\delta \theta} = 0$$

$$(2a^2 \dot{\theta})^2 - 2a^2 \sin \theta \cos \theta \phi^2 = 0$$

b) Similarly for $\phi(\lambda)$:

$$(2a^2 \sin^2 \theta \dot{\phi})^2 - 0 = 0$$

Step III. Re-arrange to get $\ddot{x}^m$ as the leading term. If this step is not straightforward, this method will not save time. A direct computation of the Christoffel symbols may be more
suitable.

a) \( \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0 \)
b) \( \ddot{\phi} + 2 \cot \theta \dot{\phi} \dot{\theta} = 0 \)

Step IV Read out the Christoffel symbols \( \Gamma^m_{\alpha \beta} \) by comparing the above equations to

\[ \ddot{x}^m + \Gamma^m_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta = 0 \quad (*) \]

Result:

\[
\begin{align*}
\Gamma^\theta_{\phi \phi} &= -\sin \theta \cos \theta \\
\Gamma^\phi_{\phi \phi} &= \Gamma^\phi_{\theta \theta} = \cot \theta
\end{align*}
\]

There is no factor of 2 because the velocity terms in Eqm. (*) are \( \Gamma^m_{\phi \phi} \dot{\phi}^2 + \Gamma^m_{\theta \phi} \dot{\phi} \dot{\theta} + \cdots \).