Lecture 5

The mathematical bridge from Newton's equations of motion to geometry.

Lecture 5.

Consequences of

Extremal proper time: (i) compatibility between metric and parallel transport [MTW §13.3] and (ii) geodesic deviation and curvature.

Jacobi's eq'n of geodesic deviation [MTW 11.3, Box 11.4];

(iii) the geodesic Lagrangian method [MTW 14.4]

Mathematical Comment about "geometry"

1) Hamilton's principle yields extremal curves whose tangents

$$\mathbf{u} = e_\nu \frac{dx^\nu}{d\tau}$$

have components that satisfy

$$\frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta = 0, \text{ where } u^\alpha = \frac{dx^\alpha}{d\tau}$$

OR

$$e_\nu u^\beta \frac{\partial}{\partial x^\beta} u^\nu + e_\nu \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta = 0$$

$$e_\nu \nabla_{\mathbf{u}} u^\nu + (\nabla_{\mathbf{u}} e_\alpha) u^\alpha = 0$$

$$\nabla_{\mathbf{u}} u^\alpha e_\alpha = 0$$

$$\boxed{\nabla_{\mathbf{u}} \mathbf{u} = 0}$$

Thus we have

Hamilton's Principle — SCHILD'S LADDER (METRIC COMPATIBILITY)

metric $\rightarrow$ geodesics $\rightarrow$ parallel xport

$$g_{\mu\nu} \rightarrow \frac{\delta L}{\delta x^\mu} = 0 \rightarrow \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$\Leftrightarrow 0 = g_{\alpha\beta,\mu} - g_{\alpha\nu}\Gamma^\nu_{\beta\mu} - g_{\nu\beta}\Gamma^\nu_{\alpha\mu} \quad (\equiv g_{\alpha\beta;\mu})$$

$$\Leftrightarrow \nabla_\mu(e_\alpha \cdot e_\beta) = e_\alpha \cdot \nabla_\mu e_\beta + (\nabla_\mu e_\alpha)\cdot e_\beta$$

These are the compatibility conditions between a metric and its parallel transport.

2) The nature of this parallel transport is determined by the Riemannian curvature.

From the viewpoint of physics, curvature is gravitation, i.e.

curvature $= 0 \Leftrightarrow$ gravitation is absent

or curvature $\neq 0 \Leftrightarrow$ gravitation is present

Gravitation leaves its imprints on the states of geodesic motion,

$$\nabla_{\mathbf{u}} \mathbf{u} = 0,$$

via Jacobi's equation of geodesic deviation

$$\nabla_{\mathbf{u}} \nabla_{\mathbf{u}} + R(\mathbf{n}, \mathbf{u})\mathbf{u} = 0,$$

where

$$R(\mathbf{n}, \mathbf{u}) = [\nabla_{\mathbf{n}}, \nabla_{\mathbf{u}}] - \nabla_{[\mathbf{n}, \mathbf{u}]}$$

is the Riemann curvature.

Let us recall the line of reasoning leading to this equation, and thereby determine its physical and/or geometrical significance.

Curvature = relative acceleration of geodesics
= tidal force

Geodesic Deviation

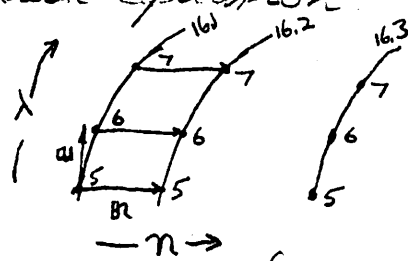
(The effect of a gravitational field gradient on freely orbiting particles)

1. Consider a one parameter family ("congruence") of geodesics, $P(\lambda, n)$, such that the curves corresponding to fixed parameter n are geodesics whose tangents are

$$\mathbf{u} = \left.\frac{\partial}{\partial \lambda}\right)_n$$

They satisfy the geodesic equation

$$\nabla_{\mathbf{u}} \mathbf{u} = 0$$



2. Let

$$\mathbf{n} = \left.\frac{\partial}{\partial n}\right)_\lambda$$

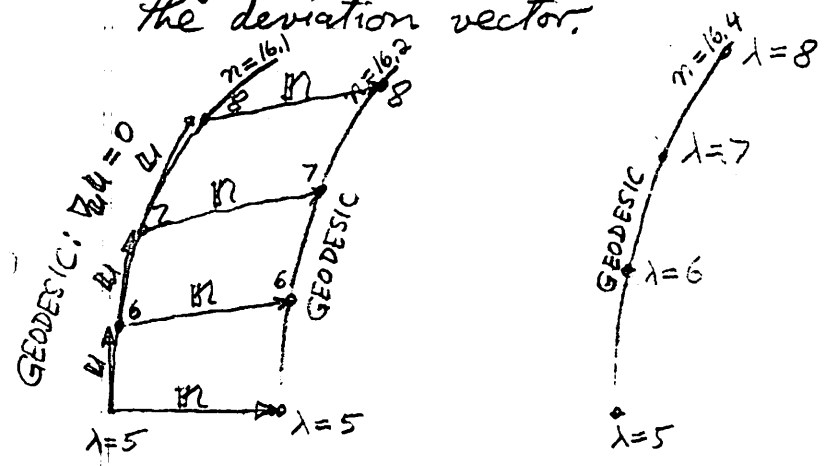
be the vector which measures differences in coordinates

$$\mathbf{n}[x^i] = \left.\frac{\partial x^i}{\partial n}\right)_\lambda$$

or in any other function

$$\mathbf{n}[f] = \left.\frac{\partial f}{\partial n}\right)_\lambda$$

between corresponding (same λ) points on geodesics. This vector $\mathbf{n}$ is called the deviation vector.



From the definition of the vectors $\mathbf{u}$ and $\mathbf{n}$ as partial derivatives it follows that $\mathbf{n}$ and $\mathbf{u}$ commute,

$$[\mathbf{n}, \mathbf{u}] = 0.$$

3. Let us now consider the curvature in

the plane spanned by λ and n.
In this plane

$$\nabla_{[n,u]} = 0$$

Consequently,

$$R(n,u) = \nabla_n \nabla_u - \nabla_u \nabla_n - \overset{0}{\overset{\shortparallel}{\nabla_{[n,u]}}}$$

For geodesics one has $\nabla_u u = 0$. Thus

$$R(n,u)u = -\nabla_u \nabla_n u .$$

We are assuming that the manifold for our geodesics has <u>no</u> torsion: Thus

$$\nabla_u n - \nabla_n u = [u,n] \overset{!}{=} 0$$

The fact that u and n commute implies therefore that

$$\nabla_n u = \nabla_u n$$

One obtains therefore the final result

$$\boxed{\nabla_u \nabla_u n + R(n,u)u = 0},$$

the equation of <u>geodesic deviation</u>.
(Calculus of Var'n: "Jacobi Eq'n")

In Calculus of Variation this equation is also called the Jacobi equation. It is the equation satisfied by the difference between two neighboring ("infinitely close") extremals.
See e.g. Gelfand & Fomin sect 26
and also MTW Ch. 11

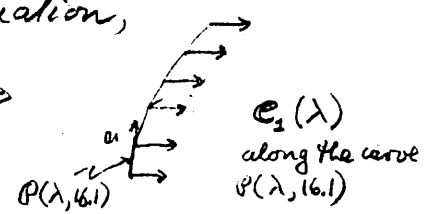
4. The best way of seeing the content of the equation of geodesic deviation is to choose an appropriate vector basis at each point.

At one point of any of the geodesics (say on the $n = 16.1$ geodesic) choose any basis

$$\{e_1, e_2, \dots, e_n\}$$

which may contain u as one of the basis vectors.
Extend this basis to all points of the geodesic by using these basis vectors as <u>initial conditions</u> for solving the parallel transport differential equation,

$$\nabla_u e_i = 0 \Rightarrow$$



Using these basis of vectors, which are parallel (constant) along the curve write

$$\mathbf{n} = n^i \mathbf{e}_i$$

Thus

$$\nabla_{\mathbf{u}} \mathbf{n} = \nabla_{\mathbf{u}}(n^i)\mathbf{e}_i$$

$$= \frac{d n^i}{d\lambda}\mathbf{e}_i$$

and the equation of geodesic deviation becomes

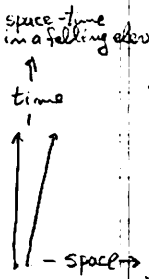
$$\mathbb{R}(\mathbf{w}, \mathbf{v}, \mathbf{u}) = \mathbf{e}_i w^j R^i{}_{jkl} v^k u^l$$

$$\frac{d^2 n^i}{d\lambda^2}\mathbf{e}_i + \mathbb{R}(n^j \mathbf{e}_j, \mathbf{u})\mathbf{u} = 0$$

$$\mathbb{R}(\mathbf{e}_k u^k, n^j \mathbf{e}_j, \mathbf{u}) = \mathbf{e}_i (R^i{}_{kjl} u^k u^l) n^j$$

or

$$\boxed{\frac{d^2 n^i}{d\lambda^2} + R^i{}_{uju}\, n^j = 0}$$



In the absence of curvature we see that the deviation vector depends linearly on λ,

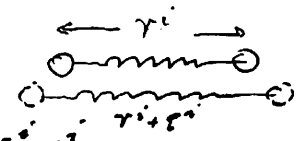
$$n^i = a^i \lambda + b^i$$

Evidently the curvature term in the equation measures precisely the extent to which this <u>linear</u> deviation law for

straight lines fails.
This term evidently measures the curvature of nearby (small n-16.1) geodesics relative to the one taken as a base line (n=16.1).

In Newtonian theory $\lambda \propto$ time and the equation yield the <u>relative acceleration</u> between adjacent freely orbiting particle.
One therefore infers from this fact that <u>curvature</u> is an expression of <u>gravitation field gradient</u>.

Consider



$$n^i = r^i + \xi^i$$

with $\frac{dr^i}{d\lambda} = 0 \quad \forall \lambda$.

Let $f^i = -k^i{}_j \xi^j$ be the restoring force

Then

$$\frac{d^2\xi^i}{d\lambda^2} + R^i{}_{ujl}(r^j \xi^l) = -\frac{1}{m} k^i{}_j \xi^j$$

so that

$$\frac{d^2\xi^i}{d\lambda^2} + \left(\frac{1}{m} k^i{}_j + R^i{}_{ujl}\right)\xi^j = -R^i{}_{ujl}\, r^j$$

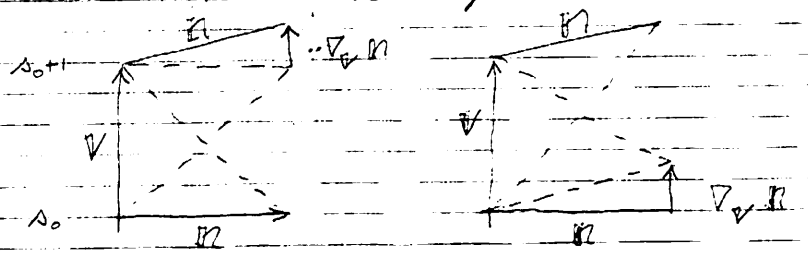
is the eq'n governing the displacement ξ^i

Rotation = Tidal Acceleration

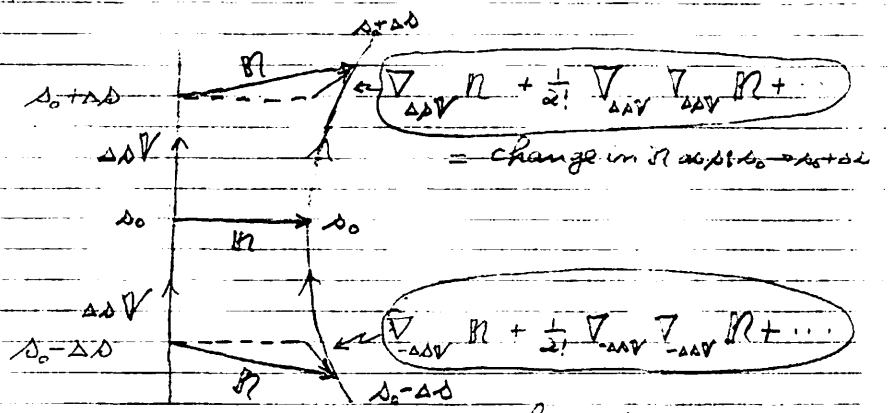
Curvature (i.e. tidal forces) manifests itself in two seemingly different ways: (i) Rotation of a vector upon being carried around a closed loop and (ii) relative acceleration between a pair of nearby geodesics.

We shall show that these two are in fact different aspects of one and the same curvature.

Everything hinges directly on the idea of parallel transport which is expressed by the two mnemonic diagrams



Consider two close-by geodesics and the evolution of the connecting vector n between them



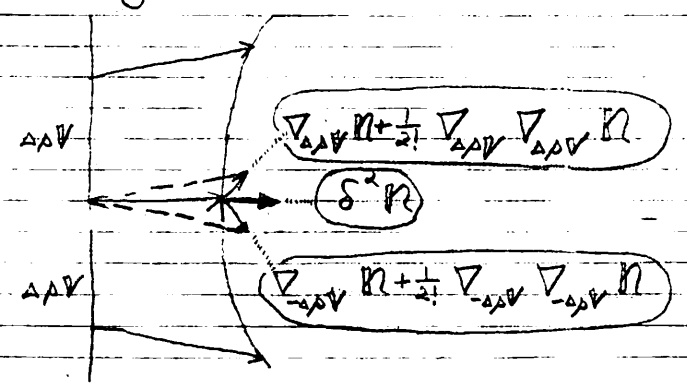
$$\nabla_{\Delta s V} n + \frac{1}{2!}\nabla_{\Delta s V}\nabla_{\Delta s V} n + \cdots$$

= change in n as s: $s_0 \to s_0+\Delta s$

$$\nabla_{-\Delta s V} n + \frac{1}{2!}\nabla_{-\Delta s V}\nabla_{-\Delta s V} n + \cdots$$

= change in n as s: $s_0 \to s_0-\Delta s$

or equivalently



$$\nabla_{\Delta s V} n + \frac{1}{2!}\nabla_{\Delta s V}\nabla_{\Delta s V} n$$

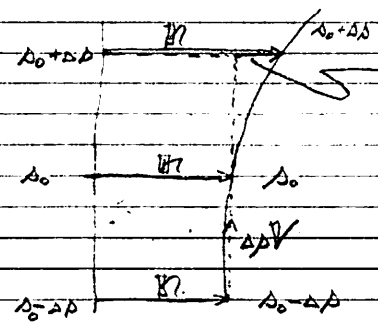
$$\delta^2 n$$

$$\nabla_{-\Delta s V} n + \frac{1}{2!}\nabla_{-\Delta s V}\nabla_{-\Delta s V} n$$

Adding the two, one obtains the total change in n as s: $s_0-\Delta s \to s_0+\Delta s$

$$\boxed{\delta^2 n = \frac{2}{2!}(\Delta s)^2 \nabla_V \nabla_V n}$$

Without changing the relative acceleration between the two geodesics, we can tilt the right hand geodesic clockwise around s_0 until the two geodesics are parallel at $s_0-\Delta s$. In that case the picture at the top of P4.11 becomes



$$\nabla_{\Delta s\mathbb{V}}\nabla_{\Delta s\mathbb{V}}\mathbb{n} = \delta^2\mathbb{n}$$

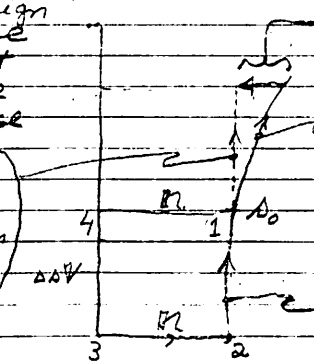
or

$$-(\Delta s)^2\nabla_{\mathbb{V}}\nabla_{\mathbb{V}}\mathbb{n} = -\delta^2\mathbb{n}$$

EQUAL

$$-\delta^2\mathbb{n}\ \left[= -R(\Delta s\mathbb{V},\mathbb{n})\Delta s\mathbb{V}\right]$$

minus sign because transport was done clockwise



parallel translate of $\Delta s\mathbb{V}$ from 1 along $1\to2\to3\to4$

$\Delta s\mathbb{V}$ = tangent at s_0

PARALLEL because curve is a geodesic : $\nabla_{\mathbb{V}}\mathbb{V}=0$

$\Delta s\mathbb{V}$ = tangent at $s_0-\Delta s$

Comparing the two pictures on this page we obtain

$$\frac{\delta^2\mathbb{n}}{(\Delta s)^2} = -R(\mathbb{V},\mathbb{n})\mathbb{V}$$

or

$$\boxed{\frac{\delta^2\mathbb{n}}{(\Delta s)^2} + R(\mathbb{n},\mathbb{V})\mathbb{V} = 0}$$ as it should be

Properties of Curvature Tensors

Curvature tensors derived from metric induced connection have additional symmetries above and beyond those which all curvature tensors have

Consider the curvature operator

$$\mathbb{R}(s,t) = [\nabla_s, \nabla_t] - \nabla_{[s,t]}$$

Letting it operate on functions and recalling that

$$\nabla_s f \equiv s[f] \equiv \partial_s f$$

we see that

$$\mathbb{R}(s,t) f = 0$$

In particular

$$\mathbb{R}(s,t)\, u\cdot v = 0$$

Expanding the left hand side with the help of the compatibility condition

$$\nabla_s u\cdot v = u\cdot \nabla_s v + (\nabla_s u)\cdot v$$

One obtains

$$0 = \{\nabla_s\nabla_t - \nabla_t\nabla_s - \nabla_{[s,t]}\} u\cdot v =$$

$$= \nabla_s[u\nabla_t v + (\nabla_t u) v] - \nabla_t[u\nabla_s v + (\nabla_s u) v]$$

$$- u\,\nabla_{[s,t]} v - (\nabla_{[s,t]} u)\cdot v$$

$$= (\nabla_s u)(\nabla_t v) + u\,\nabla_s\nabla_t v + (\nabla_s\nabla_t u) v + \nabla_t u\cdot\nabla_s v$$

$$- t \leftrightarrow s$$

$$- u\,\nabla_{[s,t]} v - (\nabla_{[s,t]} u) v$$

The $(\nabla_s u)(\nabla_t v)$ term gets cancelled:

$$= u\left(\nabla_s\nabla_t - \nabla_t\nabla_s - \nabla_{[s,t]}\right) v + v\left(\nabla_s\nabla_t - \nabla_t\nabla_s - \nabla_{[s,t]}\right) u$$

$$= u\cdot\mathbb{R}(v,s,t) + v\cdot\mathbb{R}(u,s,t)$$

$$= u\cdot e_i R^i{}_{jkl} \langle\omega^j|v\rangle\langle\omega^k|s\rangle\langle\omega^l|t\rangle + u \leftrightarrow v$$

$$= (u_i R^i{}_{jkl} v^j + v_i R^i{}_{jkl} u^j)\langle\omega^k|s\rangle\langle\omega^l|t\rangle$$

$$= u^i v^j (R_{ijkl} + R_{jikl})\, s^k t^l$$

This holds $\forall\ \{u^i\}, \{v^j\}\ \{s^k\}$ and $\{t^l\}$

Consequently

$$\boxed{R_{ijkl} = -R_{jikl}}$$

Consolidating everything, yields the following tree of ideas

Newton's equations of motion

$\downarrow$ ← Equivalance Princeple

Geodesics and metric
$\updownarrow$
Parallel transport
$\downarrow$
Curvature

= "GEOMETRY"

Geodesic Lagrangian Method

of computing Christoffel Symbols.

The most efficient way of computing connection coefficients relative to a coordinate basis is via the Geodesic Lagrangian Method.

Example (Connection on S^2)

Given: Metric on S^2: $ds^2 = a^2 d\theta + a^2 \sin^2\theta \, d\varphi^2$

Step I: Set up the variational integral

$$I = \int \left(a^2 \dot{\theta}^2 + a^2 \sin^2\theta \, \dot{\varphi}^2 \right) d\lambda \equiv \int L \, d\lambda$$

The most efficient way of computing connection coefficients relative to a coordinate Basis:

Geodesic Lagrangian method of computing Christoffel Symbols

Given: Metric on S^2 $\quad ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi$

Step I. Setup the variational integral

Step II Vary the curve components, one at a time, in their dependence on λ

a) First vary $\theta(\lambda)$ keeping $\varphi(\lambda)$ fixed:

$$\theta(\lambda) \to \theta(\lambda) + \delta\theta(\lambda)$$
$$\varphi(\lambda) \to \varphi(\lambda) + \delta\varphi(\lambda)$$

The requirement that $\delta I = 0$ for such a variation results in

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0$$

$$(2a^2\dot\theta)^{\cdot} - 2a^2 \sin\theta\cos\theta\, \dot\varphi^2 = 0$$

b) Similarly for $\varphi(\lambda)$:

$$(2a^2 \sin^2\theta\, \dot\varphi)^{\cdot} - 0 = 0$$

Step III Rearrange to get $\ddot{x}^\mu$ as the leading term. If this step is not straight forward, this method will not save time. A direct computation of the Christoffel symbols maybe more

suitable,

a) $\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = 0$

b) $\ddot{\varphi} + 2\cot\theta\,\dot{\varphi}\dot{\theta} = 0$

Step IV Read out the Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$ by comparing the above equations to

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\alpha\beta}\,\dot{x}^{\alpha}\dot{x}^{\beta} = 0 \qquad (*)$$

Result:

$$\boxed{\begin{aligned} \Gamma^{\theta}{}_{\varphi\varphi} &= -\sin\theta\cos\theta \\ \Gamma^{\varphi}{}_{\theta\varphi} &= \Gamma^{\varphi}{}_{\varphi\theta} = \cot\theta \end{aligned}}$$

There is no factor of 2 because the velocity terms in Eq'n * are $\Gamma^{\mu}{}_{\theta\varphi}\,\dot{\theta}\dot{\varphi} + \Gamma^{\mu}{}_{\varphi\theta}\,\dot{\varphi}\dot{\theta} + \ldots$