Lecture 7

Momentum and Energy: Their definition and conservation

[Taylor & Wheeler, ch. 7]
Proof: a) What is the relation between $\mathbf{p}$, momentum, and $\mathbf{v}$, velocity of the particle?

A1: $\mathbf{p} = f(\mathbf{v}) \mathbf{v}$

Why? (i) Isotropy of space

(ii) $\mathbf{p}$ is unique

(iii) $f(\mathbf{v}) = f(\sqrt{\mathbf{v} \cdot \mathbf{v}})$

because of isotropy of space

b) Principle of Relativity $\Rightarrow f(\mathbf{v})$ is the same for in all inertial reference frames.

c) To determine $f$, consider the elastic collision between two identical particles viewed in two inertial frames.

\[ \begin{align*}
B & \to A \quad \frac{1}{2} \Delta y = \Delta y \\
A' & \to B' \quad \frac{1}{2} \Delta y = \Delta y \\
A & \to B \quad \frac{1}{2} \Delta y = \Delta y \\
\end{align*} \]

LAB MOVING FRAME

c) Two pictures are $R \Rightarrow$ symmetrical

Isotropy of space

Symmetry $\Rightarrow$ pictures are congruent.
(2) Exhibit components of A is MOVING and LAB frame.

(2.1) In MOVING frame

A's velocity = \( \frac{\Delta y}{\Delta t} = \mu \)

\( \Delta t \) = time for A to move from bottom to top of collision

A's four vector = \( (\Delta t, \Delta y, 0) \)

(2.2) In LAB frame

A's four vector = \( \left( \frac{\Delta E}{\sqrt{1-\beta^2}}, \frac{\Delta x}{\sqrt{1-\beta^2}}, \Delta y, 0 \right) \)

P.t.R. & isotropy of space.

A's spatial velocity = \( (\beta, \frac{\Delta y}{\Delta t} \sqrt{1-\beta^2}, 0) \)

= \( (\beta, \mu \sqrt{1-\beta^2}, 0) \)

(3) Conservation of momentum in LAB frame
(e) (cont'd)

(ii) $X$ component in LAB frame:

$$f\left(\beta^2 + \mu^2(1-\beta^2)\right) \cdot \beta + O = f\left(\sqrt{\beta^2 + \mu^2(1-\beta^2)}\right) \beta + O$$

is trivially true; no information!

(iii) $Y$ component to be conserved $P_y$ before $P_y$ after.

$$f\left(\sqrt{\beta^2 + \mu^2(1-\beta^2)}\right) \mu\sqrt{1-\beta^2} - f(\mu)\mu = -f\left(\sqrt{\beta^2 + \mu^2(1-\beta^2)}\right) \mu\sqrt{1-\beta^2}$$

$$\Rightarrow f\left(\sqrt{\beta^2 + \mu^2(1-\beta^2)}\right) = \frac{f(\mu)}{\sqrt{1-\beta^2}}$$

let $\mu = 0$

whose solution is $f(\beta) = \frac{f(0)}{\sqrt{1-\beta^2}}$

Newtonian correspondence as $\beta \to 0$ implies

$$1 = \frac{P_x}{P_y} = \lim_{\beta \to 0} \frac{m \mu \beta}{f(\beta) v_y} \quad \rightarrow \quad f(0) = m$$

CONCLUSION

$$\text{momentum} = \frac{m}{\sqrt{1-\beta^2}}, \beta$$

Note: if $f(\mu)$ is independent of $\mu$, then one implicitly assumes Newtonian approximation which does not apply to relativistic mechanics.
A. Momentum in Relativity (and d)

1. We found (Symmetry, Relativity) that the quantity

$$\frac{m}{\sqrt{1-v^2}} \text{ velocity} = \text{momentum}$$

is conserved in elastic collisions. It also is conserved in inelastic collisions -- in all collisions.

2. Energy - Momentum 4-vector

a. We have seen that unit tangent to the world line of particle

$$u^\alpha = \left(\frac{1}{\sqrt{-g}}, \frac{\dot{x}^\mu}{c}, \frac{\dot{x}^\mu}{c}, \frac{1}{c} \right)$$

is a 4-vector [transforms like \(x^\alpha\) under a Lorentz transformation]

b. To connect 4-velocity to momentum, rewrite \(u^\alpha\) in a new form

(1) \(d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = (c^2 - 1)dt^2\)

(2) Thus \(u^\alpha = \left(\frac{1}{\sqrt{-g}}, \frac{\dot{x}^\mu}{c}, \frac{\dot{x}^\mu}{c}, \frac{1}{c} \right)\) four-velocity

c. Combine (1) \& (2)

momentum \(= m\) (space part of 4-momentum)

\[\mathbf{p} = m \mathbf{u} = \frac{m^2 c^2}{\sqrt{1-v^2}}\]

Given: (i) Principle of Relativity

(ii) Conservation of spatial momentum (during a collision of particles)

Conclusion: Energy of particles is conserved

proof: Let \(\{u^\alpha\}\) = 4-velocity of a particle

Use transformation law for \(u^\alpha\) to get that of momentum for each particle

\[p_x = m u_x = m \left(\bar{u}_x \cosh \theta + \bar{u}_y \sinh \theta\right)\]

\[= \bar{p}_x \cosh \theta + \frac{m}{\sqrt{1-\beta^2}} \sinh \theta\]

\(\bar{p}\) Consider a collision of particles as viewed in lab frame as well as in moving frame

\[\sum \mathbf{p}_{\text{after}} = \mathbf{p}_{\text{before}}\]
3. Consider a collision of particles as viewed in LAB frame as well as in MOVING frame:

\[
0 = \sum_i^2 \mathbf{P}_x(i) \mid \text{after} - \sum_i^2 \mathbf{P}_x(i) \mid \text{before} = \left(\sum_i \mathbf{P}_x(i) \mid \text{after} - \sum_i \mathbf{P}_x(i) \mid \text{before}\right) \cos \theta
\]

\[
+ \left(\sum_i \frac{m_i}{\sqrt{1 - \beta^2}} \mid \text{after} - \sum_i \frac{m_i}{\sqrt{1 - \beta^2}} \mid \text{before}\right) \sin \theta
\]

Conclusion:
\[
\sum_i \frac{m_i}{\sqrt{1 - \beta^2}} \text{ is also conserved.}
\]

3. Go to the Newtonian limit:
\[
\frac{mc^2}{\sqrt{1 - \beta^2}} = m c^2 \left(1 + \frac{1}{2} \frac{\beta^2}{c^2} - \frac{1}{8} \frac{\beta^4}{c^4} + \ldots\right)
\]

\[
= mc^2 + \frac{1}{2} m v^2 + \ldots = \text{energy of particle}
\]

Summary: Momentum conservation + Principle of Relativity \Rightarrow Energy conservation Q.E.D.

\[\sum_{i=1}^{N} \mathbf{P}_i = \mathbf{P}_{\text{total}} = \sum_{i=1}^{N} \mathbf{P}_i\]
Proposition (Energy-momentum 4-vector)

1. Definition
   \[ P_i = \left\{ p_i^0 = \frac{m^2}{\sqrt{1 - v^2}}, \frac{m^2 v_i}{\sqrt{1 - v^2}} \right\} \text{ for each particle} \]

2. Magnitude
   \[ \sqrt{(p_i^0)^2 - p_i^2} = m \]

3. Projection onto the zero-axis = \( E \) (energy)

\[ \sum_{i=1}^{N} P_i \text{ before} = P_{\text{tot before}} \]
\[ = P_{\text{tot after}} = \sum_{j=1}^{M} P_j \]

4. Every application of the law of conservation of momentum in a collision is a statement about a polygon built of 4-vectors in spacetime.