Lecture 7

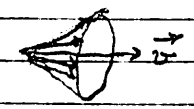
Momentum and Energy: Their definition and conservation

[Taylor & Wheeler, ch. 7]

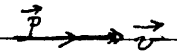
proof: ⓐ Q: What is the relation between
$\vec{p}$, momentum,
and
$\vec{v}$, the velocity,
of the particle?

A: $\vec{p} = f(v)\,\vec{v}$

Why? (i) isotropy of space



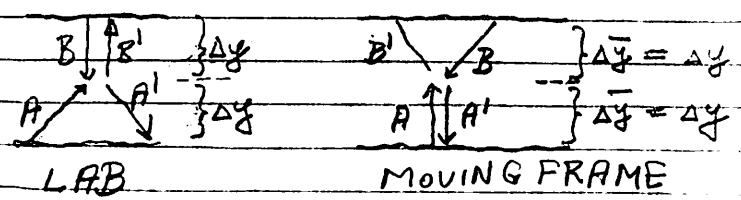
(ii) $\vec{p}$ is unique



(iii) $f(v) = f\left(\sqrt{v_x^2+v_y^2+v_z^2}\right)$
because of isotropy of space

ⓑ Principle of Relativity $\Rightarrow$ $f(v)$ is the <u>same</u> fn in all inertial reference frames.

ⓒ To determine f, consider the elastic collision between two identical particles viewed in two inertial frames



particles are identical
P. of R.
isotropy of space
$\Rightarrow$ two pictures are symmetrical

symmetry $\Rightarrow$ pictures are congruent.

(d) Exhibit components of A is MOVING and LAB frame.

(i) In $\boxed{\text{MOVING}}$ frame

A's velocity $= \frac{\overline{\Delta y}}{\overline{\Delta t}} = \mu$ $\qquad$ $\overline{\Delta t}$ = time for A to move from bottom to pt of collision.

A's four vector $= (\overline{\Delta t}, 0, \overline{\Delta y}, 0)$

(ii) In $\boxed{\text{LAB}}$ frame

A's four vector $= \left(\frac{\overline{\Delta t}}{\sqrt{1-\beta^2}}, \frac{\beta \Delta t}{\sqrt{1-\beta^2}}, \overline{\Delta y}, 0\right)$

P. of R. & isotropy of space $\Rightarrow$

$$\boxed{\begin{aligned} \text{A's spatial velocity} &= \left(\beta, \frac{\overline{\Delta y}}{\overline{\Delta t}}\sqrt{1-\beta^2}, 0\right) \\ &= \left(\beta, \mu\sqrt{1-\beta^2}, 0\right) \end{aligned}}$$

(e) Conservation of momentum in Lab frame.

(e) (cont'd)

(i) X component in LAB frame: $\boxed{\sum p_x\Big|_{before} = \sum p_x\Big|_{after}}$

$$\underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\cdot \underset{v_x}{\underset{\parallel}{\beta}}}_{A} + \underbrace{0}_{B} = \underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\beta}_{A'} + \underbrace{0}_{B'}$$

is trivially true; no information!

(ii) Y component to be conserved $\sum p_y\Big|_{before} = \sum p_y\Big|_{after}$

$$\underbrace{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\mu\sqrt{1-\beta^2}}_{A} \underbrace{- f(\mu)\mu}_{B} = \underbrace{-f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right)\mu\sqrt{1-\beta^2}}_{A'} + \underbrace{f(\mu)\mu}_{B'}$$

$$\Rightarrow \boxed{f\left(\sqrt{\beta^2+\mu^2(1-\beta^2)}\right) = \frac{f(\mu)}{\sqrt{1-\beta^2}}}$$ let $\mu = 0$

whose solution is $f(\beta) = \frac{f(0)}{\sqrt{1-\beta^2}}$

(f) Newtonian correspondence as $v \to 0$ implies

$$1 = \frac{p_y}{p_y} = \lim_{v\to 0} \frac{m\, v_y}{f(v)\, v_y} \longrightarrow f(0) = m$$

CONCLUSION

$$\boxed{\text{momentum} = \frac{m}{\sqrt{1-\beta^2}}\vec{\beta}}$$

Note: if $f(v)$ = independent of v then one implicitly assumes Newtonian approximation, which does not apply to relativistic Mechanics.

~~6.17~~ 7.5

A. Momentum in Relativity (cont'd)

1. We found (Symmetry, Relativity) that the quantity

(1) $$\boxed{\frac{m}{\sqrt{1-\beta^2}}\,\text{velocity} = \text{momentum}}$$

is conserved in elastic collisions. It also is conserved in inelastic collisions -- in all collisions.

2. Energy-Momentum 4-vector (four velocity) $\frac{m\,\beta_x}{\sqrt{1-\beta^2}}$

a. We have seen that unit tangent to the world line of particle,

$$\boxed{u^\alpha = \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}}$$

is a 4-vector [transforms like x^α under a Lorentz xformation]

b. To connect 4-velocity to momentum rewrite u^α in a new form

(i) $d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \Rightarrow d\tau^2 = dt^2(1-\beta^2)$

(ii) Thus

(2) $$\boxed{u^\alpha = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\vec{\beta}}{\sqrt{1-\beta^2}}\right)}$$ four velocity

c. Combine (1) & (2)

momentum = m (space part of 4-vel.)

i.e. $$\boxed{\vec{p} = m\vec{u}} = \frac{m\vec{\beta}}{\sqrt{1-\beta^2}}$$

$\vec{p} = \frac{m\vec{\beta}}{\sqrt{1-\beta^2}}$

$\{u^\alpha\} = \left\{\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right\} = \left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\vec{\beta}}{\sqrt{1-\beta^2}}\right\}$ $\Rightarrow \vec{p} = m\vec{u}$

~~6.18~~ 7.6

Theorem (Conservation of energy)

Given: (i) Principle of Relativity

(ii) Conservation of spatial momentum (during a collision of particles)

Conclusion: Energy of particles is conserved

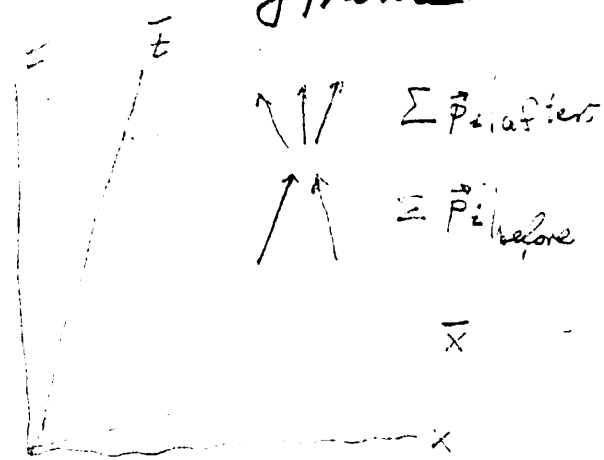
proof: ① Let $\{u^\alpha\}$ = 4-velocity of a particle

Use transformation Law for u^α to get that of momentum for each particle

$$p_x = m u_x = m(\bar{u}_x \cosh\theta + \bar{u}_t \sinh\theta)$$

$$= \bar{p}_x \cosh\theta + \frac{m}{\sqrt{1-\beta^2}} \sinh\theta$$

② Consider a collision of particles as viewed in lab frame as well as in moving frame



② Consider a collision of particles as viewed in LAB frame as well as in MOVING frame; ~~6.19~~ 7.7

$$0 = \sum_i P_{x(i)}\Big|_{after} - \sum_i P_{x(i)}\Big|_{before} =$$

conservation of momentum in LAB
in MOVING FRAME

$$= \left(\sum_i \bar{P}_{x(i)}\Big|_{after} - \sum_i \bar{P}_{x(i)}\Big|_{before}\right)\cosh\theta$$

$$+\left(\sum_i \frac{m_{(i)}}{\sqrt{1-\beta_{(i)}^2}}\Big|_{after} - \sum_i \frac{m_{(i)}}{\sqrt{1-\beta_{(i)}^2}}\right)\sinh\theta$$

Conclusion:

$\sum_i \frac{m_{(i)}}{\sqrt{1-\beta_{(i)}^2}}$ is also conserved.

③ Go to the Newtonian limit:

$$\frac{mc^2}{\sqrt{1-\beta^2}} \doteq mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}-\frac{1}{8}\frac{v^4}{c^4}+\cdots\right)$$

$$= mc^2 + \frac{1}{2}mv^2 + \cdots = \text{energy of particle}$$

Summary: Momentum conservation + Principle of Relativity $\Longrightarrow$ Energy conservation Q.E.D.

SKIP. Go to 7.8

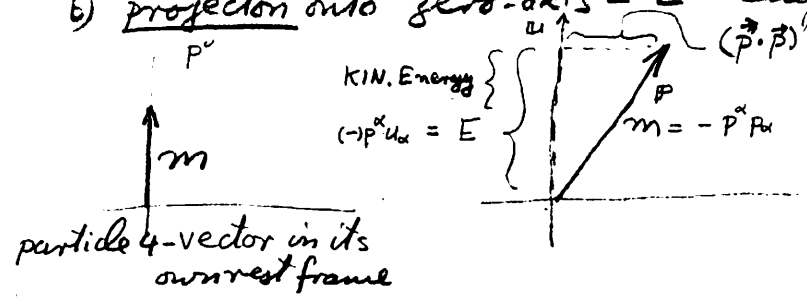
$$\boxed{\frac{mc^2}{\sqrt{1-\beta^2}} = \text{Energy of particle}}$$ is conserved ~~6.20~~ QED

Proposition

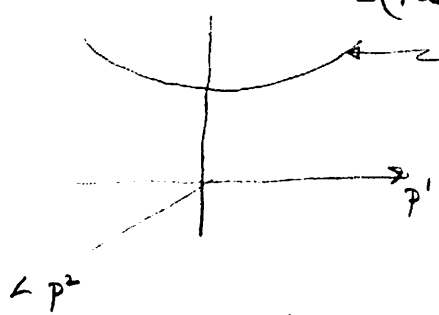
Energy-momentum 4 vector:

a) Definition: $\{p^\alpha\} \equiv \left\{\frac{m}{\sqrt{1-\beta^2}}, \frac{m\vec{\beta}}{\sqrt{1-\beta^2}}\right\} = \{E, \vec{p}\}$ Each component is conserved in a collision.

b) projection onto zero-axis $= E = \text{energy} = p^\alpha u_\alpha$



c) (Magnitude)$^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = E^2 - p\cdot p = m^2$
$= (\text{rest mass})^2$



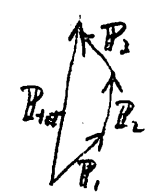
$p_0^2 - p^2 = m^2$ (mass shell

$p_0 = m\cosh\chi$
$p_1 = m\sinh\chi\sin\theta\cos\varphi$
$p_2 = m\sinh\chi\sin\theta\sin\varphi$
$p_3 = m\sinh\chi\cos\theta$

d) Total vector = sum of individual particle 4-vectors

$$P^\alpha_{tot} = \sum_{\substack{\text{all}\\ \text{particles}}} p^\alpha_i$$

$$\mathbb{P}_{tot} = \sum_{i=1}^{N} \mathbb{P}_i$$



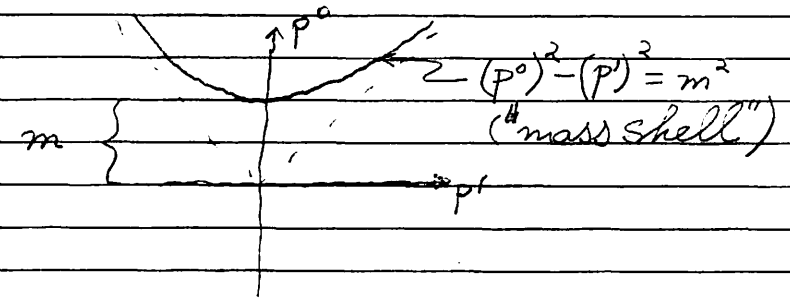
Proposition (Energy-momentum 4-vector)

a) Definition

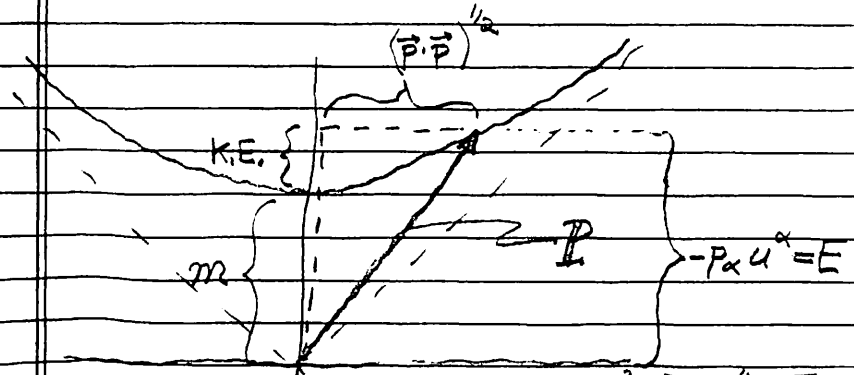
$$\mathbb{P}: \{p^\alpha\} = \left\{ \frac{m}{\sqrt{1-\beta^2}}, \frac{m\vec{\beta}}{\sqrt{1-\beta^2}} \right\} \text{ for each particle}$$

b) $(\text{Magnitude})^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$

$$= m^2 = (\text{rest mass})^2$$



c) Projection onto the zero-axis $= E$ (= energy)



Kinetic Energy $= E - m = \frac{m}{\sqrt{1-\beta^2}} - m \approx \frac{1}{2} m\beta^2 + \frac{3}{8} m\beta^4 + \frac{5}{16} m\beta^6 + \ldots$

e) Every application of the law of conservation of momenergy in a collision is a statement about a polygon built of 4-vectors in spacetime

$$\boxed{\sum_{i=1}^{N} \mathbb{P}_i \Big|_{\text{before}} = \mathbb{P}_{\text{tot}} \Big|_{\text{before}} = \mathbb{P}_{\text{tot}} \Big|_{\text{after}} = \sum_{j=1}^{M} \mathbb{P}_j' \Big|_{\text{after}}}$$