LECTURE 8

Flow of Particles:

The particle flux-density 3-form

[MTW: Box 4.4 (The math of J^*)

Box 5.2 (The physics of J^*)

Box 15.1 F

Particle 4-current: density and flux J

vs

For Lecture 9 { Flow of 4-momentum

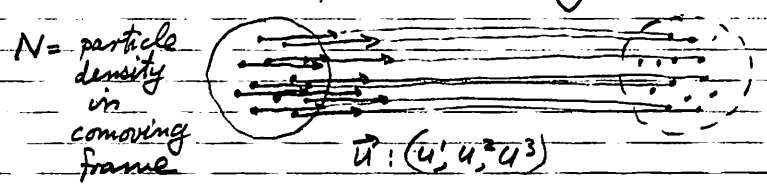
Up to now we have only considered the response of individual particles to the geometry of a manifold.

We would like to extend those considerations to more general configurations characterized by their momentum and energy (= "momenergy")

Thus we would like to formulate the manner in which geometry determines how matter moves, i.e. momenergy flows through spacetime.

Expressed in terms of the stress energy tensor, the means will also be available to identify how the flow of momenergy brings about the curving of geometry. This identification is expressed by the set of Einstein's field equations.

Consider a set of particles in a region of space and time large enough so that one can talk about their density, flux, pressure, and so on, but small enough to that these particles can be said to have the same four velocity $\mathbf{u} = u^\mu \mathbf{e}_\mu$.



Set of particles with instantaneous common velocity $\vec{u}$ and comoving density N.

The basic building block for the description of the flow of some conserved quantity, such as momenergy or charge, is the current-density (scalar valued) three form

$$^*\mathbf{S} = N u^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{1}{3!} \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma,$$

It is an antisymmetric tensor of rank $\binom{0}{3}$.

This tensor is associated with a fluid having four velocity $\mathbb{u} = u^\mu \mathbb{e}_\mu : \{u^0, u^1, u^2, u^3\}$ $(u \cdot u = -1)$ and invariant particle density N, which is the density in the comoving (="proper") frame where $\{u^\mu\} = \{1, 0, 0, 0\}$

Side Comment:

This tensor is constructed from the Levi Civita tensor whose components are related to those in a Lorentz frame by frame by

$$\boxed{\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,[\alpha\beta\gamma\delta]}$$

Here $[\alpha, \beta, \gamma, \delta]$ is the antisymmetric permutation ("Levi-Civita) symbol whose value are 1, −1, or 0.

The "volume expansion" factor $\sqrt{-g}$ is simply the Jacobian determinant

$$\sqrt{-g} \equiv \sqrt{-\det g_{\mu\nu}} = \det\left[\frac{\partial x^\alpha}{\partial y^\mu}\right]$$

This relation follows from

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\,\eta_{\alpha\beta}\,\frac{\partial x^\beta}{\partial y^\nu}$$

$$= \sum_\alpha \sum_\beta \Lambda_{\alpha\mu}\,\eta_{\alpha\beta}\,\Lambda_{\beta\nu}$$

$$= (\Lambda^T \eta \Lambda)_{\mu\nu}$$

Thus

$$\det g = \det \Lambda^T \det \eta \det \Lambda$$

$$= -(\det \Lambda)^2$$

or $\boxed{\det\left(\frac{\partial x^\beta}{\partial y^\nu}\right) = \sqrt{-\det g}}$

END of SIDE COMMENT

What does the scalar valued 3-form express?

Its value on vectors A, B, C is

$$^*S(A,B,C) = \sqrt{-g}\begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \langle\omega^0, A\rangle & \langle\omega^1, A\rangle & \langle\omega^2, A\rangle & \langle\omega^3, A\rangle \\ \langle\omega^0, B\rangle & \langle\omega^1, B\rangle & \langle\omega^2, B\rangle & \langle\omega^3, B\rangle \\ \langle\omega^0, C\rangle & \langle\omega^1, C\rangle & \langle\omega^2, C\rangle & \langle\omega^3, C\rangle \end{vmatrix}$$

The entries $\langle \omega^\alpha, A\rangle = A^\alpha$, etc are the components of the vector A, etc.

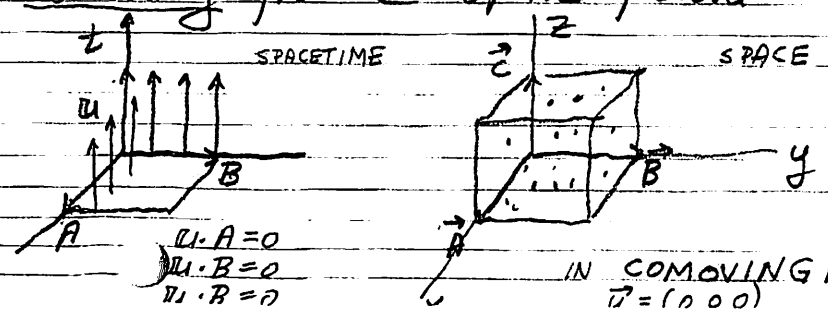
a) Let A, B, and C be <u>spacelike</u> displacement vectors. The frame independence(!) of the scalar $\boxed{{}^*S(A,B,C)}$ allows us to choose a frame relative to which

$$\langle \omega^0, A\rangle = \langle \omega^0, B\rangle = \langle \omega^0, C\rangle = 0 \qquad A,B,C \text{ are spacelike.}$$

This yields

$$ {}^*S(A,B,C) = N u^0 \, |g|^{1/2} \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} = N u^0 \text{ (volume)} \qquad (*)$$

b) Let the three vectors $\vec{A}, \vec{B}, \vec{C}$ span the proper volume $\vec{A}\cdot\vec{B}\times\vec{C}$ in the <u>comoving</u> frame of the fluid.

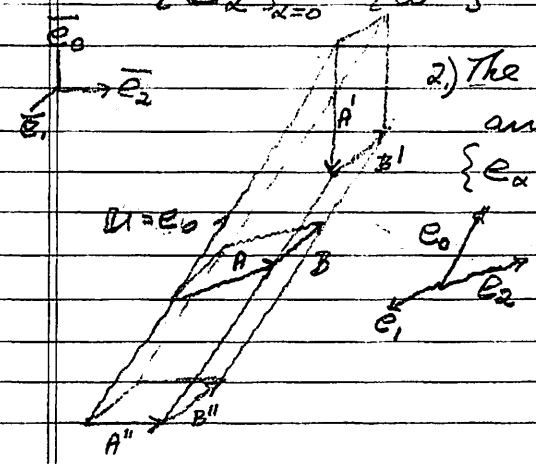


Spacetime geometry of 3-D simplices:

Consider the spacetime history of an element of fluid moving relative to the LAB. Let u = 4 velocity of that element.

1) The LAB basis and its dual basis we take to be

$$\{\bar{e}_\alpha\}_{\alpha=0} \qquad \{\bar{\omega}^\beta\} \qquad \langle \bar{\omega}^\beta, \bar{e}_\alpha\rangle = \delta^\beta_\alpha$$



2) The MOVING basis and its dual are $\{e_\alpha\}$ and $\{\omega^\beta\}$ where $e_0 = u$

There are 8 particles which identify the 3-D cubical volume element. These particles trace out worldlines having the common 4-velocity and they determine the <u>world tube</u> pictured in the diagram above.

There are three important simplices associated with the world tube of particles

<u>Spatial LAB simplex</u> determined by the ordered triple of spacelike vectors A'', B'', C'':

$$\langle \bar{\omega}^0 | A'' \rangle = \langle \bar{\omega}^0 | B'' \rangle = \langle \bar{\omega}^0 | C'' \rangle = 0$$

<u>Spatial COMOVING simplex</u> determined by

A, B, C:

$$0 = A \cdot \mathbb{u} = \langle \omega^0, A \rangle$$
$$0 = B \cdot \mathbb{u} = \langle \omega^0, B \rangle$$
$$0 = C \cdot \mathbb{u} = \langle \omega^0, C \rangle$$

<u>Timelike LAB simplex</u> determined by

A', B', C'

$$\langle \bar{\omega}^1, A' \rangle = \langle \bar{\omega}^2 | A' \rangle = \langle \bar{\omega}^3 | A' \rangle = 0$$

$$\langle \bar{\omega}^0 | B' \rangle = \langle \bar{\omega}^1, B' \rangle = 0$$

$$\langle \bar{\omega}^0 | C' \rangle = \langle \bar{\omega}^1 | C' \rangle = 0$$

Let us exhibit the evaluated $^*S$

$$^*S(\ ,\ ,\ ) = N u^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{1}{3!} \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$

$$= \sqrt{-g} \det \begin{vmatrix} N u^0 & N u^1 & N u^2 & N u^3 \\ \langle \omega^0 | & \langle \omega^1 | & \langle \omega^2 | & \langle \omega^3 | \\ \langle \omega^0 | & \langle \omega^1 | & \langle \omega^2 | & \\ \langle \omega^0 | & \langle \omega^1 | & \langle \omega^2 | & \langle \omega^3 | \end{vmatrix}$$

For these three cases:

$$^*S(A'', B'', C'') = \quad \#\text{ particles} = \#\text{ worldlines}$$

$$= \sqrt{-g}\, \det \begin{vmatrix} N \bar{u}^0 & N \bar{u}^1 & N \bar{u}^2 & N \bar{u}^3 \\ 0 & A''_1 & A''_2 & A''_3 \\ 0 & B''_1 & B''_2 & B''_3 \\ 0 & C''_1 & C''_2 & C''_3 \end{vmatrix}$$

$$= \sqrt{-g}\, N \bar{u}^0 \ \vec{A}'' \times \vec{B}'' \cdot \vec{C}''$$

$$= \sqrt{-g}\, N \bar{u}^0 \ (\text{LAB volume})$$

$$\therefore \boxed{N \bar{u}^0 = \frac{\#}{\sqrt{-g}\,(\text{LAB volume})} = \text{LAB particle density}}$$

$$*S(A,B,C) = \#\text{ particles} = \#\text{ of worldlines}$$

$$= \sqrt{-g}\,\det\begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ 0 & A_1 & A_2 & A_3 \\ 0 & B_1 & B_2 & B_3 \\ 0 & C_1 & C_2 & C_3 \end{vmatrix}$$

$$\sqrt{-g}\,N u^0\ \vec{A}\times\vec{B}\cdot\vec{C}$$

$$= N\cdot 1 \times \text{COMOVING ("proper") volume}$$

$$\boxed{N = \frac{\#}{\text{comoving volume}} = \text{COMOVING particle density}}$$

$$*S(A',B',C') =$$

$$= \sqrt{-g}\,\det\begin{vmatrix} N\bar{u}^0 & N\bar{u}^x & N\bar{u}^y & N\bar{u}^z \\ \langle\bar{\omega}^0, A\rangle & 0 & 0 & 0 \\ 0 & 0 & B_2 & B_3 \\ 0 & 0 & C_2 & C_3 \end{vmatrix}$$

$$= -\langle\bar{\omega}^0, A\rangle\, N\bar{u}^x\, (\text{area})^x$$

$$\boxed{N\bar{u}^x = \frac{\#}{(\text{LAB Time})\left(\frac{\text{LAB}}{\text{area}}\right)^x} = \left(\text{LAB particle flux into x-direction}\right)}$$

$$\# = *S(A',B',C') = \det\begin{vmatrix} N\bar{u}^0 & N\bar{u}^x & N\bar{u}^y & N\bar{u}^z \\ \langle\bar{\omega}^0, A\rangle & 0 & 0 & 0 \\ 0 & 0 & \langle\bar{\omega}^2, B\rangle & \langle\bar{\omega}^3, B\rangle \\ 0 & 0 & \langle\bar{\omega}^2, C\rangle & \langle\bar{\omega}^3, C\rangle \end{vmatrix}$$

$$= -\underbrace{\langle\bar{\omega}^0, A\rangle}_{\substack{\text{elapsed}\\ \text{LAB}\\ \text{time}}}\, N\bar{u}^x\, \underbrace{(\text{area})^x}_{(\text{lab area})^x}$$

Conclusions:

1. $N = \dfrac{\#}{(\text{comoving vol})} = $ COMOVING particle density
2. $N\bar{u}^0 = \dfrac{\#}{(\text{LAB volume})} = $ LAB particle density
3. $N\bar{u}^x = \dfrac{\#}{(\text{LAB time})\left(\frac{\text{LAB}}{\text{area}}\right)^x} = $ LAB $\begin{pmatrix}\text{particle}\\ \text{flux}\end{pmatrix}$ into x-direction
4. $1\text{-}3 \Rightarrow$ $N\mathbf{u} \equiv \mathbf{S} = $ particle density-flux 4-vector.
5. Let $\underset{\sim}{\Sigma} = \underset{\sim}{\omega}^\mu \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma$ be the <u>volume 1-form</u> determined by A, B, C.

Then

$$\langle \underset{\sim}{\Sigma}, N\mathbf{u}\rangle = N u^\mu \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma$$

$$= S(\underset{\sim}{\Sigma}) = *S(A,B,C) = \#\text{ of particles.}$$