LECTURE 8

Flow of Particles:

The particle flux-density 3-form

MTW: Box 4.4 (The math of J^+)

Box 5.2 (The physics of J^+)

Box 15.1 F

Particle 4-current: density and flux

vs

Flow of 4-momentum
Up to now we have only considered the response of individual particles to the geometry of a manifold. We would like to extend these considerations to more general configurations characterized by their momentum and energy (= "monenergy").

Thus we would like to formulate the manner in which geometry determines how matter moves, i.e., monenergy flows through spacetime. Expressed in terms of the stress energy tensor, the means will also be available to identify how the flow of monenergy brings about the curving of geometry. This identification is expressively the set of Einstein's field equations.

Consider a set of particles in a region of space and time large enough so that one can talk about their density, flux, pressure, and so on, but small enough to that these particles can be said to have the same four-velocity $\mathbf{U} = u^\mu \mathbf{e}_\mu$.

Set of particles with instantaneous common velocity $\mathbf{U}$ and comoving density $N$.

The basic building block for the description of the flow of some conserved quantity, such as monenergy or charge, is the current-density (scalar valued) three-form

$$\mathbf{S} = N u^\mu \varepsilon_{\mu\rho\sigma\delta} \omega^\rho \omega^\sigma \omega^\delta,$$

which is an antisymmetric tensor of rank $(0,3)$. 
This tensor is associated with a fluid having four-velocity \( u^\mu = (u^0, u^1, u^2, u^3) \) and invariant particle density \( N \), which is the density in the comoving (= "proper") frame where \( \bar{u}^\mu = (1, 0, 0, 0) \).

\[
\begin{align*}
\mathcal{E}^{\alpha \beta \gamma \delta} &= \frac{\partial \bar{x}^\alpha}{\partial y^\mu} \frac{\partial \bar{x}^\beta}{\partial y^\nu} \frac{\partial \bar{x}^\gamma}{\partial y^\rho} \frac{\partial \bar{x}^\delta}{\partial y^\sigma} \\
&= \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \sum_{\gamma=0}^{3} \sum_{\delta=0}^{3} \Lambda_{\alpha, \beta} \Lambda_{\gamma, \rho} \Lambda_{\delta, \sigma} \\
&= (\Lambda^T \eta \Lambda)_{\mu \nu}
\end{align*}
\]

Thus
\[
\det g = \det \Lambda^T \det \eta \det \Lambda
\]

or
\[
\det \left( \frac{\partial x^\mu}{\partial y^\nu} \right) = \sqrt{-\det g}
\]

\[\text{END OF SIDE COMMENT}\]

What does the scalar valued 3-form express?

Its value on vectors \( A, B, C \) is

\[
\begin{pmatrix}
N_0 \nu^1 & N_1 \nu^2 & N_2 \nu^3 \\
\langle \omega^0, A \rangle & \langle \omega^1, A \rangle & \langle \omega^2, A \rangle \\
\langle \omega^0, B \rangle & \langle \omega^1, B \rangle & \langle \omega^2, B \rangle \\
\langle \omega^0, C \rangle & \langle \omega^1, C \rangle & \langle \omega^2, C \rangle
\end{pmatrix}
\]
The entries $\langle \omega^\alpha A \rangle = A^\alpha$, etc. are the components of the vector $A$, etc.

1) Let $A$, $B$, and $C$ be spacelike displacement vectors. The frame independence of the scalar $S(A, B, C)$ allows us to choose a frame relative to which $\langle \omega^\alpha A \rangle = \langle \omega^\alpha B \rangle = \langle \omega^\alpha C \rangle = 0$ are spacelike. This yields

$$S(A, B, C) = \frac{\epsilon_{\alpha\beta\gamma}}{g^{1/2}} \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} = N u^\alpha \text{ (volume)}$$

(1)

2) Let the three vectors $A$, $B$, $C$ span the proper volume $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$ in the comoving frame of the fluid.

Consider the spacetime history of an element of fluid moving relative to the LAB. Let $u$ be a velocity of the element.

1) The LAB basis and its dual basis we take to be

$$\mathbf{e}_0 = \mathbf{e}_0, \quad \mathbf{e}_0 = \delta_0^\alpha$$

2) The moving basis and its dual are

$$\mathbf{e}^1 = \mathbf{e}_0, \quad \mathbf{e}^1 = \delta^1_\alpha$$

where

$$\mathbf{u} = \mathbf{e}_0$$

There are 2 particles which identify the 3-D cubical volume element. These particles trace out worldlines having the common 4-velocity and they determine the world tube pictured in the diagram above.
There are three important simplices associated with the world tube of particles:

Spatial LAB simplex determined by the ordered triple of space-like vectors:

\[ A'' \, B'' \, C'' \]

\[ \langle \omega_0^0 A'' \rangle = \langle \omega_0^0 B'' \rangle = \langle \omega_0^0 C'' \rangle = 0 \]

Spatial COMOVING simplex determined by:

\[ A, B, C \]

\[ o = A \cdot u = \langle \omega_0^0 A \rangle \]
\[ o = B \cdot u = \langle \omega_0^0 B \rangle \]
\[ o = C \cdot u = \langle \omega_0^0 C \rangle \]

Timelike LAB simplex determined by:

\[ A', B', C' \]

\[ \langle \omega_0^1 A' \rangle = \langle \omega_0^1 B' \rangle = \langle \omega_0^1 C' \rangle = 0 \]

\[ \langle \omega_0^2 A' \rangle = \langle \omega_0^2 B' \rangle = 0 \]

\[ \langle \omega_0^1 C' \rangle = \langle \omega_0^1 C' \rangle = 0 \]

Let us exhibit the evaluated \( S \):

\[ S(\, \, ) = \frac{1}{N u^0} \epsilon_{\mu \nu \rho \sigma \omega} \frac{1}{3!} \omega_\mu \omega_\nu \omega_\rho \omega_\sigma \omega_\omega \]

\[ = \frac{1}{3!} \text{det} \begin{bmatrix} N u^0 & N u^1 & N u^2 & N u^3 \\ \omega_0^0 & \omega_0^1 & \omega_0^2 & \omega_0^3 \\ \omega_0^0 & \omega_0^1 & \omega_0^2 & \omega_0^3 \\ \omega_0^0 & \omega_0^1 & \omega_0^2 & \omega_0^3 \end{bmatrix} \]

For these three cases:

\[ S(\, \, ) = \# \text{particles} \times \# \text{worldline} \]

\[ = \frac{1}{3!} \text{det} \begin{bmatrix} N u^0 & N u^1 & N u^2 & N u^3 \\ 0 & A'' & A'' & A'' \\ 0 & B'' & B'' & B'' \\ 0 & C'' & C'' & C'' \end{bmatrix} \]

\[ = \frac{1}{3!} \text{LAB volume} \]

\[ = \frac{1}{3!} N u^0 \left( \text{LAB volume} \right) \]

\[ = \frac{1}{3!} N u^0 \left( \text{LAB volume} \right) = \text{LAB particle density} \]
\[ S(A, B, C) = \# \text{ particles} = \# \text{ of worldlines} \]

\[ S(A, B, C) = \begin{vmatrix} N u^0 & N u^1 & N u^2 & N u^3 \\ 0 & A_1 & A_2 & A_3 \\ 0 & B_1 & B_2 & B_3 \\ 0 & C_1 & C_2 & C_3 \end{vmatrix} \]

\[ \sim N \cdot 1 \times \text{COMOVING ("proper") volume} \]

\[ N = \frac{\# \text{ comoving} \cdot \text{COMOVING particle}}{\text{volume}} \]

\[ S(A', B', C') = \begin{vmatrix} N u^0 & N u^1 & N u^2 & N u^3 \\ \langle \omega^0 A \rangle & 0 & 0 & 0 \\ 0 & 0 & B_2 & B_3 \\ 0 & 0 & C_2 & C_3 \end{vmatrix} \]

\[ = -\langle \omega^0 A \rangle N u^1 \text{ (area)} \]

\[ N u^1 = \text{comoving particle density} \]

\[ N = \frac{\# \text{ (comoving vol)}}{\text{volume}} \]

\[ N u^1 = \frac{\# \text{ (LAB volume)}}{\text{LAB particle density}} \]

\[ N u^x = \frac{\# \text{ (LAB time)(LAB area)}}{\text{LAB (particle flux) into x-direction}} \]

\[ 1 \Rightarrow 4, N u E = S = \text{particle density - flux} 4\text{-vector} \]

\[ \Rightarrow 5, \text{ Let } \Sigma = u^m e_{m \alpha \beta \gamma} A^\alpha B^\beta C^\gamma \text{ (0) be the volume 1-form determined by } A B C. \]

\[ \langle \Sigma, N u \rangle = N u^m e_{m \alpha \beta \gamma} A^\alpha B^\beta C^\gamma \]

\[ \Rightarrow S(\Sigma) = -S(A, B, C) = \# \text{ of particles.} \]