MINKOWSKI BESSEL MODES*

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ABSTRACT

The global Minkowski Bessel (M-B) modes, whose explicit form allows the identification and description of the condensed vacuum state resulting from the operation of a pair of accelerated refrigerators, are introduced. They span the representation space of the unitary representation of the Poincare group on 2-D Lorentz space-time. Their three essential properties are: (1) they are unitarily related to the familiar Minkowski plane waves; (2) they form a unitary representation of the translation group on two dimensional Minkowski spacetime. (3) they are eigenfunctions of Lorentz boosts around a given reference event. In addition the global Minkowski Mellin modes are introduced. They are the singular limit of the M-B modes. This limit corresponds to the zero transverse momentum solutions to the zero rest mass wave equation.

Also introduced are the four Rindler coordinate representatives of each global mode . Their normalization and density of states are exhibited in a (semi-infinite) accelerated frame with a finite bottom. In addition we exhibit the asymptotic limit as this bottom approaches the event horizon and thereby show how a mode sum approaches a mode integral as the frame becomes bottomless.

I. MOTIVATION AND SUMMARY.

There are reasons to believe that the quantum mechanics of a relativistic system with infinitely many degrees of freedom, e.g. the Klein-Gordon wave field, manifests itself in a qualitatively different way relative to a linearly uniformly accelerated frame than to inertial frames. Consider different inertial frames. They are all equivalent. This is expressed by the fact that the ground state of a relativistic wave field is the same relative to these inertial frames. Thus all inertial refrigerators produce the same quantum state, the familiar Minkowski vacuum. By contrast a pair of refrigerators accelerating linearly into opposite directions produce a different quantum state. It can perhaps best be described as a "condensed vacuum state" [1,2] . The peculiar feature of such a state is that even though it manifests itself in each coaccelerating frame as a no particle state [3,4] i.e. as a vacuum, in an inertial frame it manifests itself as liquid light in the form of a superfluid [1,2] .

What is the most direct way of identifying such a quantum state? One certainly could use the quantized Minkowski plane wave modes. But this use lacks directness. A superior way, it turns out, is to quantize the global Minkowski Bessel (M-B) modes. Their existence has in part already been known for some time [4,5] , but their simple global construction and properties as well as their usefulness as a working tool do not yet seem familiar to theoretical physics. The purpose of this note is to remedy this gap.

Minkowski Bessel modes are the global extensions of Sommerfeld's cylinder waves to Minkowski space-time. These modes allow one to relate at a glance (a) the wave field dynamics (e.g. emission and absorption) and its quantum properties (e.g of the ground state) in an accelerated coordinate reference frame to (b) those in an inertial frame.

Linearly uniformly accelerated observers produce worldlines in Minkowski space-time which in Euclidean space would correspond to circles. This correspondence extends not only to coordinate

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systems (i.e. Rindler coordinates[7], a type of Fermi-Walker transport[8] induced coordinate system, correspond to polar coordinates) but also to the wave equation and its solutions. Thus, corresponding to the Klein-Gordon $(K-G)$ wave equation one has the Helmolz equation. An inquiry into the $K-G$ wave field (solutions) relative to an accelerated frame demands therefore that one exhibit that which in Euclidean space corresponds to Sommerfeld's construction of cylinder waves from plane waves. The Minkowski space-time analogue corresponding to Sommerfeld's cylinder harmonics are the Minkowski Bessel modes. This correspondence prevails in regards to all major properties of these modes except one: space-time has a causal structure characterized by observer induced future and past even horizons which partition space-time into four coordinate charts. See Figure 1.

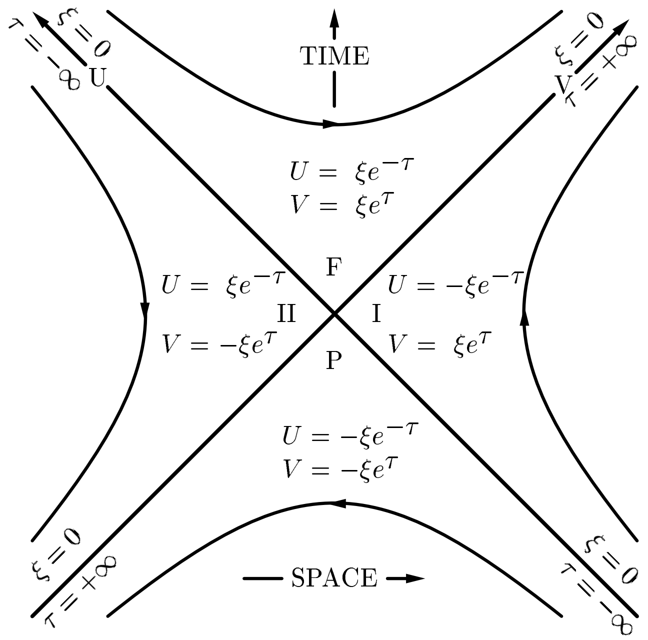


FIGURE 1

The four cooordinate neighborhoods (="Rindler" sectors) induced by the world line of an accelerated observer. The asymptotes $U \equiv t-x = 0$ and $V \equiv t+x = 0$ divide Minkowski space-times into two causally disjoint accelerated frames I and II. A relativistic wave field is thereby partitioned into a pair of mutually exclusive and jointly exhaustive subsystems.

Euclidean space has no such structure. The presence of these event horizons has a dramatic effect on the Minkowski Bessel modes. Because these horizons divide space-time into the four Rindler coordinate charts, a Minkowski Bessel mode has four coordinate representatives, the four Rindler modes for each of the four Rindler sectors I, II, F and P. The Minkowski Bessel modes together with each of their four coordinate representatives are pictured in Figures 2a and b.

II. GLOBAL PROPERTIES.

A Minkowski Bessel mode is a linear super-position of those plane wave modes,

$$\frac{1}{\sqrt{2\pi}}\exp[\mp i(\omega_k t - k_x x)] \;=\; \frac{1}{\sqrt{2\pi}}\exp[\mp i\frac{k}{2}(Ue^{\theta}+Ve^{-\theta})] \equiv P_\theta^{\pm}(kU,kV)\,, \tag{2.1}$$

which are on the same positive (upper sign) or negative (lower sign) "mass shell" given by

$$\begin{aligned}\omega_k \;&=\; k\cosh\theta\,;\, k_x \;=\; k\sinh\theta\,, -\infty<\theta<\infty;\\ k \;&=\; |k| = \sqrt{k_y^2+k_z^2+m^2}\,.\end{aligned} \tag{2.2}$$

The new coordinates

$$U \;=\; t \,-\, x\,,\;\; V \;=\; t \,+\, x$$

are the retarded and advanced times ("null coordinates") repectively.

A Minkowski Bessel mode in two dimensions is given by

$$\begin{aligned}B_\omega^{\pm}(kU,kV) \;&=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[\mp i(\omega_k t - k_x x)]e^{-i\omega\theta}d\theta\\ &=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[\mp ik(Ue^{\theta}+Ve^{-\theta})/2]e^{-i\omega\theta}d\theta\\ &=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}P_\theta^{\pm}(kU,kV)e^{-i\omega\theta}d\theta\end{aligned} \tag{2.3}$$

This is the defining property . Thus we have

Property 1

The global Minkowski Bessel modes $B_\omega^{\pm}(kU,kV)$ are by means of the unitary transformation $(1/2\pi)^{1/2}e^{-i\omega\theta}$ related to the plane wave modes $P_\theta^{\pm}(kU,kV)$.

The upper (resp. lower) sign refers to positive (resp. negative) Minkowski frequency modes. The two dimensional Klein-Gordon inner product,

$$(\psi,\phi)\equiv i\int_{-\infty}^{\infty}[\psi^*\,\partial_t\phi \;-\; \phi\partial_t\,\phi^*]dx \equiv i\int_{-\infty}^{\infty}\psi^*\overleftrightarrow{\partial}_t\,\phi dx \tag{2.4}$$

can be used to verify the sign of the Minkowski frequency of a mode. The inner product of two plane wave modes $P_\theta^{\pm}$ and $P_{\theta'}^{\pm}$ is

$$(P_\theta^{\pm},P_{\theta'}^{\pm}) \;=\; \pm\,2\delta(\theta-\theta')\,. \tag{2.5}$$

One readily sees the well-known fact that a unitary transformation such as $(1/2\pi)^{1/2}e^{-i\omega\theta}$ preserves the $K-G$ inner product. This can be verified by inserting Eq. (2.3) into Eq. (2.4) one obtains

$$(B_\omega^{\pm},\,B_{\omega'}^{\pm}) \;=\; \pm2\delta(\omega\,-\,\omega') \tag{2.6}$$

From the point of view of quantum theory the upper plus (resp. lower minus) sign of the related modes $P_\theta^{\pm}$ and $B_\omega^{\pm}$ refer to a field whose quanta are absorbed (resp. emitted). The absorption-emission distinction is the same for $P_\theta^{\pm}$ and $B_\omega^{\pm}$. Consequently the quantization of the Klein-Gordon field in terms of the set of global Minkowski plane wave modes $P_\theta^\pm$ is equivalent to that in terms of the global Minkowski Bessel modes $B_\omega^\pm$.

Property 2

The Minkowski Bessel modes form a (reducible) unitary representation of the translation group acting on the two dimensional Lorentz space-time:

$$B^\pm_{\omega-\overline{\omega}}(k(U+U_o),k(V+V_o)) \;=\; \int_{-\infty}^{\infty} B^\pm_{\omega-\omega'}(kV,kU)\; B^\pm_{\omega'-\overline{\omega}}(kU_o,kV_o)\; d\omega' \,. \tag{2.7}$$

This can be readily verifies by using Eq. (2.3).

Thus

$$B^\pm_{\omega-\overline{\omega}}(kU_o,kV_o)$$

is the unitary kernel for the space-time translation (U_o,V_o). By contrast the kernel for the plane wave representation is diagonal and is given by

$$P^\pm_\theta(kU,kV)\delta(\theta-\overline{\theta})\,.$$

It satisfies an addition law analogous to Eq. (2.7).

The plane wave modes evidently constitute irreducible representations, but the set of Bessel modes constitutes a reducible representation of the translation group.

III. TWO IRREDUCIBLE UNITARY REPRESENTATIONS.

If the group is the Poincaré group in two space-time dimensions then the $M-B$ modes yield two irreducble unitary representations. A typical group element can be realized by the 3×3 matrix

$$g(\tau,t,x) \;=\; \begin{bmatrix} \cosh\tau & \sinh\tau & t \\ \sinh\tau & \cosh\tau & x \\ 0 & 0 & 1 \end{bmatrix} \;=\; g(0,t,x)\; g(\tau,0,0)\,.$$

The two unitary representation kernals are

$$T^\pm_{\omega\omega'}(\tau,t,x) \;=\; B^\pm_{\omega-\omega'}(kU,kV)e^{-i\omega'\tau}$$

where $U \;=\; t-x$ and $V \;=\; t+x$. The group element $g(\tau,t,x)$ is the product of a pure boost

$$g(\tau,0,0) \;=\; \begin{bmatrix} \cosh\tau & \sinh\tau & 0 \\ \sinh\tau & \cosh\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and a pure translation

$$g(0,t,x) \;=\; \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & x \\ 0 & 0 & 0 \end{bmatrix}$$

by the amount $t \;=\; (U+V)/2, x \;=\; (U-V)/2$. The unitary representation kernels corresponding to the pure boost $g(\tau,0,0)$ are

$$T^\pm_{\omega\omega'}(\tau,0,0) \;=\; \delta(\omega-\omega')\; e^{-i\omega\tau}\,.$$

The kernels for the pure translation $g(0,t,x)$ are

$$T^{\pm}_{\omega'\omega''}(0,t,x) \;=\; B^{\pm}_{\omega'-\omega''}(kU,kV)$$

An arbitrary element of the Poincre group in 2-D can always be decomposed into a product to a boost, a traslation, and a boost.

In other words

$$\begin{bmatrix} \cosh(\tau-\sigma) & \sinh(\tau-\sigma) & t\cosh\tau + x\sinh\tau \\ \sinh(\tau-\sigma) & \cosh(\tau-\sigma) & t\sinh\tau + t\cosh\tau \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \; g(\tau,0,0)\; g(0,t,x)\; g(-\sigma,0,0)\,.$$

The unitary representation kernels corresponding to this generic group element are

$$\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega''\;\; T^{\pm}_{\omega\omega'}(\tau,0,0)\;\; T^{\pm}_{\omega'\omega''}(0,t,x)\;\; T^{\pm}_{\omega''\overline{\omega}}(-\sigma,0,0) \;=$$

$$= \; e^{-i\omega\tau} B^{\pm}_{\omega-\overline{\omega}}(kU,kV) e^{i\overline{\omega}\sigma}$$

If the two Lorentz boosts are equal and opposite, i.e. $\tau \,=\, \sigma$, then

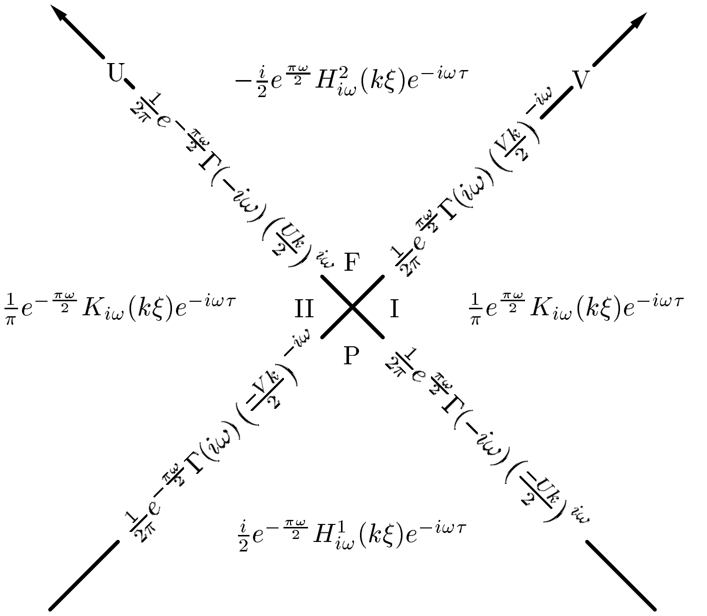
$$g(\tau,0,0)\; g(0,t,x)\; g(-\tau,0,0) \;=\; \begin{bmatrix} 1 & 0 & t\cosh\tau + x\sinh\tau \\ 0 & 1 & t\sinh\tau + x\cosh\tau \\ 0 & 0 & 1 \end{bmatrix}$$

and the corresponding unitary kernels are

$$B^{\pm}_{\omega-\overline{\omega}}(kU,kV) e^{-i(\omega-\overline{\omega})\tau} \;=\; B^{\pm}_{\omega-\overline{\omega}}(kUe^{-\tau},kVe^{\tau})\,.$$

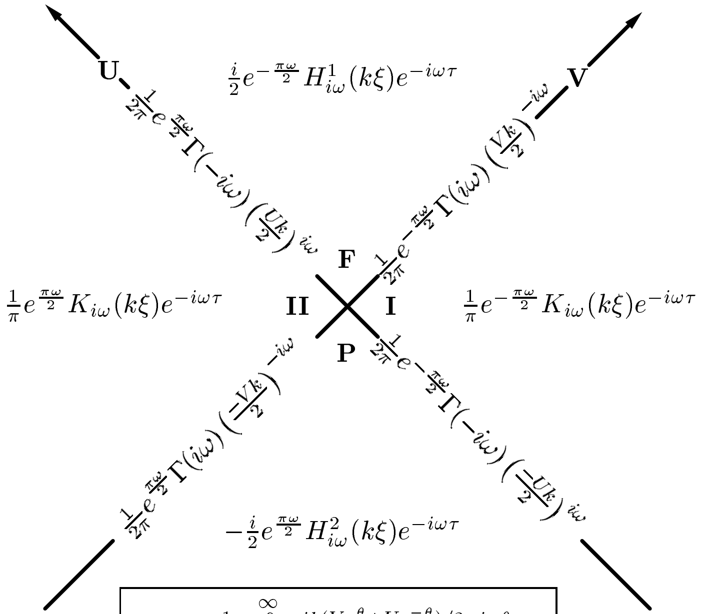
This agrees with our physical expectations which demand that these kernels refer to a translation relative to a frame Lorentz rotated (=boosted) by the amount τ .

There are two distinct representation spaces. They are spanned by $B^{+}_{\omega}(kU,kV)$ and $B^{-}_{\omega}(kU,kV)$ with $-\infty < \omega < \infty$. The two unitary representations $T^{+}_{\omega\overline{\omega}}$ and $T^{-}_{\omega\overline{\omega}}$ of the Poincare group act respectively on these two representation spaces. The $+$ sign refers to the space of positive Minkowski frequencty modes, and the $-$ sign to the space of negative frequency modes. One or the other set of modes plays the same role on the Lorentz plane in relation to the Poincare group that the familiar spherical harmonics play on the unit two sphere in relation to the rotation group.



$$\boxed{B^+_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(Ve^\theta+Ue^{-\theta})/2}e^{i\omega\theta}d\theta}$$

FIGURE (a)



$$\boxed{B^-_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(Ve^\theta+Ue^{-\theta})/2}e^{i\omega\theta}d\theta}$$

FIGURE (b)

FIGURE 2A and 2B

Global Minkowski Bessel (M-B) modes $B^\pm_\omega(kU,kV)$ and their coordinate representatives. A global M-B mode is related to its representatives in the respective Rindler charts I, II, F, and P by Eqs. (4.2) - (4.3). On the past and future event horizons, $UV=0$, these modes have the indicated form, which is obtaied by evaluating the integrals in Eqs. (6.2a) and (6.2b)

IV. LOCAL PROPERTIES

A linearly uniformly accelerated observer, whose worldline is

$$\begin{aligned} t &= \xi \sinh\tau; \quad U = -\xi\, e^{-\tau} \\ x &= \xi \cosh\tau; \quad V = \xi\, e^{\tau} \\ y &= y_0 \\ z &= z_0\,, \end{aligned}$$

induces a division of two-dimensional Minkowski space-time into four coordinate charts: I, II, F and P. See Fig. 1. Each chart is coordinatized by the local Rindler coordinates

$$\xi > 0 \text{ and } -\infty < \tau < \infty$$

$$I: \begin{cases} t = \xi\sinh\tau; x = \xi\cosh\tau \\ U = -\xi e^{-\tau}; V = \xi e^{\tau} \end{cases} \quad (\text{"right hand Rindler sector"}) \tag{4.1a}$$

$$II: \begin{cases} t = -\xi\sinh\tau; x = -\xi\cosh\tau \\ U = \xi e^{-\tau}; V = -\xi e^{\tau} \end{cases} \quad (\text{"left hand Rindler sector"}) \tag{4.1b}$$

$$F: \begin{cases} t = \xi\cosh\tau; x = \xi\sinh\tau \\ U = \xi e^{-\tau}; V = \xi e^{\tau} \end{cases} \quad (\text{"Future"}) \tag{4.1c}$$

$$P: \begin{cases} t = -\xi\cosh\tau; x = -\xi\sinh\tau \\ U = -\xi e^{-\tau}; V = -\xi e^{\tau} \end{cases} \quad (\text{"Past"}) \tag{4.1d}$$

A pair of accelerated observers confine themselves to sector I and sector II respectively. Relative to either coordinates the metric has the form

$$ds^2 = -dUdV + dy^2 + dz^2 = -\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2\,.$$

For sector F and P the metric has the form

$$ds^2 = -dUdV + dy^2 + dz^2 = \xi^2\, d\tau^2 - d\xi^2 + dy^2 + dz^2\,.$$

A global Minkoski Bessel mode, Eq. (2.3), can easily be evaluated in the accelered coordinate frame sector I. One obtains the *coordinate representative* for sector I:

$$B^{\pm}_{\omega}(kU,kV)\,|_I = (1/2\pi)\int_{-\infty}^{\infty} e^{\pm ik\xi\sinh(\theta-\tau)}\, e^{-i\omega\theta}\, d\theta = \frac{1}{\pi}\, e^{\pm\pi\omega/2} K_{i\omega}(k\xi)^{e^{-i\omega\tau}} \tag{4.2a}$$

Simlilarly for the left hand sector II, the future quadrant F, and the past quadrant P one has the respective coordinate representatives

$$B^{\pm}_{\omega}(kU,kV)\,|_{II} = (1/2\pi)\int_{-\infty}^{\infty} e^{\mp ik\xi\sinh(\theta-\tau)}\, e^{-i\omega\theta}\, d\theta = \frac{1}{\pi}\, e^{\mp\pi\omega/2} K_{i\omega}(k\xi)^{e^{-i\omega\tau}}, \tag{4.2b}$$

$$B^{\pm}_{\omega}(kU,kV)\,|_{F} = (1/2\pi)\int_{-\infty}^{\infty} e^{\mp ik\xi\cosh(\theta-\tau)}\, e^{-i\omega\theta}\, d\theta = \mp\frac{i}{2} e^{\pm\pi\omega/2} H^{\substack{2\\1}}_{i\omega}(k\xi) e^{-i\omega\tau}, \tag{4.3a}$$

and

$$B_\omega^{\pm}(kU,kV)\,|_P \;=\; (1/2\pi)\int_{-\infty}^{\infty} e^{\mp ik\xi\cosh(\theta-\tau)}\,e^{-i\omega\theta}\,d\theta \;=\; \pm\frac{i}{2}e^{\mp\pi\omega/2}H_{i\omega}^{\frac{1}{2}}(k\xi)e^{-i\omega\tau}\,. \tag{4.3b}$$

These coordinate representatives are Sommerfeld's cyclinder waves generalized from Euclidean space coordinatized by polar coordinates to Minkowski space-time coordinatized by Rindler coordinates (See Figure 2). These waves are expressed in terms of

$$K_{i\omega}(k\xi) \;=\; K_{-i\omega}(k\xi)\,,$$

the Bessel functions of imaginary argument $ik\xi$ ("MacDonald functions") and imaginary order $i\omega$. In F and P the waves are expressed in terms of

$$e^{\pi\omega/2}H_{i\omega}^{\frac{1}{2}} \;=\; e^{-\pi\omega/2}\,H_{-i\omega}^{\frac{1}{2}}(k\xi)\,,$$

multiples of the two kinds of Hankel functions. All four coordinate representatives are multiples of $e^{-i\omega\tau}$. Consequently, each one is an eigenfunction of the Lorentz boost around the reference event $t = x = 0$. But they all have the same eigenvalue. Thus one has

Property 3

$$\frac{\partial}{\partial\tau}B_{i\omega}^{\pm} \;=\; -i\omega B_{i\omega}^{\pm} \tag{4.4}$$

in all four coordinate neighborhoods, i.e. a globally defined Minkowski Bessel mode (M-B)is an eigenfunction of the Lorentz boost operation.

One can see that a Lorentz boost in an inertial frame is simply a time (τ) translation in the accelerated frame.

V. ORTHONORMALIZATION.

The utility of the Klein-Gordon ("Wronskian") orthonormality, Eq. (2.6)

$$\pm 2\delta(\omega-\omega') = i \int_{\substack{space-like\\ hypersurface}} (B^{\pm}_{i\omega})^* \overleftrightarrow{\partial}_\mu B^{\pm}_{i\omega'} \, d^1\Sigma^\mu \,, \tag{5.1}$$

of the $M-B$ modes extends beyond their identification as positive and negative Minkowski frequency modes. One can also use it to obtain a very useful normalization integral in the accelerated frame. We shall do this for a frame without any bottom $(0<\xi<\infty)$ as well as for a frame with a bottom $(b\le \xi <\infty\,,\ b>0)$. These results we shall use to obtain the asymptotic behavior of mode sums in an accelerated frame with a bottom. We take the limit as the bottom aproaches the event horizon $(\xi=0)$ and the frame becomes thereby bottomless.

A. Bottomless Frame $(0<\xi<\infty)$

The result is obtained in three steps.

(1) Evaluate the hypersurface integral, Eq. (5.1) on the (1-dimensional) space-like hypersurface $\tau=$ constant in both Rindler sectors I and II. Thus the only non-zero component of the hypersurface element

$$d^1\Sigma^\mu = g^{\mu\nu}\epsilon_{\nu\alpha}dx^\alpha = g^{\mu\nu}\sqrt{-g}[\nu,\alpha]dx^\alpha$$

is

$$d^1\Sigma^\tau = \frac{d\xi}{\xi}\,.$$

The Wronskian integral, Eq. (5.1) can therefore be written as

$$2\delta(\omega-\omega') = i\int_\infty^0 (B^+_{i\omega})^* \frac{\overleftrightarrow{\partial}}{\partial\tau} B^+_{i\omega'}|_{II}\ \frac{d\xi}{\xi} + i\int_0^\infty (B^+_{i\omega})^* \frac{\overleftrightarrow{\partial}}{\partial\tau} B^+_{i\omega'}|_{I}\ \frac{d\xi}{\xi} \tag{5.2}$$

It is unnecessary to use $B^-_{i\omega}$ because it will give the same result.

(2) Use Property 3 and insert the coordinate representatives, Eqs. (4.2a) and (4.2b) into Eq. (5.2). The result is

$$\delta(\omega-\omega') = \frac{\omega+\omega'}{\pi^2}\sinh[\frac{\pi(\omega+\omega')}{2}]e^{i(\omega'-\omega)\tau}\int_0^\infty K_{i\omega}(k\xi)K_{i\omega'}(k\xi)\ \frac{d\xi}{\xi} \tag{5.3}$$

Our interest lies in the integral, which is not determined by this equation when $\omega+\omega'=0$. This however is not a problem because $K_{i\omega}(k\xi)$ is an even function of ω, i.e.

$$K_{-i\omega}(k\xi) = K_{i\omega}(k\xi)\,.$$

Consequently

$$\delta(\omega+\omega') = \frac{\omega-\omega'}{\pi^2}\sinh[\frac{\pi(\omega-\omega')}{2}]\ e^{-i(\omega+\omega')\tau}\int_0^\infty K_{i\omega}(k\xi)K_{i\omega'}(k\xi)\frac{d\xi}{\xi}\,. \tag{5.4}$$

(3) Add Eqs. (5.3) and (5.4) to obtain the useful normalization integral

$$\int_0^\infty K_{i\omega}(k\xi)\ K_{i\omega'}(k\xi)\frac{d\xi}{\xi} = \frac{\pi^2}{2\omega\sin h\pi\omega}[\delta(\omega-\omega')+\delta(\omega+\omega')]\,. \tag{5.5}$$

The fact that both delta functions occur on the right side is a reflection of the fact that the Bessel function $K_{i\omega}(k\xi)$ is an even function of ω.

B. Frame With a Finite Bottom $(b \le \xi < \infty)$

We consider the mode $K_{i\omega}(k\xi)$ which satisfies the differential equation $(k^2 = k_y^2 + k_z^2 + m^2)$

$$\{\xi\frac{d}{d\xi}\xi\frac{d}{d\xi} + \omega^2 - k^2\xi^2\}K_{i\omega}(k\xi) = 0 \tag{5.6}$$

on the domain $b \le \xi < \infty, b > 0$. It satisfies some fixed and given homogeneous boundary condition at $\xi = b$,

$$a_1 K_{i\omega}(kb) + a_2\frac{d}{d\xi}K_{i\omega}(kb) = 0 .$$

Consequently the allowed modes have discrete frequencies ω. Let these frequencies be

$$0 < \omega_1 < \omega_2 < \cdots < \omega_n < \cdots .$$

The Sturm Liouville nature of this eigenvalue problem guarantees that these modes satisfy

$$\int_{kb}^{\infty} K_{i\omega_m}(x)K_{i\omega_n}(x)\frac{dx}{x} = c_n(kb)\delta_{mn} \tag{5.7}$$

Our objective is to obtain the normalization constant $c_n(kb)$ as $kb \to 0$. Thus one can make a transition from an accelerated form with a finite bottom $(b > 0)$ to a bottomless one $(b = 0)$. In quantum field theory or in condensed matter physics such a transition is called "going to the thermodynamic limit". Comparing Eq. (5.7) with (5.5) one writes this transition as

$$c_n(kb)\delta_{mn} \to \frac{\pi^2}{2\omega\sinh\pi\omega}\delta(\omega-\omega') \quad (as\ b \to 0) \tag{5.8}$$

This is a useful equation because one can now work with finite quantities (namely the righthand side) which in the thermodynamic limit become infinite (namely the l.h.s. when $\omega = \omega'$).

One can evaluate Eq. (5.7) exactly in terms of $K_{i\omega}$ and its derivative with respect to ω.[9] But we shall use the WKB approximation because it is more transparent. In this approximation $(k\xi \equiv x)$

$$K_{i\omega}(x) = \sqrt{\frac{\pi}{2\sinh\pi\omega}}\frac{1}{(\omega^2-x^2)^{1/4}}\,cos(\int\sqrt{\omega^2-x^2}\,\frac{dx}{x} + const) \tag{5.9}$$

whenever $k\xi << \omega$.

Consequently

$$\int_{kb}^{\infty} K_{i\omega}^2(x)\frac{dx}{x} \simeq \int_{kb}^{\omega}\frac{\pi}{2\sinh\pi\omega}\frac{1}{\sqrt{\omega^2-x^2}}\frac{1}{2}[1+cos(\cdots)]\frac{dx}{x}. \tag{5.10}$$

Upon integration the term $cos(\cdots)$ averages to zero. One obtains therefore

$$\int_{kb}^{\infty} K_{i\omega}^2(x)\frac{dx}{x} \simeq \frac{\pi}{2\omega\sinh\pi\omega}log\frac{(\omega+\sqrt{\omega^2-k^2b^2})}{kb}, \quad kb \le \omega < \infty \tag{5.11}$$

Using this normalization integral to compare Eq. (5.7) with (5.5) one obtains the desired relation for Eq. (5.8), namely

$$\lim_{b\to 0} \frac{1}{\pi} log \frac{(\omega + \sqrt{\omega^2 - k^2 b^2})}{kb} \delta_{mn} \;=\; \delta(\omega_m - \omega_n) \tag{5.12}$$

C. Mode Sums in an Accelerated Frame.

Our final objective is to establish the corresponding relation between a mode sum and a mode integral. The allowed normal mode frequencies ω are determined by the "Bohr quantization" condition applied to Eq. (5.9). One has

$$\int_{kb}^{\omega} \sqrt{\omega^2 - x^2}\, \frac{dx}{x} + const \;=\; n\pi$$

The density of states is obtained by differentiation with respect to ω and then doing the integration,

$$\frac{dn(\omega)}{d\omega} \;=\; \frac{1}{\pi} log \frac{(\omega + \sqrt{\omega^2 - k^2 b^2})}{kb}, \quad kb \le \omega < \infty \tag{5.13}$$

It follows from Eq. (5.13) that the transition from a mode sum to a mode integral is established for small kb by

$$\begin{aligned} \sum_{n=1}^{\infty} (\cdots) \stackrel{kb\ small}{\longrightarrow} \int dn(\cdots) &= \int_{kb}^{\infty} d\omega \frac{dn}{d\omega} (\cdots) \\ &= \int_{kb}^{\infty} d\omega \frac{1}{\pi} log \frac{\omega + \sqrt{\omega^2 - k^2 b^2}}{kb} (\cdots) \end{aligned} \tag{5.14}$$

Combining Eqs. (5.12) and (5.13) one obtains the expected result

$$\sum_{n=1}^{\infty} \delta_{mn} (\cdots) \stackrel{kb\ small}{\longrightarrow} \int_{kb}^{\infty} d\omega \delta(\omega - \omega_m)(\cdots)\,. \tag{5.15}$$

Equations (5.14) and (5.15) show how mode sums for a bottomless $(b = 0)$ accelerated frame are the asymptotic limit of corresponding sums for accelerated frame with a bottom $(b > 0)$.

Equations (5.12), (5.13), and (5.14) have their anologue in carteseian coordinates where the modes satisfy the differential equation

$$[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_y^2 + k_z^2]\psi = 0$$

on the domain $-\frac{L}{2} \le y, z \le \frac{L}{2}$. The corresponding equations are well known and are given by

$$\lim_{L\to\infty} (\frac{L}{2\pi})^2 \delta_{m_y n_y} \delta_{m_z n_z} = \delta(k_y - k_y')\delta(k_z - k_z') \tag{5.12$'$}$$

$$\frac{d(modes)}{dk_y dk_z} = (\frac{L}{2\pi})^2 \tag{5.13$'$}$$

$$\sum_{n_y=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} (\cdots) \to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dn_y dn_z\, (\cdots) = \lim_{L\to\infty} L^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} (\cdots) \tag{5.14$'$}$$

To achieve our final objective of relating a mode sum in a bottomless frame to a mode integral in a frame with a bottom one first multiplies Eqs. (5.13) and (5.13′),

$$\frac{d(all\ modes)}{(dk_y)(dk_z)(d\omega)} = (\frac{L}{2\pi})^2 \frac{1}{\pi} log \frac{\omega + \sqrt{\omega^2 - k^2 b^2}}{kb} \tag{5.16}$$

This is the "density of all states" in a frame whose bottom is at $\xi = b$. Secondly one uses this density to evaluate the mode sum

$$\sum_{n_y=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} \sum_{n=1}^{\infty} (\cdots) \equiv \sum (\cdots)$$

asymptotically for "small" b. Small b means

$$0 < b << g^{-1}$$

where g^{-1} is the Fermi-Walker [8] ("head start") distance of the fiducial observer whose world line is $x^2 - t^2 = \xi^{-2} = g^{-2}$. As one readily sees from the metric

$$ds^2 = -\xi^2 g^2 d\tau_{rel}^2 + d\xi^2 + dy^2 + dz^2$$

relative to the accelerated frame, this distance is at $\xi = g^{-1}$. There the coordinate time coincides with proper time $(\triangle s = \triangle \tau_{rel})$ and the proper acceleration is measured to be g. Throughout this paper we have hidden this acceleration by absorbing it with the realtivistic "boost" time and "boost" frequency into the dimensionless geometrical quantities and

$$\omega = \omega_{rel}/g$$

$$\tau = \tau_{rel} g$$

respectively. For the purpose of exhibiting the asymptotic $(b << g^{-1})$ expression of the total mode sum $\sum(\cdots)$, we shall temporarily reintroduce this acceleration explicitly. Thus we have

$$\sum(\cdots) \to \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{bg}^{\infty} \frac{d(all\ modes)}{(dk_y)(dk_z)(d\omega)} dk_y dk_z d\omega \quad (\cdots)$$

where the density of all moes is given by Eq. (5.16). Introducing Ω by means of

$$\omega = \frac{\Omega k}{g}$$

one obtains for $bg << 1$

$$\sum(\cdots) \to \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{bg}^{\infty} (\frac{L}{2\pi})^2 \frac{1}{\pi} log \frac{\Omega + \sqrt{\Omega^2 - b^2 g^2}}{bg} \frac{k}{g} dk_y dk_z d\Omega \quad (\cdots) \tag{5.17}$$

The logarithmic factor is independent of k_y and k_z. Furthermore the convergence of the $\int^{\infty} \cdots d\Omega$ integral implies that for $bg << 1$ this logarithmic factor becomes

$$log \frac{\Omega + \sqrt{\Omega^2 - b^2 g^2}}{bg} \overset{bg<<1}{\longrightarrow} log \frac{1}{bg} .$$

Consequently the total mode sum is

$$\sum(\cdots) \quad \stackrel{bg<<1}{\longrightarrow} \quad L^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{d^2k}{(2\pi)^2}\frac{d\omega}{\pi} \ln \frac{\omega+\sqrt{\omega^2-k^2b^2}}{kb}(\cdots), \tag{5.18}$$

provided $(\cdots)$ is well-behaved near $\omega = 0$.

This is the asymptotic expression for the mode integral in an uniformly and linearly accelerated frame. By contrast the corresponding familiar expression in an inertial frame is

$$\sum(\cdots) \quad \stackrel{V\ large}{\longrightarrow} \quad V \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2}\frac{dk_x}{2\pi}(\cdots).$$

Here $V = L^3$. One sees that what corresponds to going to the infinite volume limit in an inertial frame $(V \to \infty)$, corresponds to

$$V_{R-W} \equiv L^2 g^{-1} log(1/bg) \to \infty \tag{5.19}$$

in an accelerated frame. Here V_{R-W} is the Regge-Wheeler volume. It's longitudinal length is based on the flat space-time analogue of the Regge-Wheeler ("tortoise") coordinate [10], ξ^*,

$$\xi^*(\xi) = g^{-1} log \xi g. \tag{5.20}$$

The Regge-Wheeler coordinate straightens out the null cone along the acceleration direction in an accelerated frame, be it near a black hole or in flat space time,

$$ds^2 = g^2\xi^2(-d\tau_{rel}^2 \; + \; d\xi^{*2}) + dy^2 + dz^2 \,.$$

This coordinate is not proper distance. It pushes the event horizon $\xi = 0$ to $\xi^* = -\infty$. In terms of this coordinate $\xi = 0$ lies at negative spatial infinity of the accelerated frame, and the *Regge-Wheeler size* of the proper interval $[b, g^{-1}]$ is from Eq. (5.20)

$$\xi^*(g^{-1}) - \xi^*(b) = g^{-1} log(1/bg).$$

Thus the Regge-Wheeler volume of a semi-finite accelerated box with bottom at $\xi = b > 0$ is the product of this length with the transverse area L^2,

$$V_{R-W} \; = \; L^2 g^{-1} log(1/bg) \tag{5.21}$$

One concludes therefore that the "thermodynamic" limit in an accelerated frame consists of

$$\Sigma(\cdots) \quad \stackrel{bg<<1}{\longrightarrow} \quad (gV_{R-W}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{d^2k}{(2\pi)^2}\frac{d\omega}{\pi} \ln \frac{\omega+\sqrt{\omega^2-k^2b^2}}{kb} \quad (\cdots). \tag{5.22}$$

VI. THE DEGENERATE CASE

The Minkowski Bessel modes, as well as the Minkowski plane wave modes satisfy any one of th three equations,

$$\begin{aligned} \{-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-k^2\}\psi &= 0 \\ \{-4\frac{\partial^2}{\partial U\partial V}-k^2\}\psi &= 0 \\ \{\mp\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2}\pm\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}-k^2\}\psi &= 0 \end{aligned} \tag{6.1}$$

(upper sign for $I\cup II$, lower sign for $F\cup P$) depending on which coordinates one uses. These equations are the result of solving the Klein-Gordon equation so that

$$k^2 = k_y^2 + k_z^2 + \frac{m^2c^2}{h^2}$$

where the individual terms have the usual meaning.

In this paper we have exhibited the set of Minkowski Bessel modes for the non-degenerate case $k^2>0$. The degenerate case is $k^2\to 0$. This is a set of measure zero and it is in a class by itself. It corresponds to plane waves of a massless field travelling strictly along the x-direction. There are several ways of obtaining the solution corresponding to this singular limit. One of the most direct ways is to simply consider

$$\begin{aligned} M_\omega^{r\pm}(\kappa U) &\equiv B_\omega^{\pm}(2\kappa U,0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} exp(\mp i\kappa U e^{\theta})e^{-i\omega\theta}d\theta \end{aligned} \tag{6.2a}$$

We shall call these modes the "retarded" (superscript "r") Minkowski Mellin (M-M) modes because they are (with $s=e^\theta$) a Mellin synthesis of the plane waves $exp[\mp i(t-x)\mathcal{H}e^{\theta}]$. The "advanced" (superscript "a") Minkowski Mellin modes are

$$\begin{aligned} M_\omega^{a\pm}(\kappa V) &\equiv B_\omega^{\pm}(0,2\kappa V) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} exp(\mp i\kappa V e^{-\theta})e^{-i\omega\theta}d\theta \end{aligned} \tag{6.2b}$$

and they are composed of positive (upper sign) or negative (lower sign) Minkowski frequency plane waves travelling towards negative x. These Minkowski Mellin modes satisfy Eqs. 6.1 with $k^2=0$. The constant $k>0$ appearing in Eqs. (6.2) is arbirary and has been introduced for dimensional reasons. These $M-M$ have first been exhibited by R. Hughes [11]. Note that the advanced modes considered as functions of their argument are related to the retarded ones by

$$M_\omega^{r\pm} = M_{-\omega}^{a\pm}$$

The most important aspect of these $M-M$ modes is that they lack no property which the Minkowski Bessel functions have, namely

(1) They are unitarily related to the plane waves and hence are globally defined.

(2) They form a unitary representation of the translation group in two dimensional Lorentz space-time,

$$M_{\omega-\overline{\omega}}^{r\pm}(\kappa(U+U_0)) = \int_{-\infty}^{\infty} M_{\omega-\omega'}^{r\pm}(\kappa U)M_{\omega'-\overline{\omega}}^{r\pm}(\kappa U_0)d\omega'; \tag{6.3}$$

(3) They are eignfunctions of the Lorentz boosts (see below)

$$\frac{\partial}{\partial\tau} M_\omega^{j\pm} = -i\omega M_\omega^{j\pm} \quad j = a, r\,.$$

Furthermore their $K-G$ normalization, Eq. (2.4), is also the same

$$(M_\omega^{j\pm}, M_{\omega'}^{j\pm}) = \pm 2\delta(\omega - \omega') \quad j = a, r \tag{6.4a}$$

In addition

$$(M_\omega^{a\pm}, M_{\omega'}^{r\pm}) = 0 \tag{6.4b}$$

as one might have expected.

The Rindler *coordinate representatives* of the $M-M$ mode are readily obtained by inserting the expressions from Eqs. (4.1) into the definitions, Eqs. (6.2). The resulting four coordinate representatives of the retarded $M-M$ modes are

$$M_\omega^{r\pm}(\kappa U) = \frac{1}{2\pi}\Gamma(i\omega)(\kappa\xi)^{-i\omega}\, e^{-i\omega\tau} \begin{cases} e^{\pm\pi\omega/2} & in \quad I \quad and \quad F \\ e^{\mp\pi\omega/2} & in \quad II \quad and \quad P \end{cases} \tag{6.5b}$$

Here Γ is the gamma function. Except for a normalization factor $\sqrt{2}$ these expressions agree with those already known for I and II[11] .

VII. CONCLUSION

The global Minkowski Bessel modes are the Poincare group harmonics on the Lorentz 2-plane. They are analogous to the large ℓ limit of the $Y_m^\ell(\theta,\phi)$, the rotation group harmonics on the 2-sphere. The three most important ideas of this paper are expressed by *Properties* 1-3. They hold both for the $M-B$ modes and the $M-M$ modes. The two sets are related by Eqs. (6.2). The asymptotic approximation of a mode sum as a mode integral is exhibited by Eq. (5.22) for a physically interesting coordinate representative.

Finally let us mention three observations which are more technical in nature:

(1) The conventional group theoretic treatment of the MacDonald functions $K_{i\omega}$ and the Hankel functions $H_{i\omega}$ view them as a representation of translations in the Minkowski plane [12]. This does not express the true state of affairs. Actually it is the global Minkowski Bessel modes which play this role. The functions $K_{i\omega}$ and $H_{i\omega}$ only refer to the *coordinate representatives* of these $M-B$ modes in each of the respective Rindler charts ("sectors").

(2) The group composition properties, Eqs. (2.7) and (6.3) for the global $M-B$ and $M-M$ modes imply numerous composition theorems for the Bessel related functions $(K_{i\omega}, H^1_{i\omega}$ and $H^2_{i\omega})$[13] and the gamma function $\Gamma(i\omega)$[14]. These theorems are now so easy to obtain because one merely has to insert the coordinate representatives, Eqs. (4.2) - (4.3) and Eqs. (6.5) into the group composition laws.

(3) If one recalls that

$$B^{\pm}_{\omega-\omega'}(2\kappa U,0) = M^{r\pm}_{\omega-\omega'}(\kappa U)$$

and

$$B^{\pm}_{\omega-\omega'}(0,\kappa V) = M^{a\pm}_{\omega-\omega'}(\kappa V)\,,$$

then inserting the corresponding coordinate representatives into the group composition law Eq. (2.7) yields integral relations [15] between the Bessel related functions and the gamma function.

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