# ACCELERATION INDUCED SPIN AND ITS GAUGE GEOMETRY

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## ABSTRACT

Does there exist a purely quantum mechanical characterization of gravitation? To this end the fundamental results of Fulling, Davies, and Unruh are incorporated into a framework which is qualitatively different from Einstein's starting point. Instantaneous pairs of achronal Rindler frames together with the concomitant set of Minkowski-Bessel modes replace Einstein's instantaneous inertial frames and his concomitant set of plane wave modes. Because of this replacement, a particle, which is governed by its relativistic wave equation, acquires a set of *internal* quantum states at each event of spacetime. These states are organized by means of the spinor group SU(1,1) into a geometrical picture of an entity with achronal spin at each event. A unique and natural law of parallel transport of quantum states between different events is identified. This parallel transport (a) is compatible with the inner product on the space of quantum states at each event, (b) has zero curvature, and (c) determines a conserved achronal spin operator. The orientation of the axis of rotations generated by this operator is given by Planck's thermal spectrum together with the associated r.m.s. fluctuation spectrum. A purely wave mechanical line of reasoning leads to the heuristic conclusion that gravitation is to be identified with the gauge geometry of the group $[SU(1,1)]^{\infty}$.

## ACCELERATION INDUCED SPIN AND ITS GAUGE GEOMETRY

The line of reasoning which lies at the base of Einstein's gravitation theory and points towards his field equations is purely classical and is well known. It consists of pitting two simple ideas against each other:

(1) Consider the straight line paths of freely floating particles. Relative to a curvilinear coordinate frame, e.g. accelerating or rotating, the governing equations are the geodesic equations of motion with their metric coefficients.

(2) Consider particles falling freely in a gravitational field. The governing equations are Newton's equations of motion. The inertial and gravitational mass on each side of the equation cancel: They are equal. What do the resulting equations tell us? Comparing it with the geodesic equations obtained in (1), one finds that

$$\text{gravitational potential} = \text{metric coefficient}$$

From these and similar considerations [1] one concludes that

$$\text{gravitation} = \text{geometry}.$$

This is the key conclusion. It inevitably leads to Einstein's field equations.

This line of reasoning is purely classical. It is based on classical mechanics: particle world lines, the classical equations they satisfy, and so on. This line of reasoning is also purely local: The equations as well as the accelerated frames in the above gedanken experiments are local in space and in time.

Classical mechanics is expressed in terms of world lines. Events are intersections of world lines. A vector is an infinitesimal world line segment. A tangent space is a collection of vectors. A metric is a structure imposed on a tangent space. A covariant derivative relates different tangent spaces. Riemann curvature characterizes a covariant derivative. To summarize: General relativity is based on world lines, i.e. on classical physics.

The actual world rests however on a quantum framework and quantum mechanical reasoning. Unlike classical mechanics, quantum mechanics is characterized by infinite dimensions, noncommutativity, and global considerations.

Quantum mechanical reasoning is as different from classical reasoning as day is from night. Nevertheless it does contain all of classical mechanics. The classical analysis of the above free-float-free-fall gedanken experiments is at the root of Einstein's gravitation theory. If quantum mechanics is to be incorporated in a natural way with gravitation, then it must be merged at the roots and not at the trunk of gravitation. In other words, a purely quantum mechanical analysis of these gedanken experiments is necessary.

Put differently: How different would Einstein's formulation of gravitation have been, had quantum mechanics been available to him already in 1907 [2]?

Fundamental to Einstein's analysis is his formulation in terms of instantaneous Lorentz frames (one copy attached to each event of spacetime) and the concomitant locally plane wave behavior of solutions to any relativistic wave equation.

Such a formulation must however be considered deficient because it ignores the fundamental results of Fulling [3], Davies [4], and Unruh [5]. Their quantum mechanical results demand that accelerated Rindler frames and their concomitant Minkowski Bessel modes [6] of a relativistic wave equation be substituted for the Lorentz frames and their plane waves.

Moreover their results hold not only for their pair of accelerated frames at one particular reference event, but also for any other pair related to theirs by spacetime translation. The extension of this observation to all relativistic laws of physics is undoubtedly not controversial: it merely expresses translation invariance in Minkowski spacetime. Applied to their causally disjoint frames this implies the "translation invariance of achronal frames".

In spite of its simplicity this principle opens a new vista in that it quite naturally elevates the quantum states of a system to the geometrical status that used to be occupied by events and four-velocities. In other words, as we shall see below, one can characterize a system by the inner products, the parallelism, the curvature, and so on, of its quantum states. A necessary consequence of this new point of view is that the quantum states (of the Klein-Gordon wave equation, for example) are two-component $SU(1,1)$-spinor wave functions.

The pioneering investigations of quantum mechanics relative to a uniformly accelerated frame with its event horizons were performed some twenty years ago. Motivated by Fulling [3], Unruh [5] considered the quantum analogue of free particles moving in an accelerated frame, the Rindler spacetime. The result was truly astonishing: Empty (!) flat spacetime is characterized by a measurable thermal ambience which for bosons has the Planckian spectrum

$$\frac{1}{2}+\frac{1}{\exp(\hbar\omega kT)-1} \tag{1a}$$

with the Davies-Unruh temperature

$$kT=\frac{\hbar}{c}\frac{g}{2\pi}. \tag{1b}$$

The randomness of the thermal ambience is an expression of the intrinsic quantum mechanical randomness of each member of a pair of highly correlated subsystems [7].

But these Einstein-Podolski-Rosen correlations do not account for the specifically Planckian (or Fermionic, if the quanta are fermions) nature of the intensity spectrum. Even though there are a number of different ways [8,9,10,11] of establishing the existence of the thermal ambience , at the end one always obtains the same mathematical result: Relative to an accelerated frame, globally empty spacetime is characterized (for bosons) by the Planckian spectral function!

Why does nature favor this function over all the others available to it?

This question is related to a peculiar problem associated with the spacetime of a pair of accelerated frames

$$\left.\begin{array}{l} t - t_0 = \pm\xi \sinh\tau \\ x - x_0 = \pm\xi \cosh\tau \end{array}\right\} \quad \begin{array}{ll} + : & \text{Rindler Sector I} \\ - : & \text{Rindler Sector II} \end{array} .$$

Such a pair of *achronal* frames has the undesirable property that it singles out the preferred base event $(t_0, x_0)$, the intersection of the future and past event horizons of the frames.

Classically one can remove this arbitrary preference: One relates events (= intersections of particle world lines) and vectors (= infinitesimal world line segments) measured and described in one frame at, say, $(t_0, x_0)$ to the ones measured and described in another accelerated frame based at $(t'_0, x'_0)$. The mathematical implementation of this procedure is parallel transport ("connection") which is compatible with the inner product in each accelerated frame.

Quantum mechanics does not permit such a procedure. There are no deterministic world lines: At best, one has smeared out strips or tubes that only resemble world lines. In general, there are only probability or charge densities as determined by a wave function. Thus we have two opposing requirements: need for a quantum description vs. need for equal status of all accelerated frames.

In addition, there is a third problem: part of the wave function disappears behind the event horizons of any one accelerated frame.

## THREE ASSOCIATED FIBER BUNDLES

An appropriate fiber bundle framework not only overcomes these difficulties, but also highlights an important geometrical spinor structure implied by the wave equation. We shall describe it for for the Klein-Gordon equation, although one can do this for other relativistic wave equations as well.

There are three associated fiber bundles that play a central role: (1) the bundle of Rindler spacetimes, the arena for classical particle physics, (2) the bundle of Rindler single particle states, the arena for the quantum mechanics of a single particle, and

(3) the bundle of Rindler Fock states, the arena for the quantum mechanics of many particles.

(1) The bundle of Rindler spacetimes, or more simply, the Rindler bundle, is not unfamiliar. It is a continuous family, $R_z$, of pairs of achronal and oppositely accelerated frames. These accelerated frames are parametrized by the base event

$$(t_0, x_0) \equiv z \in M$$

of spacetime $M$ ("the base manifold"). A base event is the intersection of the past and the future event horizons ("null lines") of a particular pair of frames.

A Rindler spacetime ("the fiber") is the union of two disconnected sets of spacelike rays emanating from the base event $z = (t_0, x_0)$,

$$R_z = \{\text{spacelike rays in Rindler I}\} \cup \{\text{spacelike rays in Rindler II}\} \tag{3}$$

(Note that the future and the past of the base event are not part of this fiber.) It forms a representation space for the group of boosts and for the discrete group

$$Z_2 = \{\text{identity, I}\},$$

which is characterized by the inversion operation through the base event,

$$I : (t - t_0, x - x_0) \rightarrow (-t + t_0, -x + x_0).$$

The total group

$$\{\text{(boosts), I(boosts)}\} \equiv G$$

is the internal symmetry ("structure" ) group of the bundle of Rindler spacetimes.

This group consists of its two disconnected components, the invariant subgroup "(boosts)" and its coset "I(boosts)". This internal symmetry group is commutative. Thus it is (isomorphic to) the direct product of its two invariant subgroups [12],

$$G = \{\text{(boosts), I(boosts)}\} = Z_2 \times \text{(boosts)}. \tag{4}$$

(2) Associated with the Rindler bundle is the bundle of single particle Rindler states. For the Klein-Gordon wave equation this bundle is simply the "Klein-Gordon vector bundle". It is a continuous family, $K\text{-}G_z$, of spaces of solutions to the Klein-Gordon wave equation

$$\left[-\frac{1}{\xi}\frac{\partial^2}{\partial\tau^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} - k^2\right]\psi = 0, \tag{5}$$

with $k^2 = k_y^2 + k_z^2 + \frac{m^2c^2}{\hbar^2}$, on each Rindler spacetime $R_z$. Here

$$\psi = \begin{bmatrix} \psi_{\mathrm{I}} \\ \psi_{\mathrm{II}} \end{bmatrix} \tag{6}$$

is a wave function with two components whose respective domains are the two respective causally disjoint Rindler sectors I and II. Each component satisfies the same differential Eq. (5). The two-component wave function belongs to the solution space $K$-$G_z$ ("the fiber") attached to the base event

$$(t_0, x_0) = z \in M$$

of spacetime $M$ ("the base manifold").

In fact, there is one copy of such a space of quantum states (= solutions to Eq.(5)) attached to each event of spacetime.

Thus the bundle of Rindler single particle states is the same as the bundle of Rindler spacetimes in every respect. The only difference is the fiber and the manner in which the internal symmetry group acts on it. The linear space of solutions $K$-$G_z$ forms a representation space for the internal symmetry group G, Eq.(4). Its commutativity guarantees that its invariant subspaces are one-dimensional and that the concomitant spanning states

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad K_{i\omega}(k\xi)e^{-i\omega\tau}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} K_{i\omega}(k\xi)e^{-i\omega\tau} : \quad -\infty < \omega < \infty \tag{7}$$

form a preferred (unique, "symmetry induced") set of basis states for $K$-$G_z$. Such a preferred basis for all linear spaces $K$-$G_z$ implies that they all can be referred to a *single* reference space, the *standard* Klein-Gordon fiber, $K$-$G$. Its existence allows one to compare quantum states in $K$-$G_z$ with those in $K$-$G_{z'}$. We should hasten to add that this correspondence in bases is not a parallel transport. It may rather be looked at more like an analogue of Fermi-Walker transport than of anything else.

The basis states, Eq. (7), span the internal ( i.e. Rindler ) quantum states of the Klein-Gordon equation at a particular event $z = (t_0, x_0)$. This feature suggests that the intuitive machinery pertaining to a spinning particle, and its internal dynamics as expressed by its spinors, be applied to our acceleration-induced internal structure of a Klein-Gordon particle.

The number of such internal acceleration-induced quantum states of the free particle is evidently infinite. On the other hand, the number of internal spin-induced quantum states of a rotating particle is finite. In spite of this difference the application of the intuitive machinery turns out to be very fruitful: As we shall see below, the internal quantum space has an inner product, a law of parallel transport, which is compatible with this inner product, curvature,etc.

(3) From the viewpoint of the physics of many particles the most important bundle (associated with the Rindler bundle) is the Rindler Fock bundle. It is a continuous family $F_z$, of Rindler Fock spaces attached to each base event $z=(t_0,x_0)$ of spacetime. The quantum states in each space are the multiparticle excitation states of the Rindler vacuum

$$|O_{R_z}>=|O_{I_z}>\otimes|O_{II_z}>$$

attached to each $z$. This vacuum, we recall, is the direct product of the Fulling vacuum [3] for Rindler sector $I_z$ and $II_z$ respectively. Physically this product state is obtained by operating two refrigerators as they accelerate into opposite directions [13].

The multiparticle states, the elements of the "fiber $F_z$", are obtained by applying linear combinations of products of creation operators to this vacuum state. In general, such an application results in a quantum state which is thermodynamically distinct from the vacuum. Mathematically this manifests itself by the fact that, for example, the Minkowski vacuum state cannot be related by a unitary transformation to the Rindler vacuum state [14]. It follows that the set of multi-particle states, i.e. the Rindler Fock fiber $F_z$, has a considerably more sophisticated subspace structure than the corresponding quantum single particle space $K\text{-}G_z$.

Moreover, if $K\text{-}G_z$ is the fiber of a "vector" bundle, then $F_z$ is the fiber of a Whitney sum of "tensor" [15,16] bundles. In spite of this difference both bundles have the same internal symmetry group $G$, Eq.(4). The only difference lies in the manner in which its operations are represented on the two respective fibers of quantum states.

## GEOMETRY OF THE BUNDLE OF SINGLE PARTICLE RINDLER STATES

The Klein-Gordon wave equation imposes two natural structures on the Klein-Gordon vector bundle: (1) an inner product in each fiber space of solutions, the familiar Klein-Gordon inner product, and (2) a gauge structure, the quantum mechanical analogue of parallel transport. This gauge structure is compatible with the inner product.

### 1. Inner Product

The Klein-Gordon inner product for the Klein-Gordon Eq.(5) defined on $R_z$, Eq.(3), is

$$\langle\psi,\psi'\rangle=\frac{i}{2}\int_{\infty}^{0}\left(\psi_{II}^{*}\frac{\partial}{\partial\tau}\psi'_{II}-\frac{\partial}{\partial\tau}\psi_{II}^{*}\psi'_{II}\right)\frac{d\xi}{\xi}+\frac{i}{2}\int_{0}^{\infty}\left(\psi_{I}^{*}\frac{\partial}{\partial\tau}\psi' I-\frac{\partial}{\partial\tau}\psi^{*} I\psi_{I}\right)\frac{d\xi}{\xi}.$$

This inner product is defined on a global Cauchy hypersurface extending from II through $z = (t_0, x_0)$ to I. In terms of the Pauli matrix

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

this product becomes

$$\langle \psi, \psi' \rangle = \frac{i}{2} \int_0^\infty \left( \psi^\dagger \sigma_3 \frac{\partial}{\partial \tau} \psi' - \frac{\partial}{\partial \tau} \psi^\dagger \sigma_3 \psi' \right) \frac{d\xi}{\xi}. \tag{8}$$

Here

$$\psi = \begin{bmatrix} \psi_I \\ \psi_{II} \end{bmatrix} \tag{9a}$$

is the wave function with its two components defined on I and II respectively, while

$$\psi^\dagger = [\psi_I^* \psi_{II}^*] \tag{9b}$$

is its Hermitian adjoint.

This inner product, we recall, is invariant under the group $SU(1,1)$. This invariance is based on the defining property of $SU(1,1)$: It leaves invariant the Pauli matrix $\sigma_3$:

$$s^\dagger \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Leftrightarrow s \in SU(1,1).$$

Consequently, if s operates on a spinor, say Eq.(9),

$$s\psi(\xi, \tau) \equiv \overline{\psi}(\xi, \tau),$$

then one obtains

$$\overline{\psi}^\dagger \sigma_3 \overline{\psi}' = \psi^\dagger \sigma_3 \psi'$$

which verifies the SU(1,1) invariance of the inner product, Eq.(8), in each fiber space of quantum states.

## 2. Linear Gauge

A *linear gauge* for the Klein-Gordon vector bundle means a choice of a basis for $K\text{-}G_z$ varying continuously with $z$. A *gauge transformation* is a linear transformation at each $z$ – a transformation which provides a change of basis at each $z$. A *gauge* *structure* is a gauge together with all its gauge transformations. The Klein-Gordon equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + k^2\right]\psi = 0 \tag{10}$$

together with boundary conditions on its solutions imply a unique gauge structure for the Klein-Gordon vector bundle.

One can specify this structure (i) in terms of a locally defined covariant derivative or (ii) in terms of a set of globally defined solutions. We shall take the global approach because it motivates the local definition.

The global approach to the definition of the gauge structure consists of a reinterpretation of the translation representation property of the globally defined Minkowski-Bessel (= MB) modes. The motivation and the definition of these modes are as inextricably related as they are simple [6]. See Figure 1. Their three defining properties are:

(i) They satisfy the Klein-Gordon Eq.(10).

(ii) They are invariant under the boost transformations.

(iii) They are related to the plane wave solutions by a Fourier transform,

$$\begin{aligned} B_\omega^\pm(kU, kV) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mp i\omega_k t - k_x x)]e^{-i\omega\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mp ik(Ue^\theta + Ve^{-\theta})/2]e^{-i\omega\theta}d\theta. \end{aligned} \tag{11}$$

Here

$$\begin{aligned} \omega_k &= k\cosh\theta, \\ k_x &= k\sinh\theta \end{aligned} \qquad -\infty < \theta < \infty$$

$$k^2 = k_y^2 + k_z^2 + \frac{m^2c^2}{\hbar^2}$$

and

$$U = t - x, \quad V = t + x$$

are the retarded and advanced times ("null coordinates"). These two types of MB modes are unitarily (= by the Fourier transform) related to the two types of plane wave modes: those on the positive ( upper sign ) "mass shell", and those on the negative (lower sign) "mass shell".

Note that these MB modes are boost eigenfunctions indeed:

$$\left( t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} \right) B_\omega^\pm(kU,kV) = -i\omega B_\omega^\pm(kU,kV).$$

Although they are not eigenfunctions of spacetime translations, they do form a (non-diagonal) representation for them. In fact, one has

$$B_\omega^\pm(z+y) = \int_{-\infty}^{\infty} \mathcal{D}_{\omega'\omega}^{k\pm}(z) B_{\omega'}^\pm(y) d\omega'. \tag{12a}$$

Here the representation kernel is expressed in terms of the MB mode,

$$\begin{aligned} \mathcal{D}_{\omega'\omega}^{k\pm}(z) &= B_{\omega-\omega'}^\pm(kU_0, kV_0) \\ &\equiv B_{\omega-\omega'}^\pm(z). \end{aligned} \tag{12b}$$

It leaves invariant the space of positive $(+)$ and negative $(-)$ Minkowski frequency modes.

In summary, the MB modes represent boosts diagonally, and translations non-diagonally. This is just the other way around from plane wave modes: they are not eigenfunctions of boosts, but are invariant under translations. Put differently, MB modes are as ubiquitous in pairs of instantaneous accelerated frames as plane wave modes are in instantaneous Lorentz frames.

The gauge structure of the Klein-Gordon vector bundle is constructed by picking a basis of quantum states at one event, say $z = (0,0)$, and then specifying the set of corresponding parallel quantum states at another closeby event. The parallelism is dictated by the linear transformation $\mathcal{D}_{\omega'\omega}^{k\pm}(z)$, Eq. (12), on the space of MB modes. We proceed within the framework of the Klein-Gordon vector bundle as follows:

(i) Imagine the one-parameter family of Klein-Gordon solution spaces $K\text{-}G_{z(\lambda)}$, which are attached to the curve $z(\lambda)$ in Minkowski spacetime.

(ii) At each point $z(\lambda)$ identify the $B_\omega^\pm(y)$'s as a set of basis states (spanning vectors) for each Klein-Gordon fiber, $K\text{-}G_{z(\lambda)}$. This is a natural thing to do because the two (I and II) Rindler coordinate representatives of these MB modes form a basis, essentially the same as Eq.(7),

$$B_\omega^\pm(y) : \begin{bmatrix} B_\omega^\pm(y)|_{\mathrm{I}} \\ \\ B_\omega^\pm(y)|_{\mathrm{II}} \end{bmatrix} = \frac{K_{i\omega}(k\xi)}{\pi} e^{-i\omega\tau} \begin{bmatrix} e^{\pm\pi\omega/2} \\ e^{\mp\pi\omega/2} \end{bmatrix} \qquad -\infty < \omega < \infty. \tag{13}$$

Here

$$y = \left\{ \begin{array}{c} t - t_0 = \pm\xi \sinh\tau \\ x - x_0 = \pm\xi\cosh\tau \end{array} \right\} \qquad \begin{array}{ll} + : & \text{Rindler Sector I} \\ - : & \text{Rindler Sector II} \end{array}$$

is spacelike and thus is a typical point in the Rindler fiber $R_{z(\lambda)}$. Thus each linear space $K$-$G_{z(\lambda)}$ has its own basis given by Eq. (13).

(iii) Attach to the curve $z(\lambda)$ in spacetime the following one-parameter family of quantum states,

$$\int_{-\infty}^{\infty} \mathcal{D}^{k\pm}_{\omega'\omega}(z(\lambda))B^{\pm}_{\omega'}(y)d\omega' \in K\text{-}G_{z(\lambda)}.$$

Considered as vectors, these states consistute a set of basis vectors which stay parallel as one moves along the curve.

We shall now determine explicitly the relationship between these sets of parallel basis states.

Suppose our chosen curve connects the two points

$$z(0) = (0,0) \to z(\lambda) = (t_0, x_0) = z.$$

¿From Eqs.(11) and (12) we have

$$\mathcal{D}^{k\pm}_{\omega'\omega}(z(0)) = \delta(\omega - \omega').$$

Consequently, the quantum state

$$B^{+}_{\omega}(y) : \frac{K_{i\omega}(k\xi)}{\pi} e^{-i\omega\tau} \begin{bmatrix} e^{\pi\omega/2} \\ e^{-\pi\omega/2} \end{bmatrix} \in K\text{-}G_{z(0)} \tag{14a}$$

which may be looked at as a vector at $z(0) = (0,0)$, is parallel to

$$\int_{-\infty}^{\infty} \mathcal{D}^{k+}_{\omega'\omega}(z)B^{+}_{\omega'}(y)d\omega' : \int_{-\infty}^{\infty} d\omega' \frac{K_{i\omega'}}{\pi} e^{-i\omega'\tau} \begin{bmatrix} e^{\pi\omega'/2} \\ e^{-\pi\omega'/2} \end{bmatrix} \quad \mathcal{D}^{k+}_{\omega'\omega}(z), \tag{14b}$$

a quantum state at $z(\lambda) = z$.

Thus one has a linear transformation which takes linear combinations of

$$\left\{ \frac{K_{i\omega}(k\xi)}{\pi} e^{-i\omega\tau} \begin{bmatrix} e^{\pi\omega/2} \\ e^{-\pi\omega/2} \end{bmatrix} \quad -\infty < \omega < \infty \right\} \tag{15a}$$

into linear combinations of the same set. In other words, this set of quantum states (elements of the fiber $K$-$G_z$) is invariant under the transformation $\mathcal{D}^{k+}_{\omega'\omega}(z)$. Similarly, the set

$$\{B^{-}_{\omega}(y), -\infty < \omega < \infty\} : \left\{ \frac{K_{i\omega}(k\xi)}{\pi} e^{-i\omega\tau} \begin{bmatrix} e^{-\pi\omega/2} \\ e^{\pi\omega/2} \end{bmatrix} \quad -\infty < \omega < \infty \right\} \tag{15b}$$

is invariant under the transformation $\mathcal{D}^{k-}_{\omega'\omega}(z)$.

Taking linear combinations from both invariant sets, one finds that the general linear combination

$$\int_{-\infty}^{\infty} (B^+_\omega(y)a^+_\omega + B^-_\omega(y)a^-_\omega)d\omega : \psi(z(0)) = \int_{-\infty}^{\infty} \frac{K_{i\omega}}{\pi} e^{-i\omega\tau} \begin{bmatrix} e^{\pi\omega/2} & e^{-\pi\omega/2} \\ e^{-\pi\omega/2} & e^{\pi\omega/2} \end{bmatrix} \begin{bmatrix} a^+_\omega \\ a^-_\omega \end{bmatrix} \tag{16a}$$

at $z(0)$, the initial point of the curve, yields the parallel vector,

$$\psi(z(\lambda)) = \int_{-\infty}^{\infty} d\omega' \frac{K_{i\omega'}}{\pi} e^{-i\omega'\tau} \begin{bmatrix} e^{\pi\omega'/2} & e^{-\pi\omega'/2} \\ e^{-\pi\omega'/2} & e^{\pi\omega'/2} \end{bmatrix} \int_{-\infty}^{\infty} \begin{bmatrix} \mathcal{D}^{k+}_{\omega'\omega}(z) & 0 \\ 0 & \mathcal{D}^{k-}_{\omega'\omega}(z) \end{bmatrix} \begin{bmatrix} a^+_\omega \\ a^-_\omega \end{bmatrix} \tag{16b}$$

the corresponding linear combination at the end of the curve.

Thus relative to the complete set of basis states (the union of Eqs. (15a) and (15b)) the transformation ("parallel propagator")

$$\begin{bmatrix} \mathcal{D}^{k+}_{\omega'\omega}(z) & 0 \\ 0 & \mathcal{D}^{k-}_{\omega'\omega}(z) \end{bmatrix} \equiv d^k_{\omega'\omega}(z) \tag{17}$$

establishes a parallelism between the quantum states at $z(0)$ and those at $z = z(\lambda)$. In other words, this parallel propagator specifies a representation of the sought-after *linear gauge*, i.e. it specifies the orientation of parallel basis states as $z$ varies over spacetime.

An alternative, but non-geometrical way of looking at this transformation may be more familiar: it is a type of propagator which relates Cauchy data on any of the hypersurfaces through $z(0) = (0, 0)$ to the evolved data on any of the hypersurfaces through $z(\lambda) = (t_0, x_0) = z$. From the viewpoint of mathematical analysis, finding this propagator is tantamount to finding all possible solutions to the Klein-Gordon equation. This propagator contains all information about the dynamics of the qunantized Klein-Gordon particle. It is therefore difficult to overstate the importance of this propagator.

## 3. Gauge Transformation

At first sight there is an *embarras'des richesses'* in the ways of representing the parallel propagator. For every set of basis states (in each fiber $K$-$G_z$) there exists a representation of the propagator. Such flexibility might be an expendable luxury because the base manifold is Minkowski spacetime. But one suspects that in the presence of gravitation, for example, such flexibility is precisely what is needed if the parallel propagator be capable of accomodating a faithful quantum mechanical characterization of spacetime.

There are two alternative sets of basis states that play a prominent role: (i) there is the particle-antiparticle ("positive-negative Minkowsi frequency") basis, Eqs. (15), and (ii) there is the Rindler basis, Eq. (7), whose elements are invariant under the symmetry group, Eq. (4).

A particle-antiparticle basis enters because the parallel propagator has two invariant subspaces which are the respective particle and antiparticle states.

This particle-antiparticle basis is however totally useless to a pair of oppositely accelerated observers in Rindler I and Rindler II . There is no way they can determine whether the Klein-Gordon particle is in a particle state or an antiparticle state. Their causal disconnectedness prevents them from doing so. The particle vs. antiparticle property is encoded in the correlation between the measurements they perform individually.

The Rindler basis, or more precisely the one obtained by taking sums and differences,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} K_{i\omega}(k\xi)e^{-i\omega\tau}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} K_{i\omega}(k\xi)e^{-i\omega\tau} : \quad -\infty < \omega < \infty \tag{18}$$

is the natural basis for a pair of oppositely accelerated observers. These basis states span the quantum states for Klein-Gordon subsystem I independently of those for Klein-Gordon subsystem II (as governed by Eq.(5))

The existence of these two bases expresses a typical quantum mechanical duality. Either one measures the particle-antiparticle aspect of the quantum system, or one measures its Rindler I-II aspect. One can't measure both.

The transition from one basis to the other is determined by the $z$-independent gauge transformation, which is given by the transition matrix

$$V = \begin{bmatrix} e^{\pi\omega'/2} & e^{-\pi\omega'/2} \\ e^{-\pi\omega'/2} & e^{\pi\omega'/2} \end{bmatrix} \quad -\infty < \omega < \infty. \tag{19}$$

Thus starting with the particle-antiparticle gauge, Eq. (17), one obtains the Rindler gauge, appropriate for the pair of oppositely accelerating observers,

$$[D_{\omega'\omega}(z)] = [V_{\omega'}][d_{\omega'\omega}(z)][V_{\omega}^{-1}], \tag{20}$$

which also could have been obtained by merely inspecting Eq. (16b).

## ACHRONAL SPIN

An important consequence of assigning a pair of accelerated frame to each event of spacetime is that a Klein-Gordon particle acquires internal quantum states. But, unlike for a spinning particle, they comprise a state space which is infinite dimensional. At each spacetime event $z = (t_0, x_0)$ the states have a form given by Eq. (16b).

These states may be organized into a geometrical picture analogous to a particle with spin. In fact, these states form a representation of the spinor group $SU(1,1)$, more precisely, an infinite product of independent copies of $SU(1,1)$:

$$\begin{aligned} \cdots \otimes SU(1,1) \otimes SU(1,1) \otimes \cdots &= \prod_{\omega=-\infty}^{\infty} \otimes SU(1,1) \\ &\equiv [SU(1,1)]^{\infty}. \end{aligned} \tag{21}$$

Each factor acts independently on the two-component amplitude of a corresponding spectral component of a wave function, e.g. Eq. (16b). In other words, a generic element $T(z) \in [SU(1,1)]^\infty$ is *spectrum preserving* as it maps the fiber $K$-$G$ onto itself [17]. This group element, an achronal spin rotation, is therefore

$$T(z) : T_{\omega'\omega}(z) = \delta(\omega' - \omega) e^{i\overrightarrow{L}\cdot\overrightarrow{n}_{\omega}(z)\alpha_\omega(z)} : K\text{-}G_z \to K\text{-}G_z. \tag{22}$$

Here

$$\begin{aligned} &e^{iL\cdot\overrightarrow{n}_{\omega}(z)\alpha_\omega(z)} \in SU(1,1), \\ &\overrightarrow{L} : \{L_1, L_2, L_3\} = \left\{ \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \end{aligned} \tag{23}$$

are three generators of $SU(1,1)$ rotations characterized by their commutation relations

$$\begin{aligned} [L_1, L_2] &= -iL_3 \\ [L_2, L_3] &= iL_1 \\ [L_3, L_1] &= iL_2, \end{aligned}$$

and

$$\overrightarrow{n}_{\omega}(z) : \{n_{1\omega}(z), n_{2\omega}(z), n_{3\omega}(z)\} \quad -\infty < \omega < \infty$$

which specifies the rotatiion axis while $\alpha_\omega(z)$ is the amount of rotation.

The (local) isomorphism between $SU(1,1)$ and $SO(2,1)$ implies that this rotation takes place in a three dimensional Lorentz space with its indefinite inner product. Consequently,

$$\begin{aligned} \overrightarrow{L} \cdot \overrightarrow{n}_{\omega}(z) &= L_3 n_{3\omega}(z) - L_2 n_{2\omega}(z) - L_1 n_{1\omega}(z) \\ \overrightarrow{n}_{\omega}(z) \cdot \overrightarrow{n}_{\omega}(z) &= [n_{3\omega}(z)]^2 - [n_{2\omega}(z)]^2 - [n_{1\omega}(z)]^2 \\ \overrightarrow{L} \cdot \overrightarrow{L} &= L_3^2 - L_2^2 - L_1^2 \left( = \frac{1}{2}\left(\frac{1}{2} + 1\right)\right). \end{aligned} \tag{24}$$

A rotation may be

$$\begin{aligned}
\text{“timelike”} &: \vec{n}\cdot\vec{n} = +1,\\
\text{“spacelike”} &: \vec{n}\cdot\vec{n} = -1,\\
\text{or “null”} &: \vec{n}\cdot\vec{n} = 0
\end{aligned}$$

(Subscripts and arguments have been suppressed). The corresponding elements of $SU(1,1)$ are

$$\begin{aligned}
\vec{n}\cdot\vec{n} = 1 : e^{i\vec{L}\cdot\vec{n}\alpha} &= \cos\frac{\alpha}{2} + 2i\vec{L}\cdot\vec{n}\sin\frac{\alpha}{2}\\
\vec{n}\cdot\vec{n} = -1 : e^{i\vec{L}\cdot\vec{n}\alpha} &= \cosh\frac{\alpha}{2} + 2i\vec{L}\cdot\vec{n}\sinh\frac{\alpha}{2}\\
\vec{n}\cdot\vec{n} = 0 : e^{i\vec{L}\cdot\vec{n}\alpha} &= 1 + i\vec{L}\cdot\vec{n}\alpha \quad .
\end{aligned} \tag{25}$$

The group $[SU(1,1)]^\infty$ serves the same purpose for the Klein-Gordon system that the group of rotations serves for a spinning particle in a magnetic field: First of all, the group preserves the inner product in the space of quantum states. Second, there is a subgroup of symmetry operations that preserves the evolution of the system, and third there exists a conserved spin operator.

For the particle the symmetry subgroup is the set of rotations around the tilted magnetic field axes. These rotations leave the Hamiltonian invariant. Equivalently, their generator, the angular momentum along the tilted axis, commutes with the Hamiltonian of the spinning particle.

Similarly the parallel propagator $D(z)$, Eq. (20), of the quantized Klein-Gordon particle is invariant under those spectrum preserving symmetry elements $t(z) \in [SU(1,1)]^\infty$ which commute with the propagator,

$$[D(z), t(z)] = 0.$$

By inspecting Eqs. (20) and (17) and comparing them with Eq. (22) one can readily infer the symmetry elements. But first one must decide relative to which representation, “particle-antiparticle” or “Rindler”, shall the answer be given. Our focus on accelerated reference frames demands that the quantum mechanical measurements are to be performed by the pair of observers in Rindler $\mathrm{I}_z$ and $\mathrm{II}_z$. The reasoning surrounding Eq. (18) demands that the appropriate basis be that of Rindler and not the particle-antiparticle basis. Consequently one must represent the parallel propagator as in Eq.(20) ( and not Eq.(17) ) in order to find the symmetry elements t(z) under which it is invariant. By inspection one sees that the following one parameter family of transformations commutes with Eq.(20):

$$\begin{aligned}
t(z) : t_{\omega'\omega}(z) &= \delta(\omega'-\omega)V_\omega e^{iL_3\alpha}V_\omega^{-1}\\
&= \delta(\omega'-\omega)\exp[iV_\omega L_3 V_\omega^{-1}\alpha]\\
&= \delta(\omega'-\omega)[\cos\frac{\alpha}{2} + 2iV_\omega L_3 V_\omega^{-1}\sin\frac{\alpha}{2}].
\end{aligned} \tag{26}$$

Consequently, the conserved achronal spin operator is

$$S : V_\omega L_3 V_\omega^{-1}\delta(\omega'-\omega) = \left[L_3\frac{\cosh\pi\omega}{\sinh\pi\omega} + L_1\frac{1}{\sinh\pi\omega}\right]\delta(\omega'-\omega) \tag{27}$$

$$= [2L_3(N+\frac{1}{2}) + 2L_1\sqrt{N(N+1)}]\delta(\omega'-\omega) \quad -\infty < \omega < \infty.$$

Here

$$N = N(\omega) = \frac{1}{\exp 2\pi\omega - 1} \tag{28a}$$

is Planck's thermal spectral function, and

$$N(N+1) = \frac{1}{\exp 2\pi\omega - 1} + \frac{1}{(exp 2\pi\omega - 1)^2} \tag{28b}$$

is the corresponding mean squared thermal fluctuation spectrum.

The (local) isomorphism of $SU(1,1)$ with $SO(2,1)$ implies a geometrical picture for $t(z)$, Eq.(26): It is a rotation in the direct sum (more precisely, direct integral)

$$\cdots \oplus R^3 \oplus R^3 \oplus \cdots = \sum_{\omega=-\infty}^{\infty} \oplus R^3$$

of three dimensional Lorentz spaces, each one, $R^3$, spanned by the classical values of the operators $L_1$, $L_2$, and $L_3$.

The rotation $t(z)$ is (a) spectrum preserving, i.e. leaves invariant each Lorentz space $R^3$, and (b) has an axis of rotation whose projection onto the $\omega$th subspace $R^3$ is the vector

$$\vec{n}_\omega = \left\{\left[\frac{1}{\exp 2\pi\omega - 1} + \frac{1}{(\exp 2\pi\omega - 1)^2}\right]^{\frac{1}{2}}, 0, \frac{1}{2} + \frac{1}{\exp 2\pi\omega - 1}\right\}. \tag{29}$$

In other words, the vector lies in the $L_1 - L_3$ plane. The projection along the $L_3$ axis is given by the sum of: zeropoint plus Planckian thermal spectral intensity. The tilt of this vector is given by the r.m.s. fluctuation spectral intensity, which is the projection along the $L_1$ axis.

Let us return to the question posed near the beginning of this paper: "Why does Nature choose to characterize pairs of accelerated frames with a Planckian and not with some other spectrum?"

The answer: "The Planckian is Nature's way of expressing the parallel translation symmetry of flat Minkowski spacetime."

This she achieves by the fact that (1) a relativistic wave equation imparts an acceleration induced set of *internal* quantum states to each event of spacetime, and (2) there exists an achronal spin operator, S, Eq.(27), which is conserved under translations in spacetime, and which has the striking property: The diagonal spectrum of $S$ is

$$\frac{1}{2} + \frac{1}{\exp 2\pi\omega - 1}, \tag{30a}$$

namely "zeropoint plus Planckian power", while its off-diagonal spectrum is

$$\left[\frac{1}{\exp 2\pi\omega - 1} + \frac{1}{(\exp 2\pi\omega - 1)^2}\right]^{\frac{1}{2}}, \tag{30b}$$

that of the concomitant "r.m.s. fluctuations".

## CURVATURE AND ACHRONAL GAUGE POTENTIAL

Let us explore why the achronal symmetry axis is oriented along the direction given by Eqs.(29).

Consider the parallel propagator for an infinitesimal spacetime displacement. An easy calculation based on Eqs.(17), (11), and (12) yields the connection one-form

$$\Omega(z) = \frac{ik}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\{\delta(\omega - \omega' + i)dU + \delta(\omega - \omega' - i)dV\}. \tag{31}$$

This infinitesimal law for parallel transport yields the covariant derivatives of all the particle-antiparticle basis states, Eq.(13). Its curvature vanishes [18]:

$$d\Omega + \Omega \wedge \Omega = 0. \tag{32}$$

One may also view the connection differential as an achronal gauge potential

$$\Omega(z) \equiv A_U(z)dU + A_V(z)dV \tag{33}$$

for an $[SU(1,1)]^\infty$ Yang-Mills theory [19,20]. The corresponding gauge field is

$$F_{UV}dU \wedge dV = d\Omega + \Omega \wedge \Omega, \tag{34}$$

which according to Eq.(32) vanishes everywhere.

Thus Minkowski spacetime is characterized by a vanishing achronal gauge field, but one with a non-vanishing potential. This potential is rotationally symmetric around an achronal rotation axis whose direction is given by Planck's thermal spectrum together with its r.m.s. fluctuation spectrum.

The quantum mechanical reasoning summarized by Eqs.(31)-(33), leads to the heuristic conclusion

$$\text{gravitation} = [SU(1,1)]^{\infty} \text{ gauge geometry}$$

Its basis, we reiterate, is translation invariance: every pair of achronal accelerated Rindler frames is physically and hence geometrically indistinguishable from any other pair. Purely wave mechanical considerations, in our case those for a single charge governed by the Klein-Gordon equation, then lead directly to this conclusion.

## QUANTUM MECHANICAL PICTURE: A HEURISTIC SCENARIO

Classical general relativity can be characterized by by the manner in which matter and spacetime are related: "Spacetime tells matter how to move; matter tells spacetime how to curve" [21].The first relation can be illustrated by the equations for geodesic motion and the equations for geodesic deviation. The second one is captured by Einstein's equations for the moment of curvature as dictated by energy and momentum [22].

Quantum mechanics however forces a very different picture upon us. In this picture achronal spin replaces world lines as the way to characterize the motion of particles. The equations of geodesic motion and of gedesic deviation are superceded by a Heisenberg type equation of evolution that governs the orientation of the precessional motion of the infinite dimensional achronal $[SU(1,1)]^{\infty}$ spin vector.

The existence of the achronal gauge potential demands that the evolution of the achronal spin be controlled in a way analogous to how the Yang-Mills gauge potential controls the evolution of isotopic spin. Suppose one indeed characterizes spacetime in terms of its gauge potential. Then its quantum mechanical control over matter may be expressed by saying "Spacetime tells achronal spin how to precess."

If one accepts this quantum mechanical picture then one must consider, of course, its converse. It consists of the answer to the question: "How does achronal spin determine the achronal gauge field?" Even though this field is based on the gauge group $[SU(1,1)]^{\infty}$, while that of Yang and Mills is based on only $SU(2)$, there are numerous reasons to believe that the latter sheds considerable light on the former.

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[6] U.H. Gerlach, Phys.Rev. D 38, 514 (1988). The reader is invited to look in particular at Fig. 2. We take this opportunity to correct in that paper three non-obvious misprints which, however, have no effect on its results. (i) In both Eqs.(4.1c) and (4.1d) $\sinh\tau$ and $\cosh\tau$ should be interchanged. (ii) Eq.(4.3b) should read:

$$B^{\pm}_{\omega}(kU,kV)\,|_{P} \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm ik\xi\cosh(\theta-\tau)}\,e^{-i\omega\theta}\,d\theta \;=\; \pm\frac{i}{2}e^{\mp\pi\omega/2}H^{\frac{1}{2}}_{i\omega}(k\xi)e^{-i\omega\tau}\;.$$

(iii) The correct version of Eq.(5.18) as well as (5.22) is:

$$\sum(\cdots)\;\overset{bg<<1}{\longrightarrow}\;L^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{bk}^{\infty}\frac{d^2k}{(2\pi)^2}\frac{d\omega}{\pi}\times log\frac{\omega+\sqrt{\omega^2-b^2k^2}}{bk}(\cdots)$$

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[17] The reader will have noticed that with the introduction of the symmetry group $[SU(1,1)]^\infty$ we have enlarged the structure group of the bundle from G given by Eq.(4), to a larger group which includes $G$ and $[SU(1,1)]^\infty$ as subgroups. Such an enlargement can have interesting consequences. See Section 4 in N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton N.J., 1951).

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