Global rigidity of some partially hyperbolic $\mathbb{Z}^k$ actions

f - PH diffeo $: M \to M$

$$TM = \underset{\text{stable}}{E^s} \oplus \underset{\text{center}}{E^c} \oplus \underset{\text{unstable}}{E^u}$$

$\alpha : \mathbb{Z}^k \xrightarrow{\text{homom}} \mathrm{Diff}^\infty(M)$ is a PH action if

for some $a \in \mathbb{Z}^k$, $\alpha(a)$ is PH

Commutativity $\Longrightarrow$ $b \in \mathbb{Z}^k$, $\alpha(b)$ preserves $E^s \oplus E^u$

$E^H = E^s \oplus E^u$ $\longleftarrow$ HORIZONTAL α-INVARIANT DISTRIBUTION

Anosov actions, global rigidity:

- α is Anosov if E^c is trivial

- Global rigidity conjecture for Anosov actions:

KATOK-SPATZIER CONJ. : α - Anosov, and does not factor to a $\mathbb{Z}$-action $\Longrightarrow$ α is $\boxed{\text{ESSENTIALLY ALGEBRAIC}}$

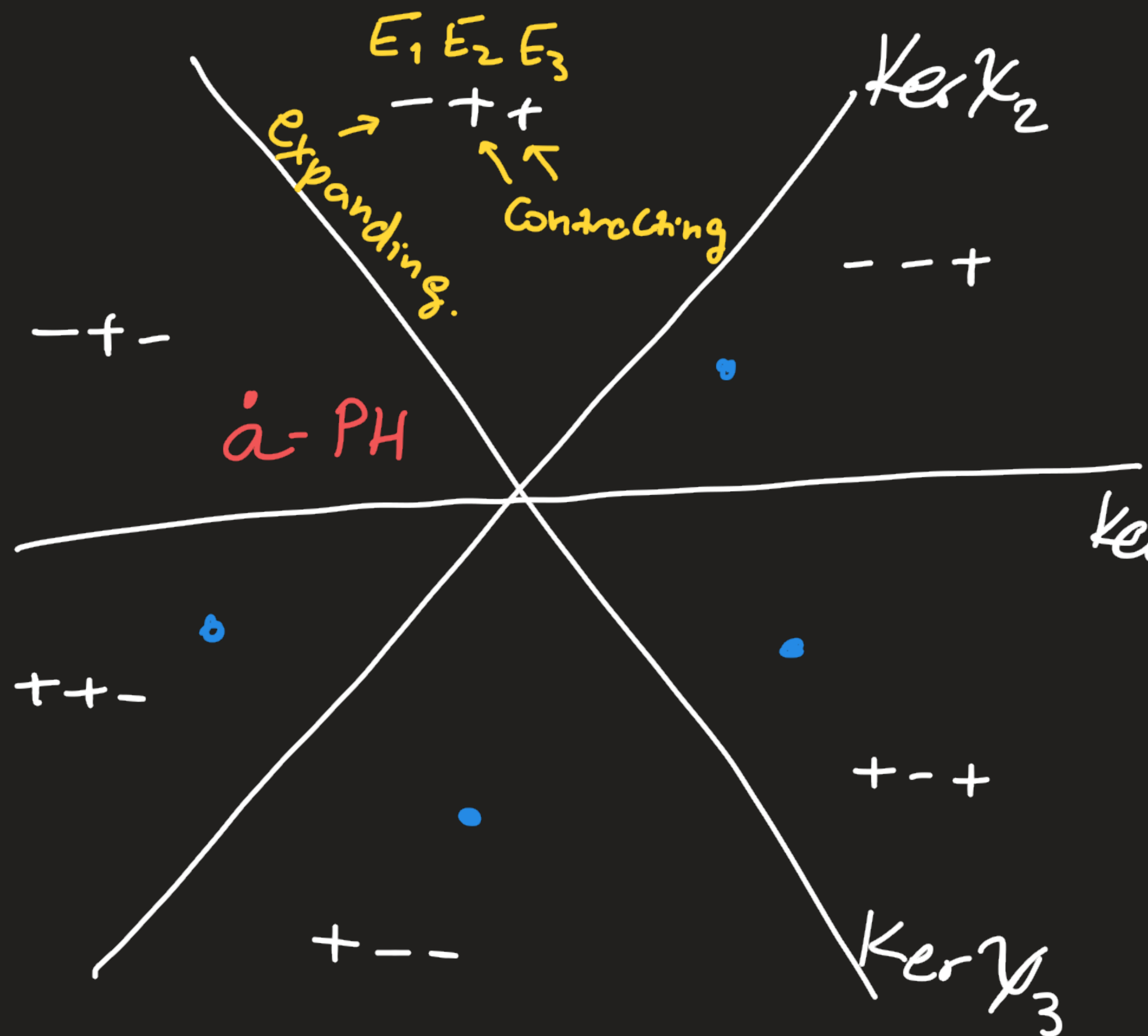
(i.e. α is C^∞ conjugate, up to a finite index subgroup in $\mathbb{Z}^k$ and up to a finite cover, to an action by affine maps on a nilmanifold)

- Solved on nilmanifolds (Fisher-Kalinin-Spatzier, FRH-Z. Wang)
- On general manifolds, assuming many Anosov elements + ... conditions .... (Kalinin-Sadovskaya, Kalinin-Spatzier, D.-Xu, Vinhage-Spatzier)

(Chyperbolic) Weyl chamber picture for α - Anosov: (PH)

PH $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(M)$, preserving fully supported measure μ

Oseledets theorem for actions $\Rightarrow$ there exist (Lyapunov) functionals $\chi_1, \dots, \chi_n : \mathbb{R}^k \to \mathbb{R}$ and $D\alpha|_{E^H}$ invariant splitting $TM|_{E^H} = E^1 \oplus \dots \oplus E^n$ such that $\chi_i(a)$ is Lyapunov exponent of $\alpha(a)$ in direction E^i.



E^i are not necessarily integrable.

But: $E_\chi := \bigoplus_{\substack{\chi_i = c\chi \\ c > 0}} E^i$ is integrable to the h-Coarse Lyapunov foliation.

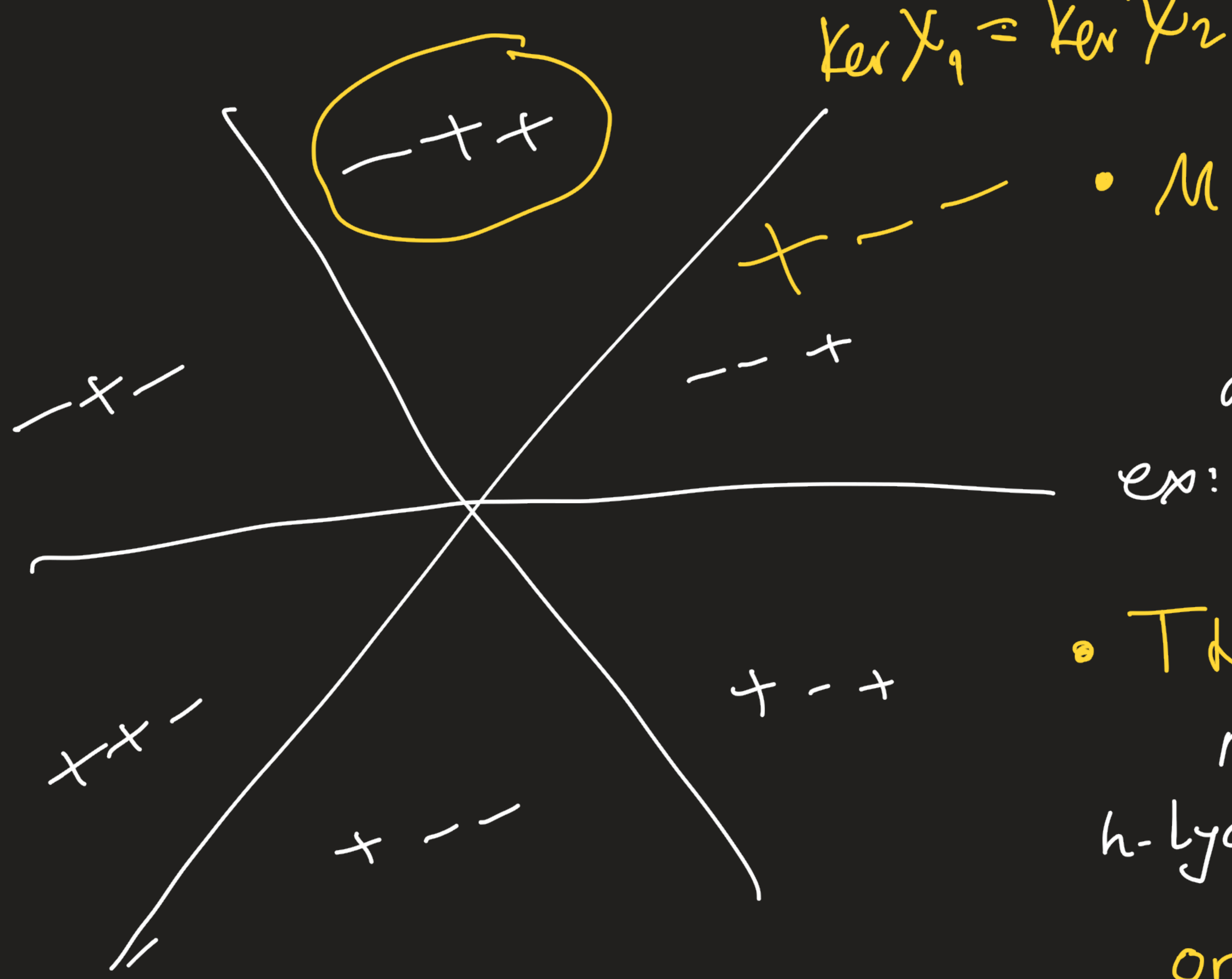
h- Coarse Lyapunov distribution

$\boxed{\text{One Anosov(PH) element in each h-Weyl chamber}} \Longrightarrow TM|_{E^H} = E_1 \oplus \dots \oplus E_r$ h-Coarse Lyapunovs

$$\begin{matrix} E_1 & \dots & E_r \\ \downarrow & & \downarrow \\ W_1 & \dots & W_r \end{matrix}$$

and $\boxed{W_i = \bigcap_{\chi_i(a) < 0} W^s_a}$ — Hölder foliation with C^∞ leaves.

Some "higher-rank" conditions:



- Maximality:

  any combination of signs appears in h-Weyl chambers

  ex: $\mathbb{Z}^{k-1}$ linear action on $\mathbb{T}^k$

- TNS: (Totally non-symplectic)

  no negatively proportional h-Lyapunov functionals

  or

  any two E_i, E_j are in E_a^s for some $a \in \mathbb{Z}^k$

Q: Can we expect global rigidity for PH $\mathbb{Z}^k$ actions, and in what sense?

Certainly not in general – without assumptions on the center dynamics

ex1:

$\gamma : \mathbb{Z}^2 \to Aut(\mathbb{T}^4)$-Anosov

$\delta : \mathbb{Z}^2 \to$ translations of $\mathbb{T}^2$

$\alpha = \gamma \times \delta : \mathbb{T}^6 \circlearrowleft$

has all elements PH w.r.t. compact center foliation by $\mathbb{T}^2$

ex2

$\alpha : \mathbb{Z}^2 \to Aut(\mathbb{T}^6)$ - ergodic with $\dim E^c = 2$

Then α also has all elements PH w.r.t. 2-dim W^c, but W^c is not a compact foliation, and α is not a product action.

VERY DIFFERENT!

More examples:

<u>ex 3</u>: $\alpha : N/\Gamma$ nilmanifold, 2-step $\circlearrowleft$ PH : E^c = center of Lie(N)

$$\begin{array}{ccl} \alpha : & N/\Gamma & \\ & \downarrow & \\ \bar{\alpha} : & \mathbb{T}^d & \text{- abelianization} \end{array}$$

$\circlearrowleft$ Anosov

Accessible

N/Γ is a non-trivial fiber bundle over $\mathbb{T}^d$

fibers = leaves of W^c

<u>ex 4</u> Isometric extensions of Anosov

$$\begin{array}{ccl} \alpha : & \mathbb{T}^d \times \mathbb{T}^k & \\ & \downarrow & \\ \bar{\alpha} : & \mathbb{T}^d & \circlearrowleft \text{ Anosov} \end{array}$$

$\alpha(a)(x, y) = (\bar{\alpha}(a)(x), y + \beta(a, x))$

where β is a cocycle

For global rigidity of PH actions: need to choose a "model" to aim at — here we aim at a product (ex 1)

Fibered PH diffeos

$f: M \circlearrowleft$ PH, $TM = E^s \oplus E^c \oplus E^u$ → W^c - center foliation

Assume: E^c is integrable, has all leaves compact with 0-holonomy

$\Downarrow$

$\boxed{M/W^c \text{ is a TOPOLOGICAL MANIFOLD}}$

Dynamics of $\bar{f}$? Pugh, Gogolev, Carrasco, ..

Bohnet-Bonatti:

$$\begin{array}{ccc} f: M & \longrightarrow & M \\ \downarrow \pi & & \downarrow \pi \\ \bar{f}: M/W^c & \longrightarrow & M/W^c \end{array}$$

$\bar{f}$ is expansive, shadowing property.

$\boxed{\text{PROP}}$ (Hiraide, Doucette)

M - topological fiber bundle over nilmanifold N, with compact fiber F, $f: M \circlearrowleft$ PH diffeo with W^c-leaves homeo to F

$\Longrightarrow$ $\bar{f}$ is Hölder conjugate to algebraic Anosov on N

Fibered PH $\mathbb{Z}^k$-actions : $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty M$ PH

$$\begin{tikzcd} \alpha : M \arrow[r] \arrow[d, "\pi"] & M \arrow[d, "\pi"] \\ \bar{\alpha} : M/W^c \arrow[r] & M/W^c \end{tikzcd}$$

Assumptions:

a) α preserves ergodic measure of full support

b) α contains a PH element in each h-Weyl chamber and each PH element is fibered PH diffeo with center W^c.

$\Rightarrow$ $\begin{cases} \bar{\alpha} \text{ is topological Anosov} \\ \text{Coarse Lyapunov distributions of } \alpha \text{ integrate to Hölder foliations with smooth leaves.} \end{cases}$

Fibered PH actions over a nilmanifold

c) M is a fiber bundle over a nilmanifold N, with compact fiber F, and PH elements of α have W^c leaves homeo to F.

$$\begin{tikzcd} \alpha : M \arrow[r] \arrow[d, "\pi"] & M \arrow[d, "\pi"] \\ \bar{\alpha} : N \arrow[r] & N \end{tikzcd}$$

Nilmanifold

π – topological

$\bar{\alpha}$ – Hölder conjugate to algebraic Anosov on N

"Higher-rank" condition on α :

d) α has no negatively proportional h-Lyapunov functionals (TNS)

"

"totally non-symplectic"

$\Updownarrow$

Any E_i, E_j are contained in some E_a^s, $a \in \Sigma^k$

TOPOLOGICAL RIGIDITY

Under assumptions a), b), c), d) for α, we have:

Ergodic inv. measure of full support

each h-Weyl chamber has a fibered PH element w.r.t W^c

$M \xrightarrow{F} N$

W^c leaves homeo to F

TNS

$\boxed{\text{Thm}}$ (D-Wilkinson-Xu)

- if $\dim F = 1 \Rightarrow \alpha$ is essentially Hölder conjugate to a product action:

(Anosov algebraic on N) $\times$ (translations on S^1)

- if - $\dim F = 2$
  - α is center bunched
  - $\exists$ α-invariant 2-form on M, which is non-degenerate on E^c

$\Rightarrow$ then α is essentially Hölder conjugate to a product over Anosov:

(Algebraic Anosov) $\times$ (action on F)

hyperbolicity of α dominates nonconformality of $D\alpha|_{E^c}$

May be removable assumption.

Improving Hölder conjugacy to smooth :

$\boxed{\text{Prop}}$ [D. - Wilkinson - Xu]

α-conservative, Hölder conjugate to a product $\alpha_1 \times \alpha_2$,

should be $\mathbb{N}$

where :
- $\alpha_1 : \mathbb{Z}^k \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ Anosov higher rank
- $\alpha_2 : \mathbb{Z}^k \to \mathrm{Diff}^\infty_\nu(S)$, S - compact smooth manifold, ν - fully supported ergodic measure
- α_2 has subexponential growth on S

Then if α has a PH element in each Weyl chamber, the Hölder conjugacy is uniformly smooth horizontally (along $\mathbb{T}^d$ coordinate).

$\Rightarrow$ $\boxed{W^c \text{ of } \alpha \text{ is a } C^\infty \text{ foliation.}}$

SMOOTH RIGIDITY

general N - in progress...

$\boxed{\text{Thm}}$ (D-Wilkinson-Xu) Let $N = \mathbb{T}^d$.

In addition to a)-d), assume α is conservative and has subexponential growth in the center direction. Then:

- if $\dim F = 1 \Rightarrow \alpha$ is essentially algebraic.

  (↳ ess. C^∞ conjugate to algebraic.)

- if $\dim F = 2$ and $\exists$ α-invariant 2-form on M non-degenerate on E^c $\Bigg\} \Rightarrow \alpha$ is essentially a product over an algebraic Anosov action on $\mathbb{T}^d$.

  (ess. C^∞ conjugate to a product.)

Some remarks and consequences

(*) By-product of our method is a local superrigidity result

$\boxed{\text{Thm}}$ (D-Wilkinson-Xu) Let $r>0$ and let α be a C^1-small perturbation of $\underbrace{(\overbrace{\text{affine Anosov on } N}^{\mathcal{I}}) \times (\mathrm{Id}_S)}$, where S is any compact manifold. Then if $\bar{\mathcal{I}}$ is TNS

$\Rightarrow$ α is a C^r product over $\bar{\mathcal{I}}$, i.e.

$$\alpha = \bar{\mathcal{I}} \times (\text{action on } S)$$

⊛ W^c cannot be pathological, if it is a center for such PH $\mathcal{L}$.

Shub-Wilkinson, Katok, examples: perturbations of $A \times Id$ on $\mathbb{T}^{n+1}$, A - Anosov on $\mathbb{T}^n$, such that W^c for these perturbations is pathological, i.e. disintegration of volume along center leaves is not abs. continuous. Moreover, there is a full volume set which intersects each center leaf in finite (bounded) number of points.

(*) A PH action on a $2n+1$-dim. Heisenberg manifold whose center coincides with the center foliation of the nilmanifold, cannot be TNS on the base.

conservative, ergodic, one PH element in each chamber

<u>Conjecture</u>: Let α be a conservative fibered PH action on a 2-step nilmanifold N such that W^c coincides with the center of N, α has subexponential growth in the center and one PH in each Weyl chamber.
If α is higher-rank $\Longrightarrow$ α is essentially algebraic.

With Wilkinson, Xu and FRH: ok if center has dim $= 1$.

(*) What about fibered PH actions on **GENERAL MANIFOLDS**?

$\boxed{\text{Thm}}$ (D-Wilkinson-Xu) α - conservative fibered PH, one PH element in each Weyl chamber, and maximal Cartan $\dim E_i = 1$ (or maximal and UQC on coarse Lyapunov distributions)

$\Longrightarrow$ Same result as in the "smooth" theorem, in case of 1 or 2-dimensional center.

Rem: - no need to assume the manifold fibers over a nilmanifold.

- UQC = "uniformly quasi-conformal"

compact Lie group

(*) Principal G-bundle extensions of Anosov TNS action on N-nilmanifold, have been classified by Nitica - Török in 2003.

The general strategy is similar to what we do for general fiber bundles here.

About proofs :

(I) $\boxed{\text{TNS implies } E^H \text{ is integrable to } W^H\text{-horizontal foliation}}$

- the proof is completely combinatorial (topological)
- Assumptions needed are :
  - α preserves ergodic μ of full support
  - PH element in each chamber
  - TNS

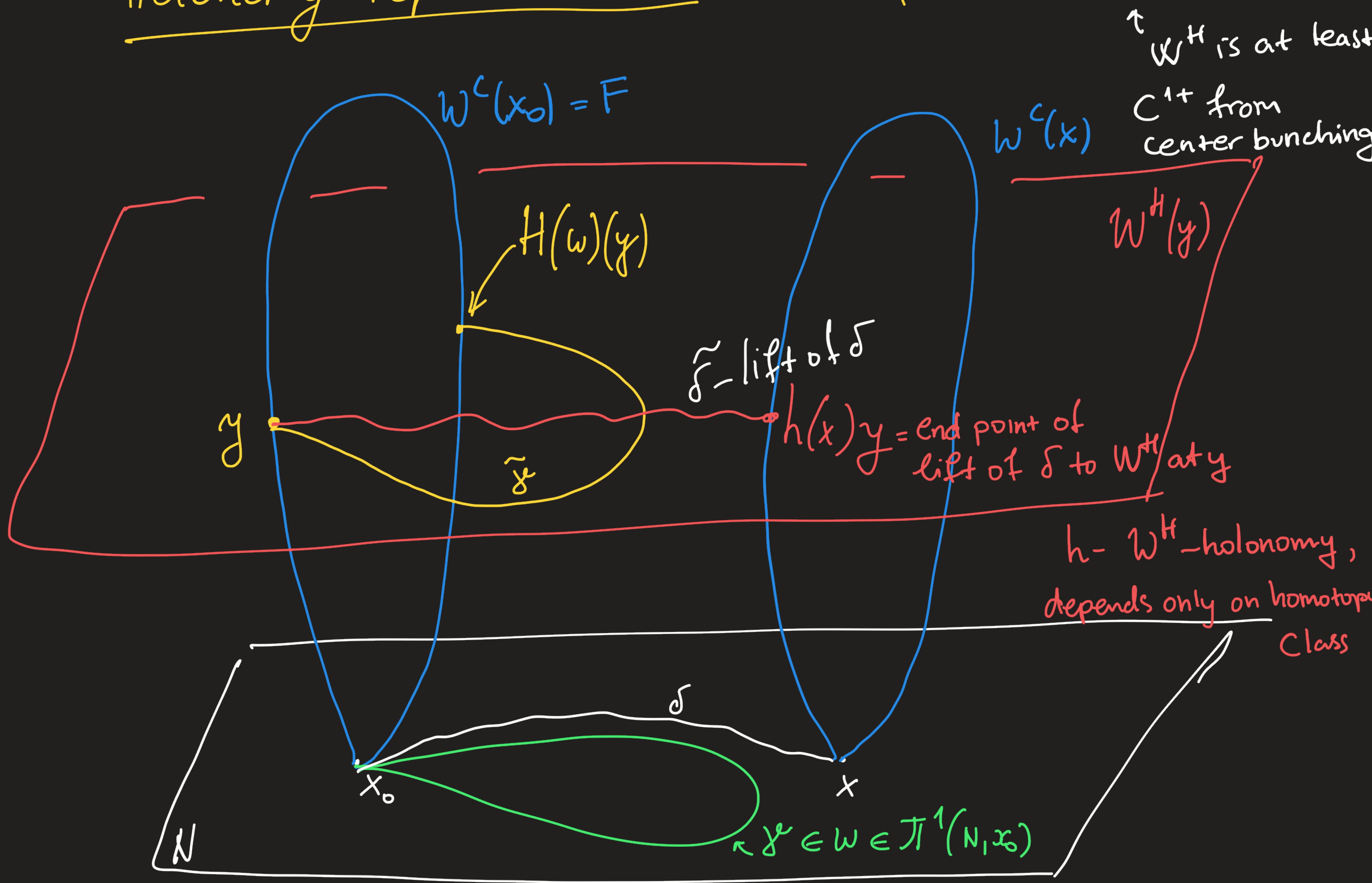
$\Rightarrow$ M is a topological $\boxed{\underline{\underline{\text{FLAT}}}}$ bundle over N, with fiber F

⊛ Flat bundles are characterised by their holonomy representations $H : \pi_1(N) \longrightarrow \mathrm{Homeo}(F)$

⊛ center bunching $\Rightarrow$ W^H is at least C^{1+}
$\Rightarrow$ $H : \pi_1(N) \longrightarrow \mathrm{Diff}(F)$

⊛ regularity of W^H $\longleftrightarrow$ regularity of $\mathrm{Im}(H)$.

Holonomy representation : $H : \pi_1(M, x_0) \to \mathrm{Diff}(F)$

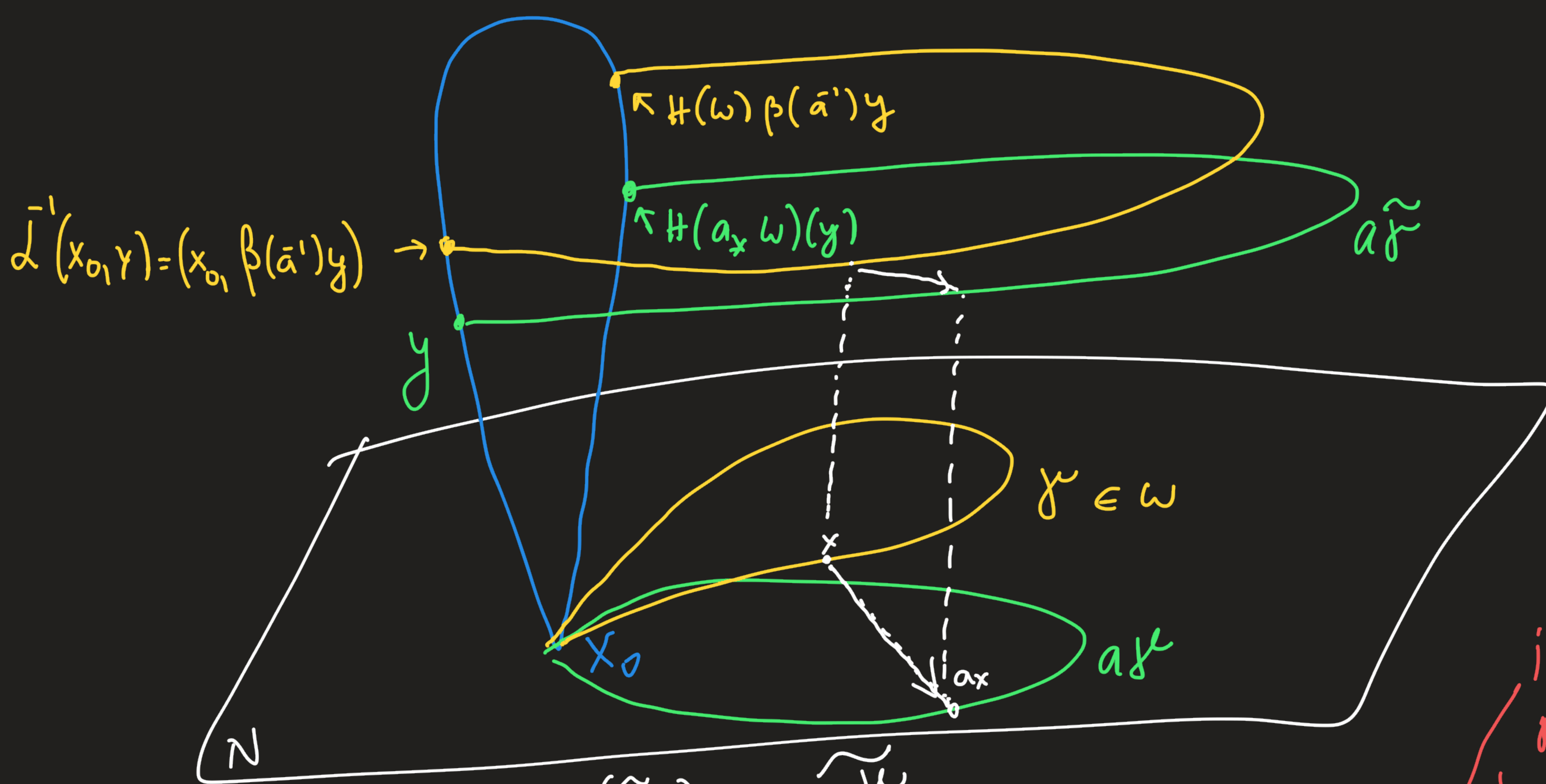


II Let $H : \pi(N, x_0) \to \mathrm{Diff}(F_0)$, x_0 = fixed point of $\bar{\alpha}$

↑ Center leaf through x_0

β - the action induced by α on F_0.

α denotes here $\alpha(a)$.



$\alpha(\tilde{\gamma}) = \widetilde{a\gamma}$

end-point of $\alpha(\tilde{\gamma})$ = end point of $\widetilde{a\gamma}$

$$\boxed{\beta(a)H(\omega)\beta(a^{-1})y = H(a_*\omega)y}$$

intertwining relation between the holonomy rep. and the action

$\Rightarrow$ Intertwining of the holonomy representation and the action is given by:

$$\boxed{\beta(a)\,H(w)\,\beta(a^{-1}) = H(a_* w)}$$

$\Rightarrow$ gives a solvable (Nilpotent-by-cyclic) group Γ in Diff(F_0)

Key point: use the relation in Γ to show that

$\mathrm{Im}(H)$ in Diff(F_0) is finite.

$\Rightarrow$ W^H is a global section of the base $\Rightarrow$ W^H leaves are compact

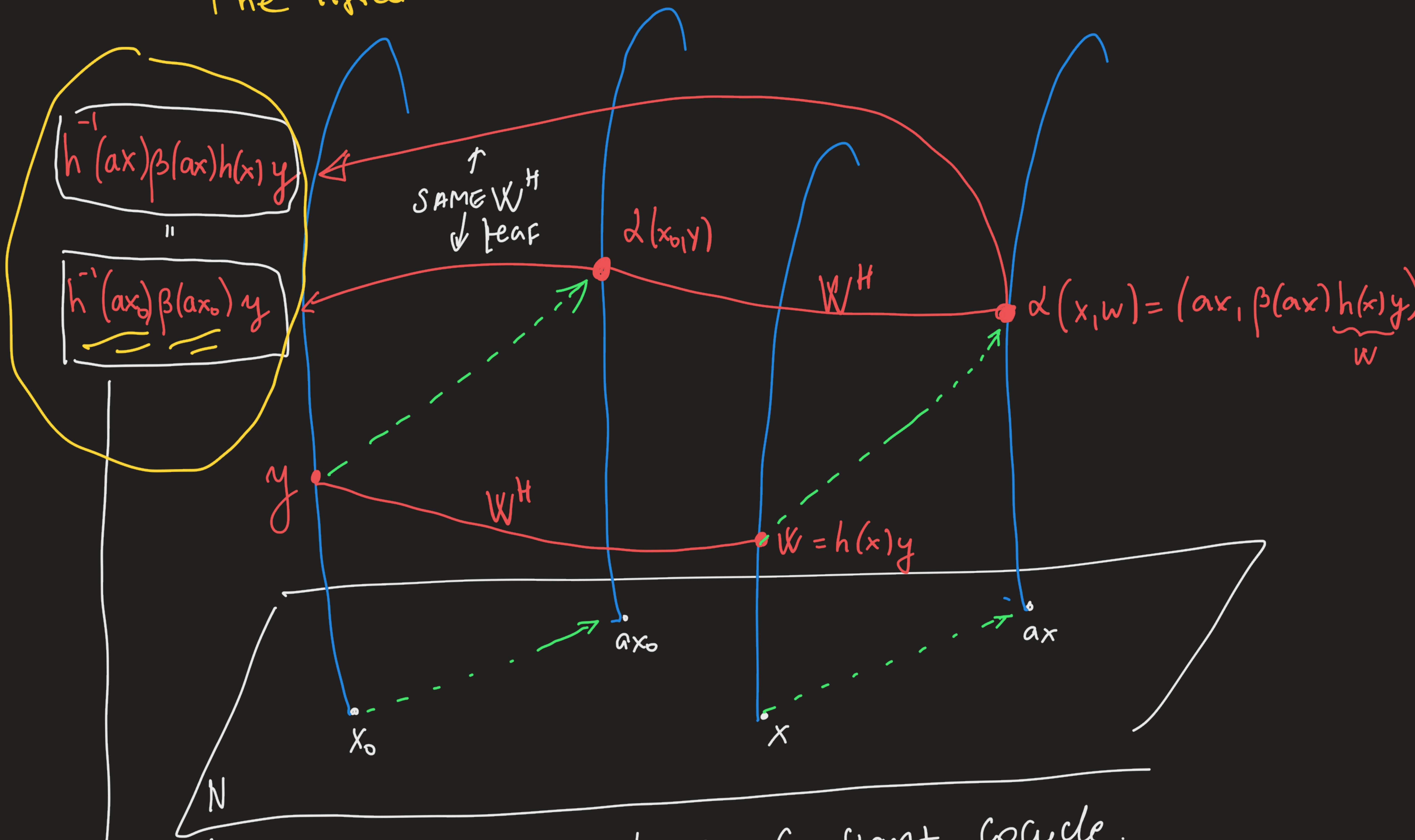
(on a finite cover)

$\Rightarrow$ α lifts to a manifold which is a product.

$\rightarrow$ $\dim F = 1$ uses rotation # : $\begin{cases} \mathrm{rot}\,\#\,(H(\langle a_* w \cdot w^{-1}\rangle)) = 0 \\ \langle a_* w \cdot w^{-1}\rangle \text{ is finite index in } \pi_1(N). \end{cases}$

$\rightarrow$ $\dim F = 2$ uses results of Franks-Handel on subgroups $\Gamma < \mathrm{Diff}(S)$ s.t. $[\Gamma, \Gamma]$ consists of distortion elements, and their study of 0-entropy actions on surfaces.

The lifted action is a product: consider horizontal holonomy



$\Rightarrow$ β is cohomologous to a constant cocycle.

III Obtaining smooth conjugacy :

- "Leafwise" version of Fisher-Kalinin-Spatzier strategy of improving Hölder conjugacy to smooth for an Anosov action.
- This uses exponential mixing of the base action. We show that it lifts to uniform "leafwise" exponential mixing for Hölder observables
- this improves regularity horizontally.
- Restricted to horizontal leaves, the conjugacy induces diffeomorphisms (this uses subexponential growth)
- $\Rightarrow W^c$ is a smooth foliation.