On Decay of Correlations for Parabolic Flows

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Dynamics on your screen

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Parabolic Flows

A parabolic flow on a manifold M is a flow whose nearby orbits diverge at polynomial rate with time. Such flows have zero entropy, with no hyperbolicity (see Katok-Hasselblatt's Survey in Handbook of Dyn. Sys., Chap. 8). Non-generic, but generic under constraints (in low dimension: flows on surfaces, billiards in polygons, ...) *Perhaps chaos is not so frequent* (E. Ghys).

Examples:

unipotent homogenous flows and their time-changes, (non-toral) nilflows and their time-changes, translation flows on surfaces and their time-changes, in particular the so-called Arnol'd flows and Kochergin flows

General Question: What are the mixing properties of such flows? How fast is the decay of correlations for smooth functions? What are the spectral properties?

Time-Changes of Unipotent Flows

For unipotent flows on compact quotients of semi-simple groups mixing with polynomial decay of correlations under a spectral gap condition holds. In fact, multiple mixing with polynomial decay of multiple correlations holds (M. Björklund, M. Einsiedler and A. Gorodnik, JEMS 2020) For $SL(2,\mathbb{R})$: unipotent flows (=horocycle flows) are mixing with polynomial decay rate of correlations (M. Ratner, ETDS 1987)

Smooth time-changes of unipotent flows.

For $SL(2,\mathbb{R})$: mixing (Marcus, Ann. of Math. 1977); with polynomial decay of correlations (F.-Ulcigrai, JMD 2012); with countable Lebesgue spectrum (F.-Ulcigrai, JMD 2012; Fayad, F., Kanigowski, 2020) Recent: polynomial 3-mixing (of all orders for a subclass) (A. Kanigowski and D. Ravotti, 2020)

Local estimates on spectral measures of smooth functions can be derived from L^2 bounds on twisted ergodic integrals: for smooth time-changes ϕ_t of horocycle flows, for all coboundaries f with smooth transfer function $u \in W^r(M)$ $(r > 7)$:

$$\| \int_0^T e^{2\pi\imath\lambda t} f \circ \phi_t dt \|_0 \leq C \|u\|_r T^{1/2}(1 + \log^+ T)$$

Theorem

(F., Ulcigrai, JMD 2012) For smooth time-changes of the classical horocycle flow, for all $\lambda > 0$, for all functions $f \in W^r(M)$ $(r > 7)$,

$$\mu_f(\lambda - \delta, \lambda + \delta) \leq C_r \|f\|_r \frac{\delta |\log \delta|}{\lambda^2} .$$

Question

Is the density of spectral measures of smooth functions locally bounded (away from the origin) ?

General semi-simple groups: no results on the rate of mixing; Lebesgue/absolutely continuous spectrum (L. Simonelli, ETDS 2018) for time-changes of uniquely ergodic/ergodic unipotents (the latter under a condition on the time change function).

The difference is related to the fact that unipotents of $SL(2,\mathbb{R})$ are horospherical, hence they are renormalized by the geodesic flow.

Question

Are smooth time-changes of general (uniquely) ergodic unipotent flows polynomially mixing? Do they have countable Lebesgue spectrum? Are they polynomially mixing of all orders ?

Question

Is the horocycle flow of a negatively curved metric on a compact surface polynomially uniquely ergodic, mixing? Does it have countable Lebesgue spectrum?

Time-Changes of Nilflows

Nilflows are never weak-mixing since they always have a toral (equicontinuous) factor. However, they are *relatively mixing* with polynomial speed.

Theorem

(A. Avila, G. F., D. Ravotti, C. Ulcigrai) For every ergodic nilflow there exists a dense set of smooth time-changes which are either measurably trivial, hence isomorphic to the nilflow, or mixing.

Generalizes Avila, F., Ulcigrai (JDG, 2011), Ravotti (ETDS, 2019).

Theorem

(G. F. and A. Kanigowski) For Diophantine Heisenberg nilflows, there exists a residual set of time-changes which are either trivial or polynomially mixing (for the bounded type case) / stretched polynomially mixing (for a full measure Diophantine case).

Stretched polynomial: $(1+|t|)^{-1/(1+\log^{\delta}(1+|t|))}$ (for any $\delta > 1/2$).

Question

Does there exists a residual set of time changes of (Heisenberg) nilflows such that, for all Diophantine nilflows, mixing (non-trivial) time changes have polynomial decay of (multiple) correlations ? Are mixing time-changes (polynomially) mixing of higher order?

Question

What is the largest class of smooth time changes of (Heisenberg) nilflows such that there is a dichotomy between trivial time changes and (weakly) mixing time-changes ? How large is the set of mixing time-changes? Is it dense or generic ?

Question

What is the spectral type of mixing time-changes of nilflows?

Estimates on twisted integrals (local estimates on spectral measures) can be derived by Venkatesh's argument from bounds on correlation decay and from effective unique ergodicity.

Time-Changes of Translation Flows

Translation Flows are generically uniquely ergodic (Masur, 1982; Veech 1982; Kerckhoff, Masur and Smillie, 1986) with polynomial speed (Zorich, 1994-1996; Kontsevich and Zorich, 1997; F., 2002; Athreya and F., 2008)

They are generically weakly mixing (Avila, F. 03), with polynomial speed (Bufetov and Solomyak, 2019; F., 2019) (polynomial bounds on twisted ergodic integrals, polynomial local estimates on spectral measures, polynomial decay of Cesaro averages of correlations).

Non-singular time-changes of translations flows are never mixing (A. Katok, 1980).

Singular time-changes: Arnol'd flows (asymmetric, symmetric), Kochergin flows. Asymmetric Arnol'd flows are typically mixing with logarithmic speed (Khanin and Sinai, 1992; Ulcigrai, 2007; Ravotti, 2017)

Symmetric Arnol'd flows are typically weak mixing (Ulcigrai, 2009), typically not mixing (Scheglov, 2009; Ulcigrai, 2011), exceptionally mixing (Chaika and Wright, 2019)

Kochergin flows are mixing (Kochergin, 1975). Polynomial decay of correlations for special flows related to toral flows with one (mild) polynomial singularity (Fayad, 2001)

Theorem

(B. Fayad, G. F., A. Kanigowski, 2020) A full measure set of Diophantine smooth/analytic Kochergin flows on the torus (with a single degenerate rest point) have countable Lebesgue spectrum.

Remark: The property of countable Lebesgue spectrum is historically associated with the K-property (it follows easily from the definition of K flows by Von Neumann criterion). Known to hold for classical horocycle flows since Parasyuk's 1953 paper.

Questions

Are decay of correlations for all mixing Kochergin flows polynomial? Do spectral measures of smooth functions satisfy polynomial local estimates? Are there mixing Kochergin or Arnol'd flows with purely singular spectrum?

For Arnol'd flows (non-mixing) on genus 2 surfaces, the spectrum is purely singular (Chaika, Fraczek, Kanigowski, Ulcigrai, 2019)

Mixing for time-changes (from F.-Ulcigrai, F.-Kanigowski)

A time-change ϕ_t^V of a flow ϕ_t^U is the flow on M defined as

$$\phi_t^V(x) := \phi_{\tau(x,t)}^U(x) \quad \text{for all } (x,t) \in M \times \mathbb{R}\,.$$

The function $\tau : M \times \mathbb{R} \to \mathbb{R}$ is a cocycle over ϕ_t^V, that is

$$\tau(x, t+s) = \tau(\phi_t^V(x), s) + \tau(x,t)\,.$$

In terms of generators $V = \alpha U$ with $\alpha(x) = \frac{\partial \tau}{\partial t}(x, 0)$. Hence

A measure μ on M is ϕ_t^V-invariant if and only if $\alpha\mu$ is U-invariant

For instance, if U is a homogeneous flow $\mu = \alpha^{-1}\mathrm{vol}$.

Mixing by shear. Let ϕ_t^V be a flow on (M,μ) (generated by the vector field) V and let ϕ_s^X be a flow on $(M,\alpha\mu)$. We have

$$\langle f\circ\phi_t^V, g\rangle_\mu = \langle f\circ\phi_t^V, \frac{g}{\alpha}\rangle_{\alpha\mu} = \frac{1}{S}\int_0^S \langle f\circ\phi_t^V\circ\phi_s^X, (\frac{g}{\alpha})\circ\phi_s^X\rangle_{\alpha\mu} ds$$

by integration by parts

$$\begin{aligned}\langle f\circ\phi_t^V, g\rangle_\mu &= \langle \int_0^S f\circ\phi_t^V\circ\phi_s^X ds, (\frac{g}{\alpha})\circ\phi_S^X\rangle_{\alpha\mu} \\ &\quad - \int_0^S \langle \int_0^s f\circ\phi_t^V\circ\phi_r^X dr, X(\frac{g}{\alpha})\circ\phi_s^X\rangle_{\alpha\mu}\end{aligned}$$

It is enough to estimate, for $s\in[0,S]$, the (L^2) size of

$$\int_0^s f\circ\phi_t^V\circ\phi_r^X dr\,.$$

Let us consider the (push-forward) curves

$$\gamma_{x,t}(s) = \phi_t^V \circ \phi_s^X(x) = \phi_t^V(\gamma_x)(s)\,.$$

Now (assuming that $\gamma'_{x,t}(s) = \hat{V}\left(\frac{d\gamma_{x,t}}{ds}(s)\right) > 0$) we have

$$\int_0^s f \circ \phi_t^V \circ \phi_r^X \, dr = \int_0^s (\gamma'_{x,t}(r))^{-1} \gamma'_{x,t}(r) f \circ \phi_t^V \circ \phi_r^X \, dr$$

Again integrating by parts

$$\int_0^s f \circ \phi_t^V \circ \phi_r^X \, dr = (\gamma'_{x,t}(s))^{-1} \int_{\gamma_{x,t}|_{[0,s]}} f \, \hat{V} + \int_0^s \frac{\gamma''_{x,t}(r)}{(\gamma'_{x,t}(r))^2} \left(\int_{\gamma_{x,t}|_{[0,r]}} f \, \hat{V} \right) dr$$

Assume now that the curves $\gamma_{x,t}$ are close to orbits of the flow ϕ_t^V. Since $\hat{V}$ is a dual 1-form to V, the integral

$$\int_{\gamma_{x,t}|_{[0,r]}} f\,\hat{V}$$

is bounded when f is a V-coboundary. For general zero-average functions: estimates in terms of ergodic integrals of ϕ_t^V.

Estimates for coboundaries are sufficient to prove mixing (but not effective mixing), local estimates on spectral measures and absolute continuity of the spectral type.

We have thus reduced estimates on correlations to the following:

1. upper bounds on $(\gamma'_{x,t})^{-1} \to 0$ (stretch estimates),
2. upper bounds on $\gamma''_{x,t}/(\gamma'_{x,t})^2 \to 0$ (distortion estimates).

Tangent dynamics. Under the hypothesis that

$$[V, X] = aV$$

we derive that

$$\begin{aligned}\frac{d\gamma_{x,t}}{ds}(s) &= (\phi_t^V)_*(X) \circ \phi_t^V \circ \phi_s^X(x) \\ &= \left(\int_0^t a \circ \phi_\tau^V \circ \phi_s^X(x) d\tau\right) V_{\gamma_{x,t}(s)} + X_{\gamma_{x,t}(s)} .\end{aligned}$$

For instance, if $V = \alpha U$ (time-change the "unipotent" U) and ϕ_s^X is (Haar) volume preserving:

$$[U, X] = 0 \to [V, X] = [\alpha U, X] = -\frac{X\alpha}{\alpha} V \quad (\textit{nilflows}, \textit{surface flows}) ,$$

$$[U, X] = U \to [V, X] = [\alpha U, X] = (1 - \frac{X\alpha}{\alpha}) V \quad (\textit{horocycle flows}) ,$$

Note that $\frac{X\alpha}{\alpha}$ has zero average (non-uniformly parabolic), while $1 - \frac{X\alpha}{\alpha}$ has non-zero average with respect to the ϕ_t^V-invariant measure $\mu = \alpha^{-1}\text{vol}$ (uniformly parabolic).

L. Simonelli's example (DCDS, 2018) Twisted horocycle flow: $V = U + \alpha\Theta$ on $\Gamma\backslash SL(2,\mathbb{R}) \times \mathbb{T}$, hence

$$[V, X] = U - X\alpha\Theta = V - (X\alpha + \alpha)\Theta\,,$$

$$(\phi_t^V)_*(X) = tV - \left(\int_{-t}^{0} (X\alpha + \alpha) \circ \phi_\tau^V(x) d\tau\right)\Theta$$

$$\approx t(V - (\int_M \alpha d\text{vol})\Theta)\,,$$

D. Ravotti's example (CMP, 2019) : $V = U + \beta Z$ with U, Z unipotents in $sl(3,\mathbb{R})$ (Z central): for X unipotent and $c \neq 0$,

$$[V, X] = -(c + X\beta)Z\,, \quad [V, Z] = -(Z\beta)Z$$

hence, under the condition $Z\beta = -V\log\lambda$ (ϕ_t^V has an invariant volume of density λ with respect to the Haar volume), it follows

$$(\phi_t^V)_*(X) = \frac{1}{\lambda}\left(\int_{-t}^{0} -\lambda(c + X\beta) \circ \phi_\tau^V(x) d\tau\right) Z + X$$

Proof of (effective) mixing for time-changes requires lower bounds on ergodic integrals on sets of (effective) large measure. It also requires upper bounds on distortion, hence lower bounds on large sets have to be comparable to (uniform) upper bounds.

For renormalizable flows (such as classical horocycle flows, Heisenberg nilflows) effective lower bounds can be proven by the technique of Bufetov functionals (finitely additive measure on rectifiable curves which provide sharp approximations to ergodic averages along the flow).

For non-renormalizable flows, such as higher step nilflows, growth can be derived for non-trivial time changes (not coboundaries) by a Gottschalk-Hedlund argument. Control of distortion is possible only for special functions (trigonometric polynomials). No effective results are in sight at the moment.

Proof of spectral estimates and properties is always based on estimates on twisted integrals or correlations of coboundaries. In particular, the absolute continuity of the maximal spectral type is derived from the L^2 properties of correlations of coboundaries.

The **maximal spectral type** is absolutely continuous (AC) iff all spectral measures of the one-parameter unitary group induced the flow on L^2 are AC.

The spectral measure μ_f of a function $f \in L^2$ is a complex measure on the real line which is the Fourier transform of the correlation $\langle f \circ \phi_t^V, f \rangle$ (as a function of $t \in \mathbb{R}$).

The simplest (standard) way of proving AC spectral type is to prove that correlations are L^2 for a dense set of functions (coboundaries).

In the uniform parabolic case (time changes of horocycle flows; F.-Ulcigrai, 2012): estimate of correlations of coboundaries by $1/t$ (square integrable function);

In the non-uniform parabolic case (Kochergin flows; Fayad, F., Kanigowski, 2020): correlations of coboundaries are estimated by $(1/t)^{1/2+\eta}$, a square integrable function, only after averaging oscillations at shorter time scales (a small power of t).

The shear is everywhere of order $(1/t)^{1/2-\eta}$ and the resulting estimates (not sufficient) are bootstrapped thanks to equidistribution properties of the rotation (Denjoy-Koksma).

Countable Lebesgue spectrum

Theorem

(Fayad, F., Kanigowski, 2020) Let $\{\phi_t\}$ be a strongly continuous one-parameter unitary group on a separable Hilbert space H with absolutely continuous spectrum. For a fixed $n \in \mathbb{N}$, let us assume that for every compact set $C \subset \mathbb{R} \setminus \{0\}$ of positive Lebesgue measure there exists $\epsilon_{n,C} > 0$ such that the following holds. For every $\epsilon \in (0, \epsilon_{n,C})$ there exist vectors $f_1, \ldots, f_{n+1} \in H$ such that

$$\|\langle f_i \circ \phi_t, f_j \rangle\|_{L^2(\mathbb{R},dt)} \leq \delta_{ij} + \epsilon\,, \qquad \text{for all } i, j \in 1, \ldots, n+1\,;$$

$$\|\prod_{i=1}^{n+1} \mathcal{F}(\langle f_i \circ \phi_t, f_i \rangle)\|_{L^{\frac{2}{n+1}}(C)} > (n+1)!(1+\epsilon)^n \epsilon\,.$$

Then the spectral type of $\{\phi_t\}$ is Lebesgue with multiplicity at least $n+1$.

Construction of functions: $f_1, \dots, f_n$ with near diagonal matrix $\langle f_i \circ \phi_t^V, f_j \rangle$ of correlations with "arbitrary" diagonal terms. Consider very long and thin flow boxes R_J of base a (horizontal) interval J of length $|J| \to 0^+$ and height $T_J \to \infty$ (forward and backward). Fix $T > 0$ and consider functions $f_1, \dots, f_n$ supported on the sub-box $R_J^T \equiv J \times [-T, T] \subset R_J$, of the form

$$f_i \circ \phi_t^V(x) = \chi_i(x) \frac{d}{dt} \psi_i(t) \,, \quad (x, t) \in J \times [-T, T] \,.$$

$\chi_1, \dots, \chi_n$ have compact support in J and are orthogonal;
$\psi_1, \dots, \psi_n$ are even with compact support in $[-T, T]$.
Now for $|t| < T_J$ we have

$$\langle f_i \circ \phi_t^V, f_j \rangle = \delta_{ij} \frac{d^2}{dt^2} (\psi_i * \psi_j)$$

and on $\mathbb{R} \setminus [-T_J, T_J]$ the square integral of correlations is small (for T_J large) by the square summability of correlations for coboundaries. The function $\frac{d^2}{dt^2} \psi_i * \psi_j$ are sufficiently arbitrary to conclude.

THANKS FOR YOUR ATTENTION!